CHAPTER 6

CONDITIONS IMPLYING COMPACTNESS OF AN OPERATOR

In this chapter, we examine the behaviour of an operator satisfying a growth condition with respect to the complement of the unit circle. We discuss various conditions on a quasi-nilpotent or a Riccati operator such that it becomes zero or compact respectively.

We know that a quasi-nilpotent operator $A$ becomes zero under the following conditions:

1. $A$ is spectraloid
2. $A$ is symmetrisable with respect to a positive operator $P$ with dense range i.e. $PA = A^*P$. 
In condition (3), we have to use Reid's inequality \[ 28, \text{ Problem 23} \] to show \( A = 0 \).

An operator \( A \) satisfying condition (2) is not necessarily spectraloid. Thus a non-spectraloid quasi-nilpotent operator may be zero. Thus it is interesting to study various other conditions under which a quasi-nilpotent operator becomes a zero operator.

In this direction, we mention that if a quasi-nilpotent operator satisfies the condition \((G_1)\) then it is spectraloid and hence it becomes a zero operator.

In our first Theorem, we show that a quasi-nilpotent operator satisfying the growth condition with respect to the complement of the unit circle is also a zero operator.

**Theorem G.1**: If a quasi-nilpotent operator \( T \) satisfies the growth condition with respect to the complement of the unit circle then \( T = 0 \).

**Proof**: By hypothesis,

\[
r(T-z)^{-1} \leq |(T-z)^{-1}| \leq \left[ d(z, \sigma'(T)) \right]^{-1} = r(T-z)^{-1} \leq 1
\]

for \(|z| = 1\).
Therefore \( \| (T-z)^{-1} \| = 1 \) for \(|z|=1\)

i.e., \((T-z)^{-1}\) is normaloid for \(|z|=1\).

For a given positive integer \(n\),

if \(1, v_1, v_2, \ldots, v_{n-1}\) are \(n^{\text{th}}\) roots of unity, then

\[(T-I) = (T^{n-1}) (T-v_1)^{-1} (T-v_2)^{-1} \cdots (T-v_{n-1})^{-1}.
\]

Since \(\| (T-v_1)^{-1} \| = 1\) for \(i=1,2,\ldots,n-1\),

\(\| (T-I) \| \leq \| T^{n-1} \| \leq \| T^n \| + 1\).

\(T\) being quasi-nilpotent, \(\| T^n \| \to 0\) as \(n \to \infty\) and \(\| T-I \| \leq 1\).

By taking \(sT\) inverse of \(T\), we also have

\(\| sT-I \| \leq 1\) for \(|s|=1\) and in particular,

\(\| (T+sI) \| \leq 1\).

Therefore \(\| (T+I)x \| \leq \| x \|\) for each \(x \in H\)

or \(2\|Tx\|^2 + 2\|x\|^2 = \| (T-x)I \|^2 + \| (Tx-x) \|^2 \leq \| x \|^2 + \| x \|^2 = 2\| x \|^2\).

This gives \(\|Tx\| = 0\) for each \(x \in H\) or \(T = 0\).

In the above Theorem, the condition \((T-z)^{-1}\) is

normaloid for each point of the unit circle cannot be
relaxed to the condition \((T-z)^{-1}\) is normaloid for some \(z\). This can be seen from the following example:

**Example G.1:** As in [38, Problem 160], if \(V\) is the Volterra integration operator, and if \(T = (1+V)^{-1}\) then \(\sigma(T) = \{1\}\) and \(\|T\| = 1\). Thus \((1+V)^{-1}\) is normaloid for \(z = 1\) only, \(V\) is quasi-nilpotent, even though \(V \neq 0\).

As a direct consequence of Theorem G.1, we have the following result of J.G. Stampfli [60, Theorem 1] as a Corollary:

**Corollary G.1:** For \(T \in \mathcal{B}(H)\), if \(T-z\) is invertible and \(\| (T-z)^{-1} \| \leq 1\) for each complex number \(z\), with \(|z| = 1\) and \(r(T) < 2\) then \(T = 0\).

**Proof:** Since

\[
\left[ d(z, \sigma(T))^{-1} = r(T-z)^{-1} \leq \| (T-z)^{-1} \| \leq 1, \right.
\]

we have \(d(z, \sigma(T)) \geq 1\) for \(|z| = 1\).

Since \(r(T) < 2\) by hypothesis, this gives us that \(T\) is quasi-nilpotent.

Now \(T = 0\) by Theorem G.1.
Corollary 6.2: For \( T, s \in \mathcal{B}(H) \), if \( \| T \| < 2 \), \( s \) a unitary operator and \( \| (T+zs)^{-1} \| \leq 1 \) for each complex number \( z \), with \( |z| = 1 \) then \( T = 0 \).

Proof: Here

\[
\| (s^*T+z)^{-1} \| = \| s^*(T+zs)^{-1} \| = \| (T+zs)^{-1}s \| \leq 1 \text{ for } |z|=1
\]

and \( r(s^*T) \leq \| s^*T \| \leq \| T \| < 2 \).

Thus \( s^*T = 0 \) i.e. \( T = 0 \) by Corollary 6.1.

The next Corollary gives us a condition under which an isolated point in the spectrum of an operator becomes an eigenvalue.

Let \( \alpha \) be an isolated point of the spectrum of an operator \( T \) and let

\[
P = \frac{1}{2\pi i} \int_C R_\alpha \, dz
\]

where \( C \) is a circle with arbitrary small radius containing the point \( \alpha \) of \( \sigma(T) \) only and \( R_\alpha = (T-z)^{-1} \).

Then \( P \) is a spectral projection and \( T \) is invariant under \( R(P) \).
**Corollary 6.2**: For \( A \in \mathcal{B}(H) \), let \( \alpha \) be an isolated point of \( \sigma(A) \) and \( R(P) \) be the range space of the spectral projection \( P \) associated with \( \{ \alpha \} \). If 
\[
T = (A - \alpha)/R(P) \quad \text{and} \quad (T-z)^{-1} \text{ is normaloid for each } z, \\
\text{with } |z| = 1 \quad \text{then } \alpha \text{ is an eigenvalue of } A.
\]

**Proof**: Since \( T \) is quasi-nilpotent \([44]\), \( T \) becomes zero by Theorem 6.1, i.e. \( Tx = Ax = \alpha x \) for \( x \in R(P) \).

This shows that \( \alpha \in \sigma_p^-(A) \).

Using the notion of spectral sets, we have

**Theorem 6.2**: If \( T \) is quasi-nilpotent and \( \{ 0, 1 \} \) is a spectral set for \( T \) then \( T = 0 \).

**Proof**: We have

\[
\| (2T-I) \| = \sup \{ \| 2z-1 \| : z = 0 \text{ or } 1 \} = 1
\]

and

\[
\| (2T-I)^{-1} \| = \sup \{ \| (2z-1)^{-1} \| : z = 0 \text{ or } 1 \} = 1.
\]

Therefore \( 2T-I \) is unitary by Theorem 3-F \([59]\)

i.e. \( T \) is normal. Now \( T \) being quasi-nilpotent and normal, \( T = 0 \).

Theorem 6.1 being valid in \( C^* \)-algebra, it is applicable in the Calkin algebra. Here we have the following analogous results:
Theorem 6.3: For $T \in \mathcal{B}(H)$ and for each complex number $z$, with $|z| = 1$,

1. if $T$ is a Riesz operator and $\| (T-z)^{-1} \| = 1$ then $T$ is compact,

2. if $\rho_0(T) < 2$, $z \in \sigma_0(T)$ and $\| (\hat{T}-z)^{-1} \| = 1$ then $T$ is compact,

3. if $\| (\hat{T}-z\hat{\delta})^{-1} \| = 1$, $\| \hat{T} \| < 2$ and $\hat{\delta}$ is unitary then $T$ is compact.

As shown in [61, Theorem 4.3], if $T$ is polynomially compact such that every invariant subspace of $T$ reduces $T$ then $T$ is normal. Using this, we have

Corollary 6.4: Let $p$ be a complex polynomial whose constant term has modulus less than 2. Let $T$ be a Riesz operator such that $\| (p(\hat{T})-z)^{-1} \| \leq 1$ for each complex number $z$, with $|z| = 1$ and every invariant subspace of $T$ reduces $T$. Then $T$ is normal.

Proof: Let $a$ be a constant term of the polynomial $p(z)$. Therefore $|a| < 2$ and $\sigma_0(T) = \{0\}$ by hypothesis.
Now by spectral mapping Theorem,
\[ \sigma_e(\mathfrak{h}(T)) = \mathfrak{p}(\sigma_e(T)) = \mathfrak{p}(0) = \{0\} \]
This gives us \( r_0(\mathfrak{p}(T)) < 2 \).

Using Theorem 6.3, \( \mathfrak{p}(T) \) is compact.

Normality part will be followed from [S1, Theorem 4.3].

Since the numerical radius \( w \) on \( \mathcal{B}(\mathcal{H}) \) is a norm and discontinuous [25, Problem 173, 175], the condition \( w(AB) \leq w(A)w(B) \) for \( A, B \in \mathcal{B}(\mathcal{H}) \) is not valid [25] and also \( w_\rho \) for \( \rho > 2 \) is not a norm, it is not possible to apply directly the Theorem 6.1 for spectraloid or \( \rho \)-oid operators. Thus it will be interesting to investigate whether the normaloid condition in Theorem 6.1 can be replaced by \( \rho \)-oid for \( \rho > 2 \). In this direction, we give below a partial answer:

**Theorem 6.4**: If \( T \) is nilpotent operator of order \( n \) and \( w(T-\mathfrak{r})^{-1} \leq 1 \) for some complex number \( \mathfrak{r} \) with \( |\mathfrak{r}| = 1 \) then \( T = 0 \).

To prove this result, we need the following Lemma:
Lemma 6.1: If $T$ is a non-zero nilpotent operator of order $n > 1$, then $0$ is an interior point of $W(T)$.

Proof: Since $T^n x = T(T^{n-1} x) = 0$ for all $x \in \mathbb{H}$, we have $0 \in \sigma(T)$ and $N(T^n) = H$.

In contrary, if $0$ is a boundary point of $W(T)$, then $0$ becomes a normal eigenvalue of $T$ [27]. This gives us that ascent of $T$ is 1 [51, Lemme 3.1, p.574] or $N(2) = N(T^2) = N(T^n) = H$. i.e. $T = 0$, a contradiction.

This completes the proof.

Proof of Theorem 6.4:

Without loss of generality, we can take $a = 1$.

Let $T \neq 0$ and $w(T-1)^{-1} \leq 1$.

Since $T^n = 0$, we have

$-I = T^n - I = (T-I)(I+B)$

where $B = T + T^2 + \ldots + T^{n-1}$.

By spectral mapping Theorem, $\sigma(B) = \{0\}$. Also $B^n = 0$.

Therefore, $w(I+B) = w(I-T)^{-1} \leq 1$. 
If $B \neq 0$ then by Lemma 6.1, 0 is an interior point of $W(B)$. Hence $w(B) \neq 0$ and $w(I+B) > 1$.

This contradiction gives us $w(B) = 0$ i.e. $B = 0$.

Therefore, 

$$-I = (T-I)(I+B) = T-I$$

i.e. $T = 0$.

Using the fact that if $T$ is quasi-nilpotent operator in $Q(k)$, the class of $k$-quasi hyponormal operators, then $T^k = 0$ [24], we derive

**Corollary 6.6**: If $T \in Q(k)$ is quasi-nilpotent and $w(T-z)^{-1} \leq 1$ for some $z$, with $|z| = 1$ then $T = 0$.

**Proof**: Since $T^k = 0$ by [24], $T$ becomes zero by Theorem 6.4.