where $\sigma(n)$ denotes as usual the sum of the positive divisors of $n$.

Since $i_k$ is multiplicative for all non-negative $k$, by theorem 1.1.3 of chapter 1 we observe that $\tau^*(n)$ and $\sigma^*(n)$ are multiplicative.

Hence for $n > 1$ with $n = p_1^{s_1} \cdots p_r^{s_r}$,

$$\tau^*(n) = \prod_i \tau^*(p_i^{s_i}) = s_1 \cdots s_r \tag{3.1.1}$$

and

$$\sigma^*(n) = \prod_i \sigma^*(p_i^{s_i}) = \prod_i p_i \sigma(p_i^{s_i-1})$$

$$= \gamma(n) \sigma\left(\frac{n}{\gamma(n)}\right). \tag{3.1.2}$$

More generally if $\sigma_k^*(n)$ denotes the sum of the $k$th powers ($k \geq 0$) of the core divisors of $n$, then it is easy to prove that

$$\sigma_k^*(n) = \left\{ \gamma(n) \right\}^k \sigma_k\left(\frac{n}{\gamma(n)}\right), \tag{3.1.3}$$

where $\sigma_k(n)$ denotes the sum of the $k$th powers of the divisors of $n$.

It is obvious that $\tau^*(n) \geq 1$ when $n > 1$, while $\tau^*(n) = 1$ if $n$ is a prime.

Hence $\lim_{n \to \infty} \tau^*(n) = 1$.
We recall that if \( \tau(n) \) denotes the number of positive divisors of \( n \), then \( \tau(n) = o(n^{\delta}) \) for all positive \( \delta \). Clearly \( \tau^*(n) \leq \tau(n) \) for all \( n \) and hence

\[
(3.1.4) \quad \tau^*(n) = o(n^{\delta}).
\]

In fact we shall prove (3.1.4) explicitly in chapter 6.

With the aid of the prime number theorem we can prove

Theorem 3.1.1.

\[
\limsup_{n \to \infty} \frac{\log \tau^*(n)}{\log n} = \frac{\log 3}{3}.
\]

This result was proved and presented at the Indian Mathematical Society Conference, Kanpur [48]. Later through the Mathematical reviews, we noted that the same result in another terminology was independently proved by Knopfmacher [25].
Let \( h(t) \) be an arithmetic function which satisfies the condition that \( h(t) < t^\delta, \delta > 0 \). Define an arithmetic function \( H(n) \) as

\[
H(1) = 1 \\
H(n) = h(s_1) h(s_2) \ldots h(s_r),
\]

when \( n > 1 \) has the canonical form \( p_1^{s_1} \ldots p_r^{s_r} \).

Let \( M \) be the maximum value of the function

\[
\left\{ \frac{1}{h(x)} \right\}
\]

for all positive \( x \). With the aid of prime number theorem we can prove

**Theorem 3.1.2**

\[
\lim_{n \to \infty} \sup \frac{\log H(n) \log \log n}{\log n} = \log M.
\]

This result includes many well known results. For example

- if \( h(x) = x+1 \) then \( H(n) = \tau(n) \), the number of divisors of \( n \) and theorem 3.1.2 reduces to theorem 317 in [24].
- If \( h(x) = x \), then \( H(n) = \tau^*(n) \), number of core divisors of \( n \) and theorem 3.1.2 reduces to theorem 3.1.1 above.
When this result was referred to him, Professor M.V. Subbarao [40] brought [46] to our notice. In [46], D. Suryanarayana and R. Sitaramachandra Rao proved this theorem 3.1.2 in a similar form. In a note added in [46] at the time of proof correction, the authors point out that a generalization of the present theorem has been published independently by E. Heppner [Arch. Math. ..., 24 (1973) 63-66; M.R 47 # 8462]. Heppner's proof used less elementary methods. Through the recent Mathematical Reviews [M.R 52 # 310], we note that the present theorem (based on elementary methods) together with various other specific applications has also been published independently by J. Knopfmacher [26].

We have followed the method in [9, (pages 19-21)] to prove theorem 3.1.2. However, we do not give the details of the proof here.
In [38], Subbarao introduced exponential divisors of a positive integer \( n \). A divisor \( d \) of \( n = p_1^{s_1} \cdots p_r^{s_r} \) is said to be an exponential divisor of \( n \) if \( d = p_1^{a_1} \cdots p_r^{a_r} \) where \( a_i \) divides \( s_i \), \( i = 1, \ldots, r \). Denote the number of such divisors of \( n \) by \( \tau^e(n) \).

Then \( \tau^e(1) = 1 \)

\[ \tau^e(n) = \tau(s_1) \cdots \tau(s_r). \]

We conclude this section by obtaining a result in connection with the normal order of \( \tau^e(n) \), which we believe is new and has not appeared so far anywhere in the literature.

**Definition:**

The normal order of \( f(n) \) is \( F(n) \) if

\[ (3.1.5) \quad (1-\varepsilon) F(n) < f(n) < (1+\varepsilon) F(n). \]

for every positive \( \varepsilon \) and almost all (i.e. all but a finite) values of \( n \).

It is proved in \([24, \text{Theorem 431}]\) that the normal order of \( \omega(n) \), the number of different prime factors of \( n \) is \( \log \log n \).
Also it is proved in [24, (Theorem 432)] that if $\epsilon$ is positive, then

\[(3.1.6) \quad 2^{(1-\epsilon) \log \log n} < \tau(n) < 2^{(1+\epsilon) \log \log n}\]

for almost all numbers $n$.

With the aid of (3.1.6) we prove

Theorem 3.1.3

If $\delta$ is positive, then

\[(3.1.7) \quad 2^{(1-\delta) Y} < \tau^e(n) < 2^{(1+\delta) Y}\]

for almost all numbers $n$, where

\[Y = \log \log \log n \log \log n - \log \log 2.\]

Proof:

If $n = p_1^{s_1} \cdots p_r^{s_r}$

\[\tau^e(n) = \tau(s_1) \cdots \tau(s_r).\]

From (3.1.6) we get

\[(3.1.8) \quad 2^{(1-\epsilon) \log \log s_1} < \tau(s_1) < 2^{(1+\epsilon) \log \log s_1}\]
It is clear that $n \leq p_1 \leq 2^{s_1}$.

Therefore $s_1 \leq \frac{\log n}{\log 2}$

$\log s_1 \leq \log \log n - \log \log 2$

$= \log \log n \left(1 - \frac{\log \log 2}{\log \log n}\right)$

Since $\log \log 2$ is positive number, denote it by $x$.

Then we get

$\log s_1 \leq \log \log n \left(1 + \frac{x}{\log \log n}\right)$

$\log \log s_1 \leq \log \log \log n + \log \left(1 + \frac{x}{\log \log n}\right)$,

And since $\log(1+t) < t$ for $t > 0$, we get

$\log \log s_1 < \log \log \log n + \frac{x}{\log \log n}$

$= Z$ (say)

Hence from (3.1.8) we have

$2^{(1-\epsilon)^2 Z} \leq \tau(s_1) \leq 2^{(1+\epsilon) Z}$
Similarly we can prove that

\[
\frac{(1-\epsilon)^2}{2} \cdot \omega(n) < \tau(s_2) < \frac{(1+\epsilon)^2}{2} \cdot \omega(n)
\]

and so on.

Combining we get

\[
\frac{(1-\epsilon)^2}{2} \cdot \log \log n \cdot \omega(n) < \tau^3(n) < \frac{(1+\epsilon)^2}{2} \cdot \log \log n \cdot \omega(n)
\]

Since the normal order of \( \omega(n) \) is \( \log \log n \) we have

\[
\frac{(1-\epsilon)^3}{2} \cdot \log \log n < \tau^3(n) < \frac{(1+\epsilon)^2}{2} \cdot \log \log n
\]

Now we can find \( \delta > 0 \) for a given \( \epsilon > 0 \) such that \( (1-\delta) < (1-\epsilon)^3 < (1+\epsilon)^2 < (1+\delta) \).
Hence we get

\[ (1- \delta) \log \log n \quad < \quad \tau^6(n) \quad < \quad (1+ \delta) \log \log n, \]

And \( Z \log \log n = \{ \log \log \log n + \frac{x}{\log \log n} \} \log \log n \]
\[ = \{ \log \log \log n - \frac{\log \log 2}{\log \log n} \} \log \log n \]
\[ = \log \log \log n \log \log n - \log \log 2 \]
\[ = Y. \]

\[ (1- \delta) Y \quad < \quad \tau^6(n) \quad < \quad (1+ \delta) Y \]

Hence the proof.

Thus \( \tau^6(n) \) is about \( 2 \log \log \log n \log \log n - \log \log 2. \)

We cannot simply say that "the normal order of \( \tau^6(n) \) is the above quantity", since the inequalities (3.1.7) are of a less precise type than (3.1.5). So more roughly one may say that the normal order of \( \tau^6(n) \) is about \( \log \log \log n \log \log n - \log \log 2. \)
Section 2.

Call \( n \) core perfect (or simply \( c \)-perfect) if \( \sigma^*(n) = 2n \). A few examples of such numbers are

\[ 2^2 \cdot 3^2, \quad 2^2 \cdot 5^2, \quad 2^3 \cdot 7^2, \quad 2^5 \cdot 31^2, \quad 2^7 \cdot 127^2, \quad 2^{11} \cdot 2047^2. \]

Note that if \( n \) is squarefree, \( \sigma^*(n) = n \). Hence if \( m \) is \( c \)-perfect and \( n \) is squarefree with \( (m,n) = 1 \), then \( mn \) is also \( c \)-perfect. Thus it is sufficient to consider only squareful \( c \)-perfect numbers.
Theorem 3.2.1

A necessary and sufficient condition that $n$ be an even squareful $c$-perfect number is that $n = 2^p (2^p - 1)^2$ where $2^p - 1$ is prime.

Proof:

Sufficiency:

\[ n = 2^p (2^p - 1)^2 \]
\[ = 2^p k^2 \quad \text{where} \quad k = 2^p - 1. \]

\[ \sigma^*(n) = \sigma^*(2^p) \sigma^*(k^2) \]
\[ = 2 \sigma(2^p - 1) k \sigma(k) \]
\[ = 2 (2^p - 1) k (k + 1) \]
\[ = 2 k^2 2^p. \]

Hence $n$ is squareful $c$-perfect number.
Necessary part:

Suppose \( n \) is squareful c-perfect and is even. The prime decomposition of \( n \) is of the type

\[
n = 2^s \prod_{i=1}^{r} p_i^{s_i}
\]

= \( 2^m \) (say) where \( m \) is odd.

Clearly \( (2^s, m) = 1 \)

Now \( \sigma^*(n) = \sigma^*(2^s) \sigma^*(m) \)

i.e. \( 2n = 2(2^s - 1) \sigma^*(m) \)

i.e. \( \sigma^*(m) = \frac{2^s m}{2^s - 1} \)

\( m + \frac{m}{2^s - 1} \)

Since \( \sigma^*(m) \) and \( m \) are integers, \( \sigma^*(m) - m \) is an integer and let it be \( d \).

i.e. \( m = d(2^s - 1) \) which implies that \( d \) divides \( m \).
Thus \( \tau(m) = m + d \) where \( m \) and \( d \) are divisors of \( m \).

But \( m \) is the sum of the core divisors of \( m \).

Any prime divisor of \( m \) is a divisor of the core divisors of \( m \).

Thus \( \tau(m) = m + d \) where \( m \) and \( d \) are divisors of \( m \).

Suppose squareful \( m \) is given by

Thus \( \tau(m) = m + d \) where \( m \) and \( d \) are divisors of \( m \).

Any prime divisor of \( m \) is a divisor of the core divisor of \( m \) and hence a divisor of the sum of the core divisors of \( m \). That means every prime divisor of \( m \) is a divisor of the core divisors of \( m \) and hence a divisor of the sum of the core divisors of \( m \).

For just two core divisors \( m \) and \( d \), so \( m \) is squareful.

Thus \( \tau(m) = m + d \) where \( m \) and \( d \) are divisors of \( m \).
That is \( m = d (2^s - 1) = p^2 \), \( p \) is a prime. This gives us \( d = (2^s - 1) = p \) since \( 2^s - 1 \neq 1 \) for \( s > 1 \).

Hence \( n = 2^s (2^s - 1)^2 \) where \( 2^s - 1 \) is a prime.

Thus the proof of the theorem is over.

Theorem 3.2.2

Each squareful even \( c \)-perfect number is a multiple of an even perfect number where \( n \) is perfect if \( \sigma(n) = 2n \).

Proof:

Let \( n = 2^p (2^p - 1)^2 \) be a squareful even \( c \)-perfect number.

Now \( n = 2^p (2^p - 1)^2 \)

\[ = (2^p - 1) (2^p - 1) 2^p \]

\[ = 2 2^{p-1} (2^p - 1) (2^p - 1) \]

\[ = 2 (2^p - 1) k \]

where \( k = 2^{p-1} (2^p - 1) \) is an even perfect number.

Hence the proof.
(3.2.1) Remark:

Regarding the existence or otherwise of odd c-perfect numbers Prof. M.V. Subbarao [39] observes that "Theorem 3.2.1 says that $n$ is core perfect iff $m$ is perfect where $m = \frac{n}{\gamma(n)}$. The question of the existence of odd c-perfect numbers is thus linked up with that of the existence of odd perfect numbers."