In this chapter we characterize the completely multiplicative arithmetic functions by using Core Convolutions 1 and 2.

We prove the following results (Vide [51]):

1. Let \( f(1) \neq 0 \) and suppose that \( g \ast h(m) \neq 0 \) for all squareful \( m \). Then \( f \) is completely multiplicative iff \( f(g \ast h) = fg \ast fh \).

2. Let \( f,g,h \in A \) and suppose that \( g \ast h(m) \neq 0 \) for all squareful \( m \). Then \( f \) is completely multiplicative iff \( f(g \ast h) = fg \ast fh \).

Finally we denote the product in (1.2.5) by \( f \cup g(n) \) and prove the following:

3. Let \( f(1) \neq 0 \) and suppose that \( g \cup h(n) \neq 0 \) for all \( n \). Then \( f \) is multiplicative iff \( f(g \cup h) = fg \cup fh \).

We refer analogous work of Longford [29], at the end of the chapter.
Theorem 2.1

Let \( f(1) \neq 0 \) and suppose \( g \) and \( h \) be arithmetic functions such that \( g * h (m) \) does not vanish for squareful \( m \). Then \( f \) is completely multiplicative iff \( f(g \circ h) = fg \circ fh \).

Proof:

Let \( f \) be completely multiplicative.

Then \( f(g \circ h)(n) = f(n) \sum_{ab = n, \gamma(a) = \gamma(n)} g(a) h(b) \)

\[ = \sum_{ab = n, \gamma(a) = \gamma(n)} f(a) g(a) f(b) h(b) \]

\[ = fg \circ fh(n) \]

Conversely suppose \( f(1) \neq 0 \) and \( f(g \circ h) = fg \circ fh \).

We show first that necessarily \( f(1) = 1 \).

\[ f(1) g * h(1) = f(1) g(1) h(1) \]

\[ = f(1) (g \circ h)(1) \]

\[ = f(g \circ h)(1) \]

\[ = fg \circ fh(1) \]

\[ = f(1)^2 g(1) h(1) \]

\[ = f(1)^2 g * h(1) \]

and since \( f(1) g * h(1) \neq 0 \), it follows that \( f(1) = 1 \).