Chapter II

SOME STATIONARY COSMOLOGICAL SOLUTIONS FOR MATTER

1. Some Properties of the Godel-metric

The possibility of the existence of stationary cosmological solutions, within the framework of general relativity theory, was first demonstrated by Godel (1949). Using the field equations of general relativity theory, that is,

\[ R_{ij} - \frac{1}{2} R g_{ij} + \kappa g_{ij} = -8\pi T_{ij}, \]

it was shown that the following metric is compatible with a perfect fluid distribution:

\[ ds^2 = (dx^0)^2 + 2e^{ax^1}(dx^0)(dx^2) + \frac{1}{2} e^{2ax^1}(dx^2)^2 - (dx^1)^2 - (dx^3)^2 \]

where \( a \) is a constant. This problem was worked out with
reference to the so-called co-moving coordinate system, 

Therefore, for the Godel-metric (2.1.2), the four-velocity 
of the fluid has the components

(2.1.3) \[ v^1 = (1, 0, 0, 0) \].

The proper density and pressure have the expressions

(2.1.4) \[ \rho = \frac{a^2}{16\pi} - \frac{\lambda}{8\pi} \]

\[ p = \frac{a^2}{16\pi} + \frac{\lambda}{8\pi} \].

If \( \lambda \) is chosen to be \(-\frac{a^2}{2}\), one obtains an incoherent 
matter distribution.

For this solution, the matter of the universe is at 
rest with respect to the coordinate system used and this 
matter, in particular, can be chosen to be the dust. 
However, it is known that with respect to the same coor-
dinate system, choosing the cosmological fluid to be the dust, 
one can obtain what is known as Einstein's static universe 
from the field equations (2.1.1). Thus we infer that for 
the same \( T^{ij} \) there are two basically different solutions 
of the field equations (2.1.1).
An equation of the form \( f(x^i) = 0 \) determines a hypersurface in a space-time. For any displacement in this hypersurface we have

\[
(2-1.5) \quad f_{,j} \, dx^j = 0 .
\]

Consequently the quantities \( f_{,j} \) are the covariant components of the vector-field of normals to the hypersurface \( f(x^i) = 0 \). The condition that the hypersurfaces \( f_1(x^i) = 0 \) and \( f_2(x^i) = 0 \) be orthogonal at each common point is

\[
(2-1.6) \quad g^{ij}(f_1),_i \, (f_2),_j = 0 .
\]

Consider the hypersurfaces \( x^h = \text{const} \) and \( x^k = \text{const} \). A necessary and sufficient condition that these hypersurfaces be orthogonal at each common point is

\[
(2-1.7) \quad g^{hk} = x^h,i \, x^k,j \, g^{ij} = 0 .
\]

If this does not hold in the coordinate system \( x^i \), to see whether a new coordinate system can be so chosen that \( g^{hk} = 0 \), we proceed as follows:

The differential equation \( g^{ij} f_0,i \, f_{,j} = 0 \) where
$f^0 = f^0(x^m)$ is any real function, admits three independent solutions for $f$. If $f^1, f^2$ and $f^3$ denote these solutions and if we introduce new coordinates defined by $x^* = f^i$, then the equations $g^{ij} f^0, i f^A, j = 0$, in the new coordinate system become

$$(2-1.8) \quad g^{ij} x^0, i x^A, j = 0.$$ 

This implies that $g^{0A} = 0$. Since the invariant $g^i \neq 0$ and $|g^i_j| = 1$, from the identity $g^{0i} g^i_j = \delta^0_j$ it follows that $g^{00} \neq 0$ and $g^{01} = g^{02} = g^{03} = 0$.

Now the condition that there exist in a space-time four families of hypersurfaces $x^m = \text{const}$ such that every two hypersurfaces $x^h = \text{const}, x^k = \text{const}$ ($h \neq k$) are orthogonal at every point is that the six simultaneous differential equations $g^{ij} x^h, i x^k, j = 0$, where $h \neq k$, admit four solutions.

In general this is not possible. When it is possible we say that the space-time admits a 4-tuply orthogonal system of hypersurfaces. Then using these hypersurfaces $x^m = \text{const}$ to define a new coordinate system and following the procedure described above, we can show that the metric of the space-time, in these new coordinates, becomes
diagonal. (Eisenhart, L.P., 1949). This situation can not be obtained with respect to the space-time corresponding to the Godel universe and hence, its metric (2-1.1) can not be diagonalized.

In the space-time of Godel's universe there is no possibility of a coordinate system for which there is a distinguished universal time-coordinate and with respect to which the matter constituting the universe is at rest. This is because the velocity four-vector \( v^i = (1, 0, 0, 0) \), which constitutes the unit tangent vector-field to the world-lines of matter at rest in the co-moving coordinate system \( x^i \), can not be, in the space-time of Godel's universe, everywhere orthogonal to a one-parameter family of hypersurfaces. This can be demonstrated as follows:

Suppose that in the space-time of Godel's universe there is a family \( F \) of one-parametric hypersurfaces \( f(x^i) = \alpha \) where \( \alpha \) is a parameter. For a given value of \( \alpha \), from this equation, we obtain a member of \( F \). This member contains a world point \( x^i \) and obviously \( f'_i \) are covariant components of a normal vector to it at this point. Therefore, any arbitrary vector-field \( v^i \) which is everywhere orthogonal to the members of \( F \) must be given by
where $m$ is an arbitrary scalar function. Now, from an arbitrary vector $v_i$ let us construct the completely antisymmetric tensor

$$
(2-1.10) \quad a_{ijk} = \frac{1}{2} [v_i(v_{j,k} - v_{k,j}) + v_j(v_{k,i} - v_{i,k}) + v_k(v_{i,j} - v_{j,i})].
$$

For the special case of vector-field $v_i$ given by (2-1.9), this tensor is easily seen by direct calculation to be identically zero. Thus a covariant necessary condition that a vector-field $v_i$ be everywhere orthogonal to a one-parameter family $F$ of hypersurfaces is

$$
(2-1.11) \quad a_{ijk} = 0.
$$

(Eisenhart, L.P., 1949).

In the case of the Godel solution which is derived with respect to the co-moving coordinate system, the velocity four-vector of the material particles is

$$
(2-1.12) \quad v^i = (1, 0, 0, 0) \quad \text{or} \quad v^i = (1, 0, e^{ax^1}, 0).
$$

Obviously for this vector
(2-1.13) \[ v_{j,k} = \begin{cases} a e^{ax} & \text{for } j = 2, k = 1 \\ 0 & \text{otherwise} \end{cases} \]

and consequently the tensor \( a_{ijk} \) is

(2-1.14) \[ a_{ijk} = \begin{cases} -a e^{ax} & \text{for even permutation of } 0, 1, 2 \\ a e^{ax} & \text{for odd permutation of } 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \]

which shows that the tensor \( a_{ijk} \) does not vanish identically. Thus we have shown that the world-lines which characterize matter at rest in the co-moving system, in the Godel universe, can not be everywhere orthogonal to a one-parameter family of hypersurfaces of the space-time. In fact, if there existed such a family of hypersurfaces, one can mark off the intervals of matter geodesics between hypersurfaces and thereby construct a coordinate system which is not only co-moving but Gaussian also, in a manner similar to that used in deriving the Robertson-Walker model of the universe. Then the intervals of the matter geodesics can be used to define a universal time coordinate. But
because this is impossible in the case of Gödel's solution, there is no possibility of having a universal time coordinate. (Adler, A., Bazin, M., and Schiffer, M., 1965).

However, one can show that at every point in Gödel's universe, a positive direction of time can consistently be introduced. This is because the totality of time-like and null vectors can be divided into positive and negative vectors in such a way that if, at a given world point, \( l^i \) is a positive vector then \(-l^i\) is a negative vector. This division can be obtained by defining \( l^i \) to be a positive or a negative vector according as the inner product \( \varepsilon_{ij} l^i v^j \) is positive or negative.

For demonstrating the rotational character of matter in Gödel's universe, we use the tensor \( a_{ijk} \) given in (2-1.10). Using \( a_{ijk} \), the local angular velocity of matter relative to the compass of inertia can be represented by the following vector \( w^h \) (which is orthogonal to \( v^i \)):

\[
(2-1.15) \quad w^h = \frac{\varepsilon^{hijk}}{6(-g)^{1/2}} a_{ijk} = \frac{\varepsilon^{hijk}}{(-g)^{1/2}} w_{ij} v_k
\]

where the skew symmetric tensor \( w_{ij} \) is given by
The fact that the $w^h$ represents the angular velocity relative to the compass of inertia is seen as follows: In a local inertial frame with coordinates $x^i$, for which at the origin the matter is at rest, one obtains the following three dimensional quantities:

$$w_{AB} = \frac{1}{2} (v_{A,B} - v_{B,A})$$

where $\varepsilon_{ABC}$ is a totally skew symmetric tensor with $\varepsilon_{123} = 1$ and $v_A$ denotes the components of the three dimensional velocity vector which can be determined by the local transformations

$$n_{hk} = x^i, h x^j, k g_{ij}$$

Here and in what follows in the present thesis, $n_{hk} = \# \text{diag}(1, -1, -1, -1)$. 

The meaning of the expression 'compass of inertia' may be explained as follows: Consider an observer sitting in a laboratory of small mass which is situated in an otherwise empty space. This observer may use the asymptotically Minkowskian coordinate system fixed to the laboratory. Because the laboratory has small mass, its effect on the metric can be considered in the weak-field approximation. According to general relativity, the observer would observe that his instruments behave in accordance with the usual laws of physics. On the other hand, within the framework of general relativity, the observer might set his laboratory rotating by leaning out of a window and firing a 22-calibre rifle tangentially. Soon after the laboratory is set rotating in this manner, a delicate gyroscope inside it would start pointing in a direction almost fixed relative to the direction of motion of the receding bullet and it would rotate relative to the walls of the laboratory. Thus it appears that the small and distant bullet is more important than the massive and nearby walls of the laboratory in determining inertial frames of reference. Obviously, what is considered here better fits with an absolute space in the sense of Newton rather than with a physical space in the sense of Mach. The fixed direction indicated by the gyroscope may be called the 'compass of inertia'. (Dickey,
One may regard the distant galaxies as determining the compass of inertia with reference to the planetary system. According to the views of Mach, these two, i.e., the distant galaxies and the planetary system can not rotate relative to each other. However, any rotating cosmological solution possesses the intrinsic rotation relative to the compass of inertia and hence, such solutions are known as anti-Mach solutions. In view of what is described in the last paragraph, because of the property of the intrinsic rotation a rotating cosmological solution could be used to explain a small difference in the constant of precession, depending on whether it is derived from the observations of the distant galaxies or the planetary system. (Schucking, B., and Heckmann, O., 1958).

We note that because the covariant velocity vector $v_i$ of the co-moving matter field in the Robertson-Walker universe has the same components as the contravariant vector $v^i = (1, 0, 0, 0)$, in this model of the universe, $a_{ijk}$ and $w^h$ are obviously zero. However, for the Godel universe, $a_{ijk}$ is given by (2-1.14). Hence, by direct calculations from (2-1.2) and (2-1.15), one finds that
Because of this, we are led to believe that the co-moving matter in the Godel universe possesses a constant intrinsic angular velocity of magnitude $w = (\sqrt{2}a)$.

A few interesting points to be noted about the Godel metric are the following: First we note that if we put $a = 0$ in (2-1.2) we can transform it into the flat-metric. (Here it should be noted that $a = 0$ implies $\chi = \rho = 0$ also). Thus $a$ can be considered as a parameter which measures the deviation of the space-time from the flatness. Next, from (2-1.2) it can be observed that in the neighbourhood of the origin of the coordinate system, this line element may be approximately written as

$$ds^2 = (dx^0)^2 + 2(dx^0)(dx^2) + \frac{1}{2}(dx^2)^2 - (dx^1)^2 - (dx^3)^2.$$ 

Putting

$$dx^0 + dx^2 = dx^0, \quad dx^1 = dx^1, \quad dx^2 = (\sqrt{2})dx^2, \quad dx^3 = dx^3,$$

this can be written as

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

and putting
(2-1.23) \((\sqrt{2})dx^0 + \frac{1}{\sqrt{2}}dx^2 = d\bar{x}^0, \quad dx^1 = d\bar{x}^1, \quad dx^0 = d\bar{x}^2, \quad dx^3 = d\bar{x}^3\),

the same can be written as

(2-1.24) \[ ds^2 = (d\bar{x}^0)^2 - (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2. \]

Besides this, because of the homogeneity of the space-time of the Godel universe, one can choose the origin of the co-moving coordinate system in an arbitrary manner. Consequently, from the above consideration it is seen that, in (2-1.20), one can either regard \(x^1, x^2\) and \(x^3\) as space coordinates and \(x^0\) as time coordinate or \(x^1, x^0\) and \(x^3\) as space coordinates and \(x^2\) as time coordinate.

A flat space of \(m\) dimensions, \(S_m\), may be denoted by the fundamental form

(2-1.25) \[ dl^2 = \sum_{p=1}^{m} C_p(dz^p)^2 \]

where \(C_p = \pm 1\) (for \(p = 1, 2, \ldots, m\)). The coordinates \(z^p\) (\(p = 1, 2, \ldots, m\)) are called Cartesian coordinates. In particular, when all of the \(C\)'s are plus one, \(S_m\) is a Euclidean space of \(m\) dimensions. In order that a Riemannian space \(V_n\) of \(n\) dimensions with the fundamental form
(2-1.26) \[ ds^2 = \varepsilon_{RQ} \, dx^R \, dx^Q \]  

be a real sub-space of \( S_m \), it is necessary and sufficient that the system of equations

(2-1.27) \[ \sum_{P=1}^{m} g_{PQ} \frac{\partial z^P}{\partial x^R} \frac{\partial z^P}{\partial x^Q} = \varepsilon_{RQ} \]

admit \( m \) independent real solutions

(2-1.28) \[ z^P = z^P(x^1, x^2, \ldots, x^n), \quad (P = 1, 2, \ldots, m). \]

(Eisenhart, L.P., 1949). It has been shown by J. Rosen (1965) that the Godel metric can be embedded into a flat space of ten dimensions.

Now we shall discuss the symmetry properties of the space-time of the Godel universe. The hidden symmetries of a space-time continuum determine those of its metric tensor. The existence of a hidden symmetry of a metric tensor leads us to demand that the metric tensor remains unchanged under a continuous one-parameter group of transformations

(2-1.29) \[ \bar{x}^i = \bar{x}^i(x^j, y) \]
where \( y \) is the parameter of the group, the condition for this is

\[(2-1.30) \quad g_{mn}(x^n) \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^n} f = g_{ij}(x^k).\]

This leads us to postulate the existence of a vector-field \( l^i \) described by an infinitesimal group of transformations or motions of the space-time into itself, i.e.,

\[(2-1.31) \quad \delta x^i = x^i + l^i \delta y \quad \text{or} \quad \delta x^i = l^i \delta y,
\]

and satisfying a system of differential equations

\[(2-1.32) \quad g_{mn, i} l^i + g_{in, m} l^i + g_{mi, n} l^i = 0\]

which are known as Killing's equations. The vector-field \( l^i \) is called the Killing vector-field and the components \( l^i \) are called the generators of the infinitesimal group of motions (2-1.31).

However, so far as the symmetry properties which are expressible by a continuous group of transformations (2-1.29) are concerned, we may very well replace the finite equations (2-1.30) by the linear homogeneous equations...
(2-1.32), i.e., the Killing equations involving only the
generators $l^i$ of the group in addition to the $g_{ij}$.
Now the fundamental result is that the four dimensional
Riemannian space having some symmetries can have at most
ten-parameter group of motions into itself. When this is
the case, this Riemannian space possesses constant
Riemannian curvature. (Eisenhart, L.P., 1949). Now,
applying this result to space-time which is regarded to
be four dimensional Riemannian continuum, we see that the
universe can at most have ten symmetries of the type
considered, and corresponding to these symmetries there
can be a group of motions with at most ten generators
$(\xi)^i (\xi = 1, 2, \ldots, 10)$.

In the case of the Godel metric, to understand the
symmetries of the type considered above, we should obtain
the Killing equations and solve them. In this case these
equations become

\begin{align*}
(2-1.33) \quad & l^0,0 + e^{ax^1} l^2,0 = 0 \\
& l^1,1 = 0 \\
& ae^{ax^1} l^1 + 2 l^0,2 + e^{ax^1} l^2,2 = 0
\end{align*}
These equations have following five independent solutions
(6) i (6 = 1, 2, 3, 4, 5):

(2-1.34) \[ l^1_i = \left( \frac{\partial x_i}{\partial y_1} \right)_{y_1=0} = (1,0,0,0) \]

(2) \[ l^1_i = \left( \frac{\partial x_i}{\partial y_2} \right)_{y_2=0} = (0,1,-ax^2,0) \]
(3) \[ l_i = (\frac{\partial x^i}{\partial y_3}) y_3 = 0 = (0,0,1,0) \]

(4) \[ l_i = (\frac{\partial x^i}{\partial y_4}) y_4 = 0 = (0,0,0,1) \]

(5) \[ l_i = (\frac{\partial x^i}{\partial y_5}) y_5 = 0 \]

\[ = (-\frac{1}{a} e^{-ax^1}, \frac{x^2}{2}, -\frac{a}{4}(x^0)^2 + \frac{1}{2a} e^{-2ax^1}, 0) \]

where \( y_1, y_2, y_3, y_4 \) and \( y_5 \) are parameters. These equations represent a five-parameter group of motions. In (2-1.34), \( l_i \) indicates that any two world-lines of matter are equidistant, \( l_i \), \( l_i \), \( l_i \) and \( l_i \) indicate the homogeneity of the three-spaces \( x^0 = \text{const} \), \( l_i \), \( l_i \), \( l_i \) and \( l_i \) together indicate the homogeneity of the entire space-time and finally \( l_i \) indicates the rotational symmetry of the space-time with respect to the \( x^3 \)-coordinate. The existence of \( l_i \) \( (\epsilon = 1,2,3,4) \) ensures the existence of the following four-parameter continuous group of transformations:

(2-1.35) \[ \bar{x}^0 = x^0 + y_1, \bar{x}^1 = x^1 + y_2 \]
\begin{align*}
-x^2 &= x^2 e^{-ay_2} + y_3, \quad x^3 = x^3 + y_4
\end{align*}

where \( y_1, y_2, y_3 \) and \( y_4 \) are parameters. The transformations (2-1.35) carry the Godel metric into itself.

2. The Metric-form and the Field Equations

In the present chapter, we wish to obtain some stationary cosmological solutions similar to that of Godel. The most general line element for the space-time of the universe could be of the form \( ds^2 = g_{ij} dx^i dx^j \) where \( g_{ij} \)'s are arbitrary functions of \( x^k \). But by various suppositions regarding the physical and symmetry properties of the universe we can bring some simplicity in this general form. We shall assume that there is no interaction except through the Einstein gravitational equations

\begin{equation}
(2-2.1) \quad R_{ij} - \frac{1}{2} R g_{ij} + \lambda g_{ij} = -8\pi T_{ij}.
\end{equation}

Under these circumstances the world-lines of matter will be time-like geodesics. We may now use the world-lines of matter as the \( x^0 \)-lines and if the proper time measured along these lines also measures the \( x^0 \)-coordinate, then \( g_{00} \) becomes unity. Hence, the line element for the universe becomes.
(2-2.2) \[ ds^2 = (dx^0)^2 + 2g_{0A}dx^0dx^A + g_{AB}dx^Adx^B. \]

Next, because we know that if the universe possesses intrinsic rotation, no coordinate transformation can annihilate all the three components of \( g_{0A} \). As we wish to consider some stationary cosmological solutions like Gödel's, we shall retain at least one of the \( g_{0A} \) in the form of the line element which we choose for constructing the cosmological models. Thus we write

(2-2.3) \[ ds^2 = (dx^0)^2 + 2g_{02}dx^0dx^2 + g_{AB}dx^Adx^B \]

where \( g_{AB}dx^Adx^B \) is a three-space metric. We note that the form (2-2.3) is quite complicated and hence, following Heckmann and Schucking, we choose for the line element of the universe:

(2-2.4) \[ ds^2 = (dx^0)^2 + 2e^x(dx^0)(dx^2) + ae^{2x}(dx^2)^2 + b(dx^1)^2 + c(dx^3)^2 + 2fe^x(dx^1)(dx^2) \]

where \( a, b, c, f \) are functions of \( x^0 \).

With a view to consider in the present chapter, only
the time-independent rotating cosmological solutions, we put the functions $a$, $b$, $c$ and $f$ appearing in (2-2.4), equal to constants. The coordinate $x^0$ is time-like because it is measured along the world-lines of the particles in the cosmological model. The coordinate $x^3$ does not appear in any term of (2-2.4) except the term $c(dx^3)^2$. Moreover the signature of the metric (2-2.4) is assumed to be $-2$. Therefore, keeping in mind that our purpose is to construct a generalization of the Godel-metric, we infer that the coordinate $x^3$ should be space-like. Hence the constant $c$ should be negative. So we can absorb the constant $-c$ into the $x^3$-coordinate and write for (2-2.4),

$$(2-2.5) \quad ds^2 = (dx^0)^2 + 2e^1(dx^0)(dx^2) + ae^2(dx^2)^2 +$$

$$+ b(dx^1)^2 - (dx^3)^2 + 2fe^1(dx^1)(dx^2)$$

where $a$, $b$ and $f$ are unknown constants.

Corresponding to the line element (2-2.5), the non-vanishing covariant components of the Ricci tensor are as follows:
(2-2.6) \[
R_{00} = -\frac{1}{2\alpha}
\]
\[
R_{02} = -\frac{e^{-x}}{2\alpha}
\]
\[
R_{11} = \frac{b(2a - 1)}{2\alpha}
\]
\[
R_{12} = \frac{f(2a - 1)}{2\alpha} e^{x}
\]
\[
R_{22} = \frac{a(2a - 3)}{2\alpha} e^{2x}
\]

where and in what follows in the present chapter, the constant \( a \) is equal to \( b(a - 1) - f^2 \). Now (2-2.5) and (2-2.6) give the following expression for the scalar curvature \( R \):

(2-2.7) \[
R = \frac{(4a - 3)}{2\alpha}.
\]

Using (2-2.5), (2-2.6) and (2-2.7) we can write the field equations (2-2.1) in the following form:

(2-2.8) \[
\frac{(1 - 4a)}{4\alpha} + \lambda = -8\pi T_{00}
\]
\[ e^{x^1} \left[ \frac{(1 - 4a)}{4a} + \lambda \right] = -8\pi T_{02} \]

\[ b \left[ \frac{1}{4a} + \lambda \right] = -8\pi T_{11} \]

\[ f e^{x^1} \left[ \frac{1}{4a} + \lambda \right] = -8\pi T_{12} \]

\[ a e^{2x^1} \left[ -\frac{3}{4a} + \lambda \right] = -8\pi T_{22} \]

\[ \frac{4a - 3}{4a} - \lambda = -8\pi T_{33} \]

\[ T_{01} = T_{03} = T_{13} = T_{23} = 0. \]

3. **The Anisotropic Fluid Distribution and a Solution**

If we imagine that the fluid distribution is anisotropic, following the scheme for such a distribution due to Lichnerowicz, v(1955), we put

\[ (2-3.1) \quad \varepsilon_{ij} = v_i v_j - \sum_{A=1}^{3} v_i^{(A)} v_j^{(A)} \]

\[ T_{ij} = \rho v_i v_j + \sum_{A=1}^{3} p^{(A)} v_i^{(A)} v_j^{(A)}. \]

Here \( v^1 \) is the velocity four-vector of the particles of
the fluid and hence time-like. \( v_{i}^{(A)} \) are three different space-like four-vectors. \( \rho \) is the proper mass density and \( p(A) \) are the proper pressures in the directions of \( v_{i}^{(A)} \), respectively. The conditions imposed upon the various quantities appearing in (2-3.1) are obviously

\[
(2-3.2) \quad T_{ij}^j = 0
\]

\[
(2-3.3) \quad v_{i} v^{i} = 1, \quad v_{i}^{(A)} v^{(A)i} = -1.
\]

Because of these conditions we note that there are sixteen independent quantities on the left hand sides of (2-3.1) and that they are expressed in terms of some other sixteen independent quantities appearing on the right hand sides.

If we choose

\[
(2-3.4) \quad p(1) = p(2) = p, \quad p(3) = q,
\]

then we obtain from (2-3.1),

\[
(2-3.5) \quad T_{ij} = (\rho + p)v_{i} v_{j} - pg_{ij} + (q - p)v_{i}^{(3)} v_{j}^{(3)}.
\]
Now we may very well drop the superscript (3) and write simply $V_i$ for $V_i(3)$. Then (2-3.5) becomes

$$T_{ij} = (\rho + p)\dot{\gamma}_{ij} - p g_{ij} + (q - p) V_i V_j$$

where we require

$$T_{ij} ; j = 0$$

$$V_i V^i = 1, \quad V_i V^i = -1.$$ 

Next, let us put $V^i = \delta_3^i$. And since we are using the co-moving coordinate system, obviously we put $V^i = \delta_0^i$. Then we see from (2-2.5) and (2-3.8) that the equations (2-3.6) are satisfied. The equations (2-3.7) will be automatically satisfied if (2-3.6) satisfies the field equations (2-2.6), because we know that the four-divergence of the tensor $R_{ij} - \frac{1}{2} R g_{ij} + \gamma g_{ij}$ identically vanishes.

Since $V^i = \delta_3^i$ and $V^i = \delta_0^i$, from (2-2.5) we obtain

$$V_i = (1, 0, e^x, 0), \quad V_i = -\delta_3^i.$$ 

Hence, from (2-3.6) it follows that
Substituting these values of the components of $T_{ij}$ in (2-2.8), we find that these equations are satisfied, and we obtain

$$8\pi \rho = \frac{(4a - 1)}{4a} - \lambda$$

$$8\pi p = \frac{1}{4a} + \lambda$$

$$8\pi q = \frac{(3 - 4a)}{4a} + \lambda.$$  

In the following we shall suppose $\lambda$ to be equal to zero.

Let us consider the restrictions imposed upon the constants $a$, $b$ and $f$ due to the signature requirement that the determinant $g$ of $g_{ij}$ be negative and the physical requirement that the density $\rho$ be positive whereas the
stresses \( p \) and \( q \) be non-negative. Since

\[
g = - [b(a - 1) - f^2] e^{2x^1},
\]

we first require that

\[
b(a - 1) - f^2 > 0.
\]

And since \( f^2 > 0 \), we write this condition as

\[
b(a - 1) > f^2 > 0.
\]

Secondly, the requirement that \( p \) be finite and non-negative indicates \( \alpha = b(a - 1) - f^2 > 0 \). However, this does not give any new restriction upon \( a, b \) and \( f \). But now the requirement that \( q \) be finite and non-negative whereas \( j^p \) be finite and positive indicates that

\[
\frac{1}{4} < a < \frac{3}{4}.
\]

Because of this we note that \( (a - 1) \) must be always negative. Consequently, from (2-3.13) it follows that \( b \) must be negative, that is,
Thus the restrictions imposed upon the constants $a$, $b$ and $f$ are

$$\frac{1}{4} < a \leq \frac{3}{4}, \quad -\infty < b < 0, \quad 0 < f^2 < b(a - 1).$$

The fact that the cosmological model represented by this solution neither expands nor contracts can be recognised by considering the quantity $H = v^i$; $\dot{1}$. For the line element (2-2.5) we find that

$$H = v^i$; $\dot{1} = 0.$

Secondly, the four-vector $w^i$ representing the angular velocity of a fluid element in the model, can be evaluated from (2-1.15). It is found to be

$$w^i = \delta^i_0 x^{-1/2}.$$

That is, $w^i$ has constant non-vanishing components. In the sense of the properties indicated in (2-3.18) and (2-3.19), we say that the solution obtained represents a stationary cosmological model like that of Godel. Thus we have obtained
a cosmological model which is filled with an anisotropic fluid distribution and which is stationary like the Godel model.

Lastly, we note that the world-lines of the particles in the model, that is, the paths of the galaxies, are geodesics. This is because the equations of geodesics

\[(2-3.20) \quad v^i_{;j} v^j = 0\]

are satisfied by \( v^i = \delta^i_0 \) and the metric \((2-2.5)\).

4. **The Perfect Fluid Distribution and a Particular Case of the Solution Discussed in Section 3**

When we put \( q = p \) in \((2-3.6)\), it reduces to the usual expression for a perfect fluid distribution, that is,

\[(2-4.1) \quad T_{ij} = (\rho + p)v_i v_j - \rho \delta_{ij}.\]

Putting \( v^i = \delta^i_0 \), from \((2-2.5)\) we obtain

\[(2-4.2) \quad v_i = (1, 0, e^{-x}, 0).\]

Then \((2-2.5)\) and \((2-4.1)\) imply
Substituting these values of the components of $T_{ij}$ in the field equations (2-2.8), we find that they are satisfied provided $a = 1/2$; and we find

$$8\pi \rho = \frac{1}{4a} - \lambda$$

$$8\pi p = \frac{1}{4a} + \lambda$$

In the following we shall discuss the case with $\lambda = 0$.

As seen in the Section 3, the signature requirement demands that

$$b(a-1) > f^2 > 0$$

and the same condition also ensures a finite and non-negative
value of $p$. Since $a = 1/2$, now \( (2-4.5) \) indicates that $-\infty < b < 0$. Thus the restrictions imposed upon the constants $a$, $b$ and $f$ are

\[
(2-4.6) \quad a = \frac{1}{2}, \quad -\infty < b < 0, \quad 0 \leq f^2 < -\frac{b}{2}.
\]

We note that the particular case with $a = 1/2$ of the solution obtained in Section 3, corresponds to $q = p$. Therefore, we infer that the properties possessed by the solution of Section 3, also hold for the solution obtained in the present section.

It is observed that if $f = 0$, the solution of Section 3 corresponds to the solution discussed by Synge (1960) whereas the solution obtained in the present section corresponds to the Godel solution (1949).

5. **The Coordinate Transformations Connected with the Solution Obtained in the Present Chapter**

The stationary cosmological solution obtained in Section 3 describes a family of Synge-type universes filled with anisotropic fluid distribution and it is a special case of the solution described by the Heckmann-Schucking metric \((2-2.4)\). It is described by the line element
(2-5.1) \[ ds^2 = (dx^0)^2 + 2a x^1(dx^0)(dx^2) + ae^{x^1}(dx^2)^2 + b(dx^1)^2 - (dx^3)^2 + 2f e^{x^1}(dx^1)(dx^2) \]

where

\[(2-5.2) \quad \frac{1}{4} < a \leq \frac{3}{4}, \quad -\infty < b < \infty, \quad 0 \leq f^2 < b(a-1). \]

Now, using the coordinate transformations

\[(2-5.3) \quad x^0 = \overline{x}^0 + \frac{f}{(a-1)} \overline{x}^1 \]

\[x^1 = \overline{x}^1 \]

\[x^2 = \frac{f}{(a-1)} e^{\overline{x}^1} + \overline{x}^2 \]

\[x^3 = \overline{x}^3, \]

(2-5.1) can be transformed into the following line element:

\[(2-5.4) \quad ds^2 = (dx^0)^2 + 2e^{x^1} dx^0 dx^2 + Ae^{x^1}(dx^2)^2 + B(dx^1)^2 - (dx^3)^2 \]

where \( A = a \), \( B = b + \frac{f^2}{(1-a)} \) are constants. Because

\[(1/4) < a \leq (3/4) \quad \text{and} \quad f^2 < b(a-1), \]

it follows that
The metric (2-5.4) is equivalent to that discussed by Synge (1960). Thus it is seen that the solution (2-5.1) is simply a transform of the solution discussed by Synge (1960).

The stationary cosmological solution obtained in Section 4 describes a family of Godel-type universes filled with perfect fluid distribution and also it is a special case of the solution described by the Heckmann-Schucking metric (2-2.4). It is given by the line element

\[
(2-5.6) \quad ds^2 = (dx^0)^2 + 2e^{x_1} (dx^0)(dx^2) + \frac{1}{2} e^{2x_1} (dx^2)^2 +
\]

\[
+ b(dx^1)^2 - (dx^3)^2 + 2fe^{x_1} (dx^1)(dx^2)
\]

where

\[
(2-5.7) \quad -\infty < b < 0 \ , \ \ 0 \leq f^2 < -\frac{b}{2}.
\]

Using the coordinate transformations

\[
(2-5.8) \quad x^0 = \bar{x}^0 - 2f \bar{x}^1
\]

\[
x^1 = \bar{x}^1
\]
\[ x_2 = -2fe^{-x^1} + x^2 \]
\[ x_3 = \bar{x}^3, \]

(2-5.6) can be transformed into the following line element:

\[ (2-5.9) \quad ds^2 = (dx^0)^2 + 2e^{x_1} (dx^0 dx^1 + e^{x_2} (dx^2)^2 + B(dx^1)^2 - (dx^3)^2 \]

where \( B = b + 2e^2 \) is a constant. Because \( f^2 < -(b/2) \), it follows that

\[ (2-5.10) \quad -\infty < B < 0. \]

The metric (2-5.9) is equivalent to the Godel-metric. Thus it is seen that the solution (2-5.6) is simply a transform of the Godel-solution.

The Godel line element and Synge line element can be put into forms which apparently look different from these line elements. Thus the line element (2-5.1) can be transformed into couple of interesting forms. The transformations
(2-5.11) \[ x^0 = \frac{\bar{x}^0}{(1-N)^{1/2}} + \frac{N^{1/2}m^{1/2}}{(1-N)^{1/2}} \bar{x}^1 \]

\[ x^1 = \pm \left( \frac{N}{M} \right)^{1/2} \bar{x}^0 + \bar{x}^1 \]

\[ x^2 = \bar{x}^2 \]

\[ x^3 = \bar{x}^3 \]

\[ \delta(x^0, x^1, x^2, x^3) = \frac{\delta(x^0, \bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)}{\left( \frac{MN}{1-N} \right)^{1/2}} = (1 - N)^{1/2} \]

where \( M \) and \( N \) are given by

(2-5.12) \[ b = -\frac{M}{(1-N)} \quad f = \mp \left( \frac{MN}{1-N} \right)^{1/2} \]

carry (2-5.1) into the form

(2-5.13) \[ ds^2 = (d\bar{x}^0)^2 + 2(1-N)^{1/2} e^{\pm \left( \frac{N}{M} \right)^{1/2} \bar{x}^0} \frac{d\bar{x}^0}{dx^0} d\bar{x}^2 \]

\[ + a e^{2 \left[ \mp \left( \frac{N}{M} \right)^{1/2} \bar{x}^0 \right]} (d\bar{x}^2)^2 - M(d\bar{x}^{-1})^2 - (dx^3)^2 \]

where

(2-5.14) \[ \frac{1}{4} < a < \frac{3}{4} \quad 0 < M < \infty \quad 0 \leq N < \frac{1}{2} \]
When \( a = 1/2 \), (2-5.13) and (2-5.14) give a transform of the Godel-metric which looks apparently more general than the Godel-metric.

The transformations

\[
(2-5.15) \quad x^0 = \tilde{x}^0 \pm \frac{M^{1/2} N^{1/2}}{(1 - N)^{1/2}} \tilde{x}^1
\]

\[
x^1 = \tilde{x}^1
\]

\[
x^2 = \tilde{x}^2
\]

\[
x^3 = \tilde{x}^3
\]

where \( M \) and \( N \) are given by (2-5.12), carry (2-5.1) into the form

\[
(2-5.16) \quad ds^2 = (dx^0)^2 + 2 e^{x^1} dx^0 dx^2 + ae^{2x^1} (dx^2)^2 -
\]

\[
- M(dx^{-1})^2 - (dx^{-3})^2 \pm 2(\frac{MN}{1-N})^{1/2} dx^0 dx^{-1}
\]

where

\[
(2-5.17) \quad \frac{1}{4} < a < \frac{3}{4}, \quad 0 < M < \infty, \quad 0 \leq N < \frac{1}{2}.
\]
When $a = 1/2$, (2-5.16) and (2-5.17) give a transform of the Godel-metric which apparently looks more general than the Godel-metric. (Raval, H.M., and Vaidya, P.C., 1965).

6. Some Geometrical Properties of the Solution Considered

Now we wish to show that the solution (2-5.1) can be embedded into a flat space of ten dimensions. We know that this solution can be transformed into the form (2-5.13). It may be written as

$$
(2-6.1) \quad ds^2 = (dx^0)^2 + 2(1-N)^{1/2} e^{\frac{1}{N^2}(N^1/2 x^0 + x^1)} dx^0 dx^2 + \frac{2}{N^2} (N^1/2 x^0 + x^1) \left( dx^2 \right)^2 - \left( dx^1 \right)^2 - \left( dx^3 \right)^2.
$$

Putting

$$
(2-6.2) \quad x^1 = z^1, \quad x^2 = z^2, \quad x^3 = z^3, \quad x^0 = z^4, \quad \frac{1}{N^2}(N^1/2 x^0 + x^1) = z^5
$$

in (2-6.1), it becomes

$$
(2-6.3) \quad ds^2 = (dz^4)^2 + 2(1-N)^{1/2} e^{\frac{1}{N^2} z^5} (dz^4)(dz^2) + ae^{2z^5} (dz^2)^2 - (dz^1)^2 - (dz^3)^2.
$$
Now put $ae^{2z^5} = (u)^2$. Therefore,

\begin{equation}
(2-6.4) \quad ae^{2z^5}(dz^2)^2 = (u)^2(dz^2)^2.
\end{equation}

Consider the transformations

\begin{equation}
(2-6.5) \quad p = u \cos z^2, \quad q = u \sin z^2.
\end{equation}

Therefore,

\begin{equation}
(2-6.6) \quad \omega^2(dz^2)^2 = (dy)^2 + (dq)^2 - (du)^2.
\end{equation}

Consequently the metric (2-6.3) becomes

\begin{equation}
(2-6.7) \quad ds^2 = (dz^4)^2 + 2(1-N)^{1/2}e^{z^5}(dz^4)(dz^2) + (dp)^2 +
+ (dq)^2 - (du)^2 - (dz^1)^2 - (dz^3)^2.
\end{equation}

Now, putting

\begin{equation}
(2-6.8) \quad 2(1-N)^{1/2}e^{z^5}(dz^2)(dz^4) = (\omega)^2(dz^2)(dz^4)
\end{equation}
\[ t = w \cos \frac{1}{2} (z^2 + z^4) \]
\[ r = w \cos \frac{1}{2} (z^2 - z^4) \]
\[ v = w \sin \frac{1}{2} (z^2 + z^4) \]
\[ y = w \sin \frac{1}{2} (z^2 - z^4) \]

we find

\[ 2(1-N)^{1/2} e^{z} (dz^2)(dz^4) = (dt)^2 + (dv)^2 - (dr)^2 - (dy)^2. \]

Therefore, \((2-6.7)\) reduces to

\[ ds^2 = (dz^2)^2 + (dt)^2 + (dv)^2 - (dr)^2 - (dy)^2 + (dp)^2 + (dq)^2 - (du)^2 - (dz^1)^2 - (dz^3)^2. \]

Next, putting

\[ z_1 = \frac{z}{6}, \quad t = \frac{z}{z}, \quad v = \frac{z}{2}, \quad p = \frac{z}{3}, \quad q = \frac{z}{5}, \]
\[ r = \frac{z}{6}, \quad y = \frac{z}{9}, \quad u = \frac{z}{10}, \quad z = \frac{z}{7}, \quad z^4 = \frac{z}{4} \]
(2-6.10) may be written as

\[(2-6.12) \quad ds^2 = (dz^1)^2 + \frac{dz^2}{\alpha^2} + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 - \frac{dz^6}{\alpha^2} - (dz^7)^2 - (dz^8)^2 - (dz^9)^2 - (dz^{10})^2\]

which is a flat metric in ten dimensions. From this result it follows that the Godel-metric can be embedded into a flat space of ten dimensions. This particular result was obtained by J. Rosen (1965).

To study the internal symmetries of the cosmological solution (2-5.1), we should consider the corresponding Killing equations. Due to (2-5.4), (2-5.1) can be equivalently written as

\[(2-6.13) \quad ds^2 = (dx^0)^2 + 2e^{ax^1} dx^0 dx^2 + be^{2ax^1} (dx^2)^2 - (dx^1)^2 - (dx^3)^2.\]

The Killing equations corresponding to this line element are

\[(2-6.14) \quad l^0_0 + e^{ax^1} l^1_0 = 0\]

\[l^0_1 - l^1_0 + e^{ax^1} l^2_1 = 0\]
ae^{ax_1^1} + e^{ax_1^2} + e^{ax_1^0} + be^{2ax_1^2} = 0
\nonumber

1^0, 3 - l^3, 0 + e^{ax_1^2}, 3 = 0
\nonumber

1^1, 1 = 0
\nonumber

1^1, 2 - e^{ax_1^0}, 1 - be^{2ax_1^2}, 1 = 0
\nonumber

1^1, 3 + l^3, 1 = 0
\nonumber

1^0, 2 + be^{ax_1^2}, 2 + abe^{ax_1^1} = 0
\nonumber

e^{ax_1^0}, 3 + be^{2ax_1^2}, 3 - l^3, 2 = 0
\nonumber

1^3, 3 = 0.
\nonumber

These equations have following five independent solutions
\nonumber

(\epsilon_i)_{1, 1} (\epsilon = 0, 1, 2, 3, 4):
\nonumber

(2-6.15) \quad \frac{\partial e^{x_1}}{\partial y_0} \bigg|_{y_0=0} = (1, 0, 0, 0)
\[ (1) \quad l_1 = (\frac{\partial x}{\partial y_1}) y_1 = 0 = (0,1,-ax^2,0) \]

\[ (2) \quad l_1 = (\frac{\partial x}{\partial y_2}) y_2 = 0 = (0,0,1,0) \]

\[ (3) \quad l_1 = (\frac{\partial x}{\partial y_3}) y_3 = 0 = (0,0,0,1) \]

\[ (4) \quad l_1 = (\frac{\partial x}{\partial y_4}) y_4 = 0 = (-\frac{1}{a} e^{-ax^1},(1-b)x^2, \]

\[-\frac{a}{2}(1-b)(x^2)^2,\frac{1}{2a} e^{-2ax^1},0) \]

where \( y_0, y_1, y_2, y_3 \) and \( y_4 \) are parameters. These equations represent a five-parameter group of motions. In (2-6.15), \( l_1 \) indicates that any two world-lines of matter are equidistant, \( l_1, l_1 \) and \( l_1 \) indicate the homogeneity of the three-spaces \( x^0 = \text{const} \), \( l_1, l_1 \), and \( l_1 \) and \( l_1 \) together indicate the homogeneity of the entire space-time and finally \( l_1 \) indicates the rotational symmetry of the space-time with respect to the \( x^3 \)-coordinate. The existence of \( l_1 \) (\( \epsilon = 0,1,2,3 \)) in the solution (2-6.15) ensures the existence of the following
four-parameter continuous group of transformations

\[(2-6.16) \quad \bar{x}^0 = x^0 + y_0, \quad \bar{x}^1 = x^1 + y_1, \quad \bar{x}^2 = x^2 e^{-ay_1} + y_2, \quad \bar{x}^3 = x^3 + y_3\]

where \(y_0, y_1, y_2\) and \(y_3\) are parameters. The transformations \((2-6.16)\) carry the metric \((2-6.13)\) into itself.