CHAPTER II

THE LANDAU RESONANT INTERACTION OF OFF ANGLE WHISTLER MODE PULSES

2.1. INTRODUCTION

The Artificial stimulation of Very Low Frequency Emissions through Morse Code pulses suggests that their origin is associated with the interaction of wave packets with the particles. The study of the wave particle interactions is, therefore, very important and is expected to provide clues as to the actual triggering mechanism of the VLF emissions.

The most important wave particle interactions are resonant interactions as they lead to the most efficient exchange of energy between the waves and the particles. The general resonance condition between obliquely propagating whistler mode waves and electrons is given by $\omega - k_{\parallel} V_{\parallel} = n\Omega$ where n is either zero or a positive integer and other symbols have meanings as defined earlier in Chapter I.

The resonance characterised by $n = 1$ is known as gyroresonance and occurs when the rate of change of the phase difference between the rotating velocity vector of the electron and the rotating field vector of the wave becomes zero. Another resonance characterised by $n = 0$ is known as Landau resonance and occurs when the rate of change of the phase of the longitudinal component of the wave electric field as seen by the particle becomes zero.

Brinca (1972) considered non linear Landau wave particle interaction between electrostatic waves and a collisionless plasma and showed that side band growth would occur in this case. Since the obliquely propagating whistlers also have a component of their electric field along the geomagnetic field direction, he suggested that similar effects will be produced by them and the consequent side band growth is likely to result into the generation of VLF emissions.

Here, in this chapter, we shall concentrate our attention on the effects of the Landau resonant interaction of an obliquely propagating whistler mode wave packet with a collisionless plasma.

2.2 MATHEMATICAL FORMULATION

We consider a fully ionised collisionless plasma immersed in a uniform magnetic field B_0 such that the electron plasma frequency ω_p is much greater than the electron gyrofrequency Ω. A whistler mode wave packet containing a narrow range of frequencies denoted by ω and wave vector $\vec{k}$ is assumed to propagate in x-z plane at an angle θ with the ambient magnetic field B_0 acting along z direction (See fig.2.1). The shape of the wavepacket is assumed to be Gaussian.

The equation of motion of a charged particle in such a system can be written as

$$\frac{d\vec{V}}{dt} = \frac{e}{m}\left\{ \vec{E} + \frac{\vec{V} \times (\vec{B}_0 + \vec{b})}{c} \right\} \quad \text{---} (2.1)$$

where $\vec{V}$ is the particle velocity, $\vec{E}$ and $\vec{b}$ are the wave electric and magnetic fields and e,m and c are the electron charge, electron mass and velocity of light in free space, respectively.

Here we have ignored all the forces on electrons except those of the electromagnetic origin.

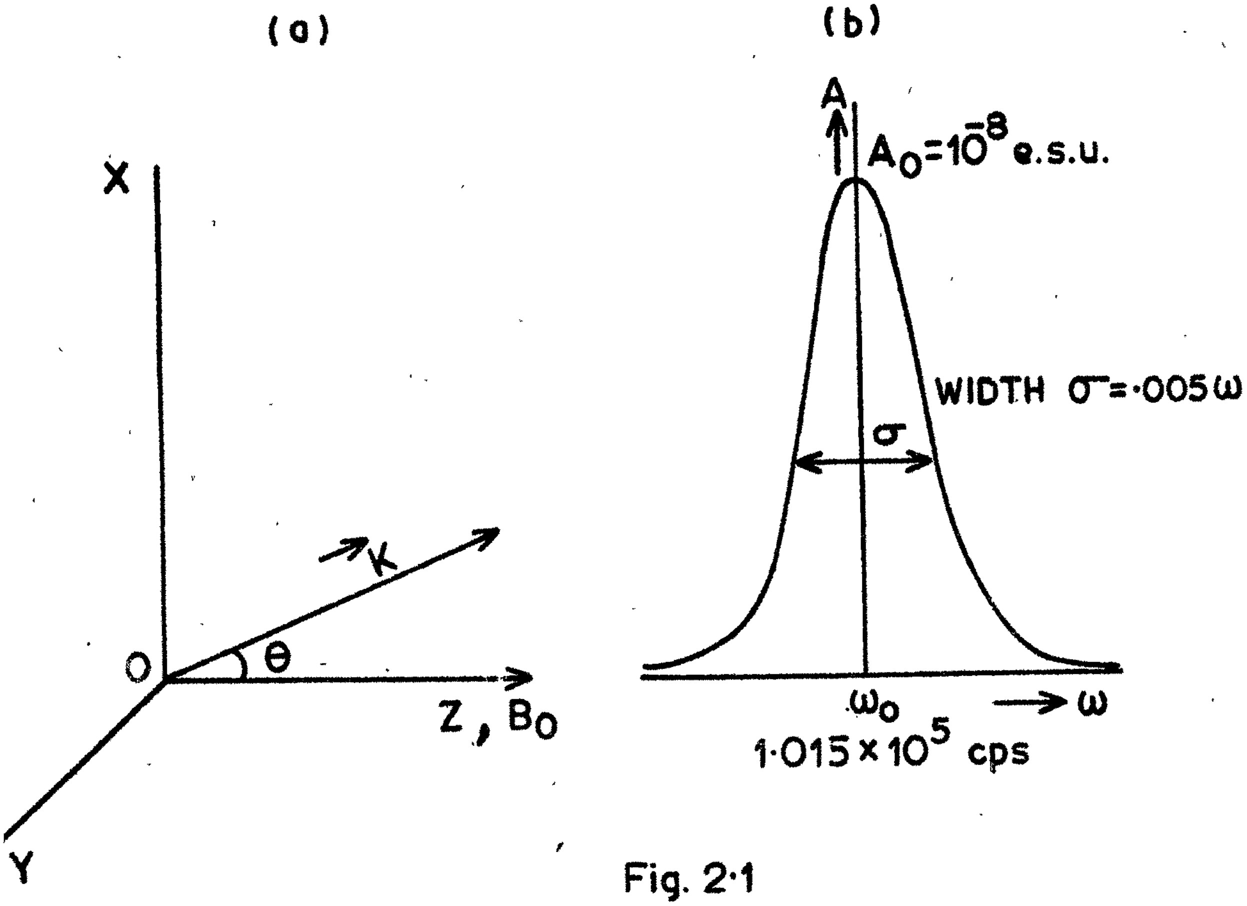

Fig. 2·1

(A) THE PULSE MOVES ALONG $\vec{k}$ MAKING AN ANGLE θ WITH $\vec{B}_0$.
(B) A DESCRIPTIVE SKETCH OF THE PULSE. A : AMPLITUDE, σ : BANDWIDTH, ω_0 : CENTRAL FREQUENCY.

Using Maxwell's equations, equation (2.1) can be reduced to

$$\frac{dv_{\parallel}}{dt} = \frac{e}{m}\left[|E_z|\cos\psi + \frac{k_{\parallel}v_{\perp}}{\omega}\left\{|E_x|\cos\psi\cos\phi + |E_y|\sin\psi\sin\phi\right\} - \frac{k_{\perp}v_{\perp}}{\omega}|E_z|\cos\psi\cos\phi\right] \qquad \text{---- (2.2a)}$$

and

$$\frac{dv_{\perp}}{dt} = \frac{e}{m}\left[\left(1-\frac{k_{\parallel}v_{\parallel}}{\omega}\right)\left(|E_x|\cos\psi\cos\phi + |E_y|\sin\psi\sin\phi\right) + \frac{k_{\perp}v_{\parallel}}{\omega}|E_z|\cos\phi\cos\psi\right] \qquad \text{---- (2.2b)}$$

where the subscripts ' $\parallel$ ' and ' $\perp$ ' refer to vector components parallel and perpendicular to the ambient magnetic field direction and $\psi\ (= \vec{k}.\vec{r} - \omega t + \psi_0)$ and $\phi (= \Omega t + \phi_0)$ are the wave phase and the particle gyration phase respectively. $|E_x|$, $|E_y|$ and $|E_z|$ represent the components of the amplitude of the wave electric field in x,y and z directions.

We are interested in the consequences of equations (2.2a) and (2.2b) over periods much greater than the gyration period of an electron. For integrating these equations, only variations in ψ and ϕ need be considered as the effects of variations in other quantities would be relatively small. We also impose the condition of Landau resonance $\omega - k_{\parallel} V_{\parallel} = 0$ during integration. We find that under these circumstances $\int \cos\psi \cos\phi \, dt$ goes to zero over an integral number of gyroperiods under the linear assumption and therefore eqn. (2.2b), on integration, gives:

$$\delta v_{\perp} = 0 \qquad \text{---(2.3b)}$$

The integration of equation (2.2a) is rather involved and we drop the terms containing the integral of $\cos\psi\cos\phi$ as their contribution over the time period of interest would be relatively small.

Thus equation (2.2a) may be written as

$$\frac{dv_{\parallel}}{dt} = \frac{e}{m}\left[|E_z| \cos\psi + \frac{k_{\parallel} v_{\perp}}{\omega} |E_y| \sin\psi \sin\phi \right] \qquad ---(2.3)$$

With the Landau resonance condition one knows that

$$\int_0^{\frac{2\pi n}{\Omega}} \cos\psi \, dt = \cos\psi_0 \frac{2\pi n}{\Omega} J_0\left(\frac{k_{\perp} v_{\perp}}{\Omega}\right),$$

and

$$\int_0^{\frac{2\pi n}{\Omega}} \sin\psi \sin\phi \, dt = \cos\psi_0 \frac{2\pi n}{\Omega} J_1\left(\frac{k_{\perp} v_{\perp}}{\Omega}\right)$$

under linear approximation.

Since a nonlinear solution of eqn. (2.3) is still complicated, the relative importance of the two terms on R.H.S can be estimated by substituting the results of integration under linear approximation.

We find that the second term on the R.H.S.of (2.3) can be neglected in comparison of its first term if

$$|E_z|\, J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) >> |E_y|\, \frac{v_\perp}{v_\parallel}\, J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)$$

If $|E_z|$ and $|E_y|$ are comparable, and both $\frac{v_\perp}{\Omega}$ and $\frac{v_\perp}{v_\parallel}$ are small, this condition is satisfied and we are left with

$$\frac{dv_\parallel}{dt} = \frac{e|E_z|}{m}\cos\psi = \frac{e|E_z|}{m}\cos(\vec{k}.\vec{r} - \omega t)$$

This may also be written as

$$\frac{dv_\parallel}{dt} = \frac{e|E_z|}{m}\cos\left[(k_\parallel v_\parallel - \omega)t + \psi_0 + \frac{k_\perp v_\perp}{\Omega}\sin\Omega t\right]$$

or

$$\frac{dv_\parallel}{dt} = \frac{e|E_z|}{m}\left[\cos\left\{(k_\parallel v_\parallel - \omega)t + \psi_0\right\}\cos\left\{\frac{k_\perp v_\perp}{\Omega}\sin\Omega t\right\} - \sin\left\{(k_\parallel v_\parallel - \omega)t + \psi_0\right\}\sin\left\{\frac{k_\perp v_\perp}{\Omega}\sin\Omega t\right\}\right]$$

Near the Landau resonance point, the variation in $(k_\parallel v_\parallel - \omega)t$ is very slow and that in $\frac{k_\perp v_\perp}{\Omega}\sin\Omega t$ is very fast. Therefore, for a time period large compared to a gyrcperiod, we can substitute the average values of cos $\left(\frac{k_\perp v_\perp}{\Omega}\right.$ Sin $\left.\Omega t\right)$ and Sin $\left(\frac{k_\perp v_\perp}{\Omega}\right.$ Sin $\left.\Omega t\right)$ (over an integral number of gyroperiods) which are $J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)$ and zero respectively. Thus the above equation will be reduced to

$$\frac{dv_\parallel}{dt} = \frac{e|E_z|}{m} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\cos\left\{(k_\parallel v_\parallel - \omega)t + \psi_0\right\}$$

If $\frac{k_\perp v_\perp}{\Omega}$ is very small compared to unity this equation can be further approximated by

$$\frac{dv_{\parallel}}{dt} = \frac{e|E_z|}{m} \cos\left\{(k_{\parallel}v_{\parallel} - \omega)t + \psi_0\right\} \quad \text{---} (2.3a)$$

We make use of this equation for computing the changes in the $v_{\parallel}$ of the particles caused by the Landau resonant interaction.

2.3 COMPUTATION OF THE CHANGES IN PARTICLE VELOCITIES

To determine the effect of waves on particles we must find how the distribution function of the particles changes during the resonance. Since the Landau resonance does not significantly affect the $v_{\perp}$ of the particles, we do not expect the particle distribution with respect to $v_{\perp}$ to change with time.

However, $v_{\parallel}$ of the particles changes with time and over long pericds, the linear approximation is not valid. Therefore, a numerical solution of equatiion (2.3a) was attempted on computer to see how the particle velocities change with time.

A wave packet with central frequency $\omega_0 = 1.015 \times 10^5$ radians per second was considered. Its band width was taken to be 0.5% of its central frequency and its shape was assumed to be Gaussian.

The electric field E_z corresponding to the central frequency ω_0 in the wave packet was taken to be 10^{-8} e.s.u..

The velocity distribution of particles was assumed to consist of two components:

i) a back ground of low energy particles such that the thermal velocities of the particles are much smaller compared to the phase velocity of the whistler mode waves, and

ii) a thin distribution of high energy particles with their $v_{\parallel}$'s comparable to the phase velocity of the whistler mode waves.

The propagation characteristics of the wave packet would be determined by the back ground component which can be considered as a cold plasma. This makes it possible for us to use the well known cold plasma dispersion relation for whistler mode:

$$\frac{c^2k^2}{\omega^2} = \frac{\omega_p^2}{\omega(\Omega\cos\theta - \omega)}$$

to compute k for a given value of ω .

The second component contains particles that undergo resonance and are therefore affected most. The distribution of these particles will change with time. To study the time evolution of the distribution function, we must first integrate equation (2.3a) to find how the velocities change with time and then develop a method to compute the effects of these velocity changes on the particle distribution function. We restrict our attention to a narrow range of particle velocities lying in the close proximity of the central resonant $v_{\parallel}^{c}$. All the quantities in equation (2.3a) were then normalised by dividing them with suitable constants of the system having dimensions same as the quantities in question. The relations connecting the normalised and the unnormalised quantities are listed below.

$$t' = t/(1/\Omega)$$

$$\omega' = \omega/\Omega$$

$$k' = k/k(\omega_0)$$

$$v' = v/\{2(\omega_0/k(\omega_0))\}$$

$$\left(\frac{eE}{m}\right)' = \left(\frac{eE}{m}\right)/\left\{\frac{\Omega^2}{k(\omega_0)}\right\}$$

Here ω_0 represents the central frequency of the wavepacket and the prime indicates that the concerned quantities are normalised. In the discussion here after, through out this chapter, we would drop the prime for the purpose of clarity of notation as we will be dealing only with the normalised quantities now onwards.

The Runge Kutta method was employed for the integration of equation (2.3a). The time increment Δt between consecutive steps was chosen to be one gyro period and the integration was carried out for a total period of about 1500 gyroperiods. The normalised value of central resonant $\vartheta_{\parallel}$ was 0.5 and the initial values for $\vartheta_{\parallel}$ were given in the range $\vartheta_{\parallel}$ = 0.496 to 0.504 with a uniform spacing of 0.0001. The range of $\vartheta_{\parallel}$ considered included both the trapped and the nearly trapped particles.

If the R.H.S of equation (2.3a) is represented by F($\vartheta_{\parallel}, t$) and if the value of F($\vartheta_{\parallel}, t$) at the end of n'th time interval is known, then $\Delta\vartheta_{\parallel}$ (the change in $\vartheta_{\parallel}$ during the next time interval) can be calculated from the equation given below:

$$\Delta\vartheta_{\parallel} = \frac{1}{6}\left\{ k_1 + 2k_2 + 2k_3 + k_4 \right\}$$

where

$$k_1 = F(t_n, \vartheta_{\parallel n})\,\Delta t ,$$

$$k_2 = F\left(t_n + \frac{\Delta t}{2}, \vartheta_{\parallel n} + \frac{k_1}{2}\right)\Delta t,$$

$$k_3 = F\left(t_n + \frac{\Delta t}{2}, \vartheta_{\parallel n} + \frac{k_2}{2}\right)\Delta t,$$

and $$k_4 = F\left(t_n + \Delta t, \vartheta_{\parallel n} + k_3\right)\Delta t$$

Here, the subscript 'n' indicates the value of the concerned quantity after n'th time interval.

The errors involved in Runge Kutta method are extremely small as they are proportional to the fifth power of the time interval Δt. Taking Δt to be of the order of 3×10^{-5} sec. the error would be approximately of the order of 10^{-23}.

The results of the integration were used to find $\delta\vartheta_{\parallel}$ (the total change in $\vartheta_{\parallel}$ over the period of integration) which was later plotted against initial $\vartheta_{\parallel}$. One such typical plot is shown in fig. (2.2) which corresponds to a time period of about 1200 gyro periods.

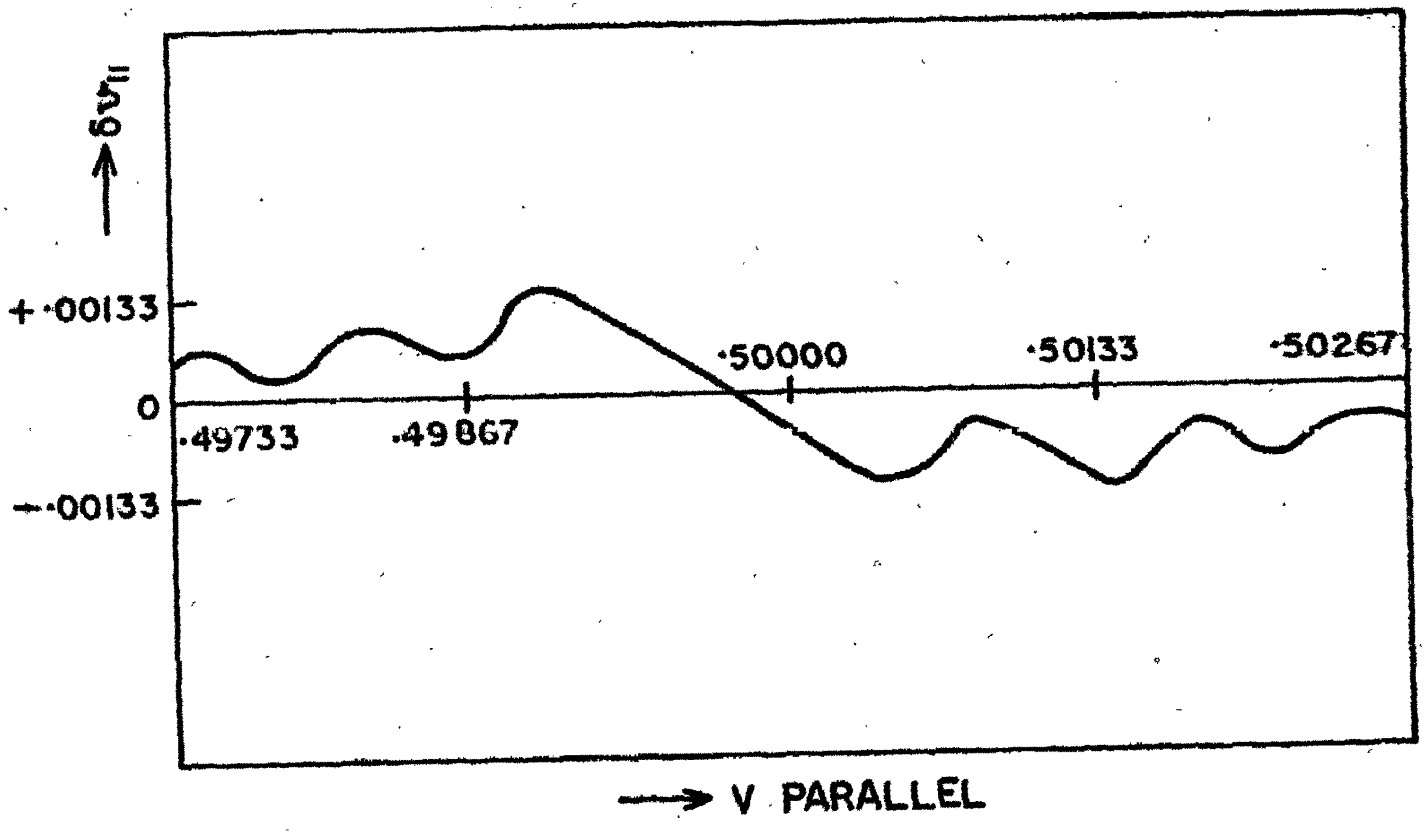

Fig. 2·2

CHANGE IN 'V' PARALLEL, '$\delta V_{\parallel}$', AGAINST '$V_{\parallel}$'
(TIME OF INTERACTION WITH THE WAVE PACKET : 1200 GP)

The figure shows that there is a point on $v_{\parallel}$ axis slightly below the central resonant $v_{\parallel}$ where the $\delta v_{\parallel}$ versus $v_{\parallel}$ curve crosses the $v_{\parallel}$ axis. This point, which corresponds to a velocity, say, v_0 , divides the whole axis into two parts. The particles with their $v_{\parallel}$ less than v_0 will be accelerated by the waves and will therefore extract energy from them. The particles with their $v_{\parallel}$ greater than v_0 are decelerated and thereby they lose their energy to the waves. The plot of $\delta v_{\parallel}$ versus $v_{\parallel}$ also shows a very interesting feature as it exhibits fluctuations in $\delta v_{\parallel}$ on either side of v_0 . These fluctuations are more or less equally spaced along the $v_{\parallel}$ axis and their amplitude decreases as we go away from $v_{\parallel} = v_0$. By looking at the shape of the fluctuations, one can qualitatively infer that the distribution of the particles should develop some sort of bunching or fine structure in the resonant range of the velocity space.

2.4 CALCULATION OF THE NEW DISTRIBUTION FUNCTION

We presume that during the interaction, the particles affected by the waves lie in a narrow range R of $v_{\parallel}$ close to the central resonant parallel velocity. The particles outside this range are considered to be unaffected by the waves and therefore their distribution function should remain unchanged. This leads to a further conclusion that, although the particles are redistributed in the region R, the total number of particles in that region should also remain unchanged. This amounts to saying that there is no diffusion of particles across the boundary of the resonant and the non-resonant regions.

Let the initial and the final velocities of a particle parallel to B_0 be $v_{\parallel 1}$ and $v'_{\parallel 1}$ and those of another be $v_{\parallel 2}$ and $v'_{\parallel 2}$. Through out this treatment the prime refers to the final value of a quantity.

If $|v_{\parallel 1} - v_{\parallel 2}|$ is very very small compared to the characteristic range of $v_{\parallel}$ over which the initial distribution function f_0 changes by a factor of e , the number of particles having $\vartheta_{\parallel}$ between $\vartheta_{\parallel 1}$ and $\vartheta_{\parallel 2}$ can be considered proportional to

$$\left| \vartheta_{\parallel 2} - \vartheta_{\parallel 1} \right| f \left\{ \frac{\vartheta_{\parallel 1} + \vartheta_{\parallel 2}}{2} \right\}$$

Similarly, if $\left| \vartheta_{\parallel 2}' - \vartheta_{\parallel 1}' \right|$ is very very small compared to the average range of $\vartheta_{\parallel}$, in the $\delta\vartheta_{\parallel}$ versus $\vartheta_{\parallel}$ graph (shown in fig. 2.2), over which $\delta\vartheta_{\parallel}$ keeps on fluctuating, the number of redistributed particles having velocities between $\vartheta_{\parallel 1}'$ and $\vartheta_{\parallel 2}'$ can be taken proportional to

$$\left| \vartheta_{\parallel 2}' - \vartheta_{\parallel 1}' \right| f' \left\{ \frac{\vartheta_{\parallel 1}' + \vartheta_{\parallel 2}'}{2} \right\}$$

From the shape of the $\delta\vartheta_{\parallel}$ versus $\vartheta_{\parallel}$ curve shown in fig. 2.2, it can be inferred that the number density of particles redistributed along the $\vartheta_{\parallel}$ - axis will exhibit alternate compressions and rarefactions in the resonant region.

This shows that if we consider ranges of velocities on $\delta v_{\parallel} - v_{\parallel}$ graph that are very narrow compared to the periodicity of the fluctuations themselves, then as a result of the interaction, they will either contract or expand as illustrated in ~~the~~ figure (2.3).

In this situation, the condition for the conservation of the number of particles in the resonant region combined with the fact that what ever particles existed initially in the range $v_{\parallel 1} - v_{\parallel 2}$ occupy the range $v_{\parallel 1}' - v_{\parallel 2}'$ after the redistribution, gives the following relationship:

$$|v_{\parallel 2}' - v_{\parallel 1}'| \, f' \left\{ \frac{v_{\parallel 2}' + v_{\parallel 1}'}{2} \right\} = |v_{\parallel 2} - v_{\parallel 1}| \, f \left(\frac{v_{\parallel 2} + v_{\parallel 1}}{2} \right)$$

which directly gives

$$f' \left(\frac{v_{\parallel 2}' + v_{\parallel 1}'}{2} \right) = \frac{|v_{\parallel 2} - v_{\parallel 1}|}{|v_{\parallel 2}' - v_{\parallel 1}'|} \, f \left(\frac{v_{\parallel 2} + v_{\parallel 1}}{2} \right) \quad (2.4)$$

from which the final distribution f' can be easily calculated if the initial distribution function f is known.

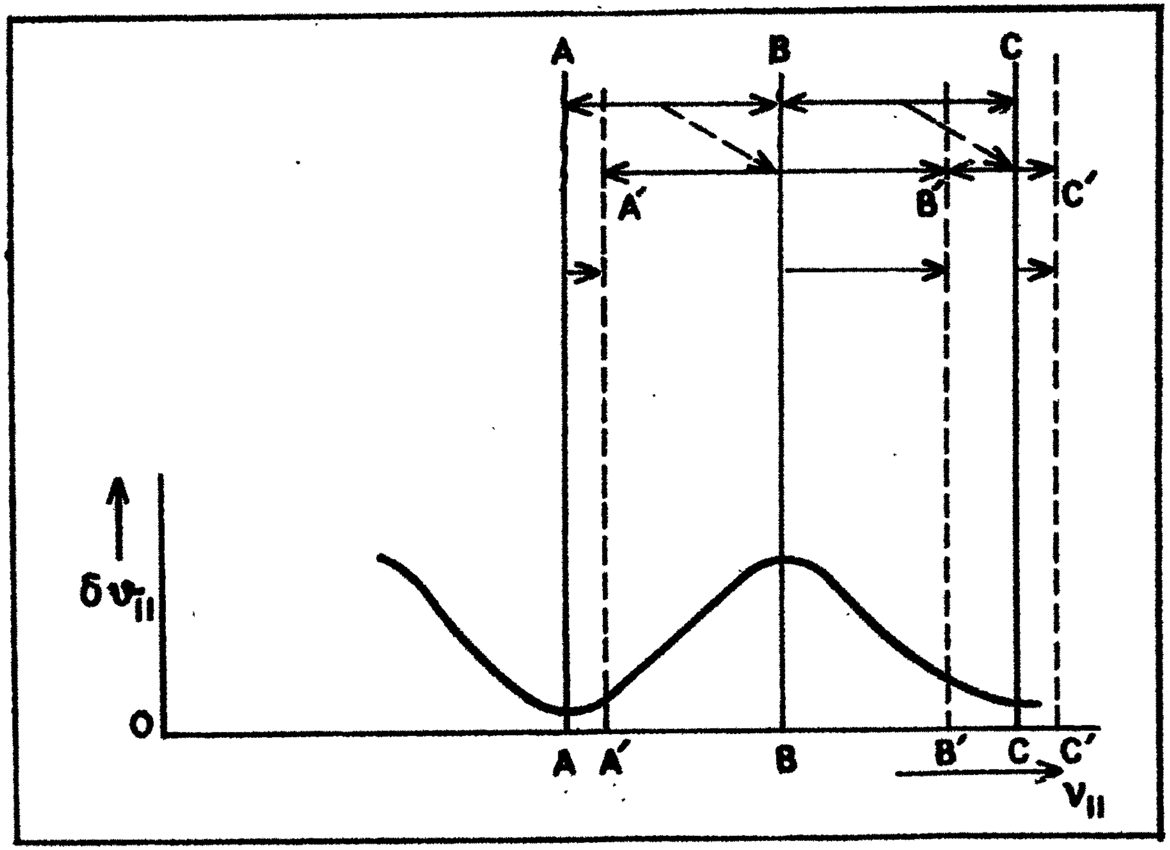

Fig. 2·3

ILLUSTRATION OF THE EXPANSION OF THE RANGE AB INTO A' B' AND THE CONTRACTION OF THE RANGE BC INTO B'C' AS A RESULT OF THE UNEVEN VARIATION IN $V_{\parallel}$ AT A, B AND C

The initial distribution function was taken to be of the following form:

$$f = f_0 \, v^{-4} \ln \left(\sin \alpha / \sin \alpha_0 \right) \qquad (2.5)$$

where f_0 is a constant and $\alpha = \tan^{-1} (v_\perp / v_\parallel)$ is the particle pitch angle. α_0 is the lower limit on the pitch angle of the particles existing in the system.

The numerical values of the different quantities used during computations are as follows:

The electron gyro frequency	Ω	$= 2.03 \times 10^5$ rad/sec.
The electron plasma frequency	ω_p	$= 11.115 \times 10^5$ rad/sec
Propagation angle	θ	$= 30^\circ$
Pitch angle	α	$= 10^\circ$
Lowest pitch angle existing in the system (i.e. the loss cone angle)	α_0	$= 6^\circ$

Figure (2.4) shows the results of the computations for the new distribution function after an interaction lasting about 1200 gyroperiods. The changes in $v_{\parallel}$ used during the calculations for this new distribution are same as shown in fig. (2.2).

The initial distribution of the particles over the range of resonant velocities has been shown by the broken curve in fig. 2.4. It looks like a horizontal straight line because the range of the resonant velocities is so narrow that the change in f over this range is too small to show up itself on the limited scale of the graph. The continuous curve shows how the particles get redistributed and develop a fine structure during the interaction.

This fine structure contains alternate positive and negative gradients of f with $v_{\parallel}$. These were also computed and are shown in figure (2.5). Positive gradient of f with $v_{\parallel}$ lead to wave growth and since the graph of $\partial f / \partial v_{\parallel}$ versus $v_{\parallel}$ shows many pronounced peaks on either side of the central resonant velocity, the growth of waves at corresponding

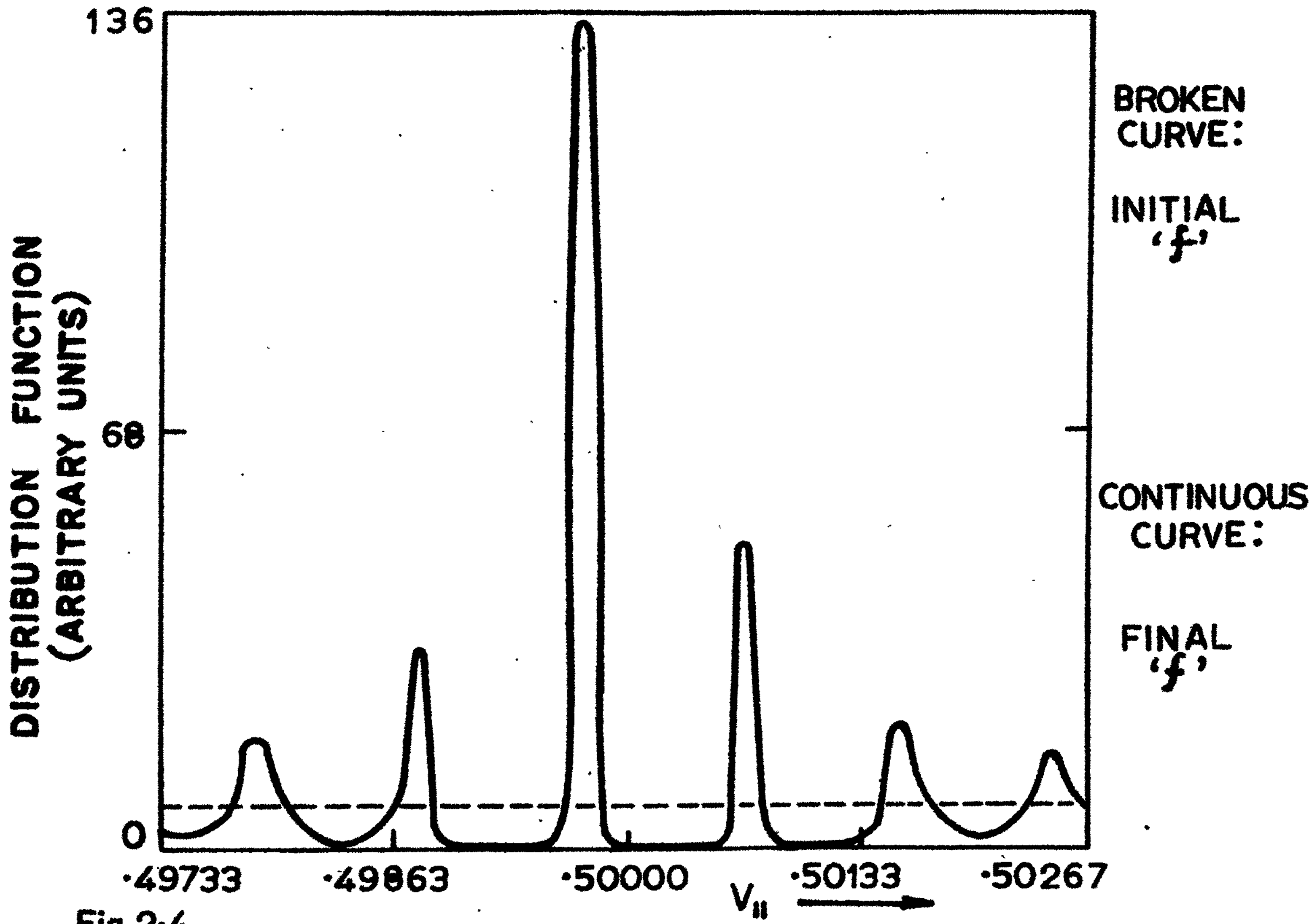

Fig.2·4

INITIAL AND FINAL DISTRIBUTION FUNCTION 'f' AGAINST '$V_{\parallel}$'

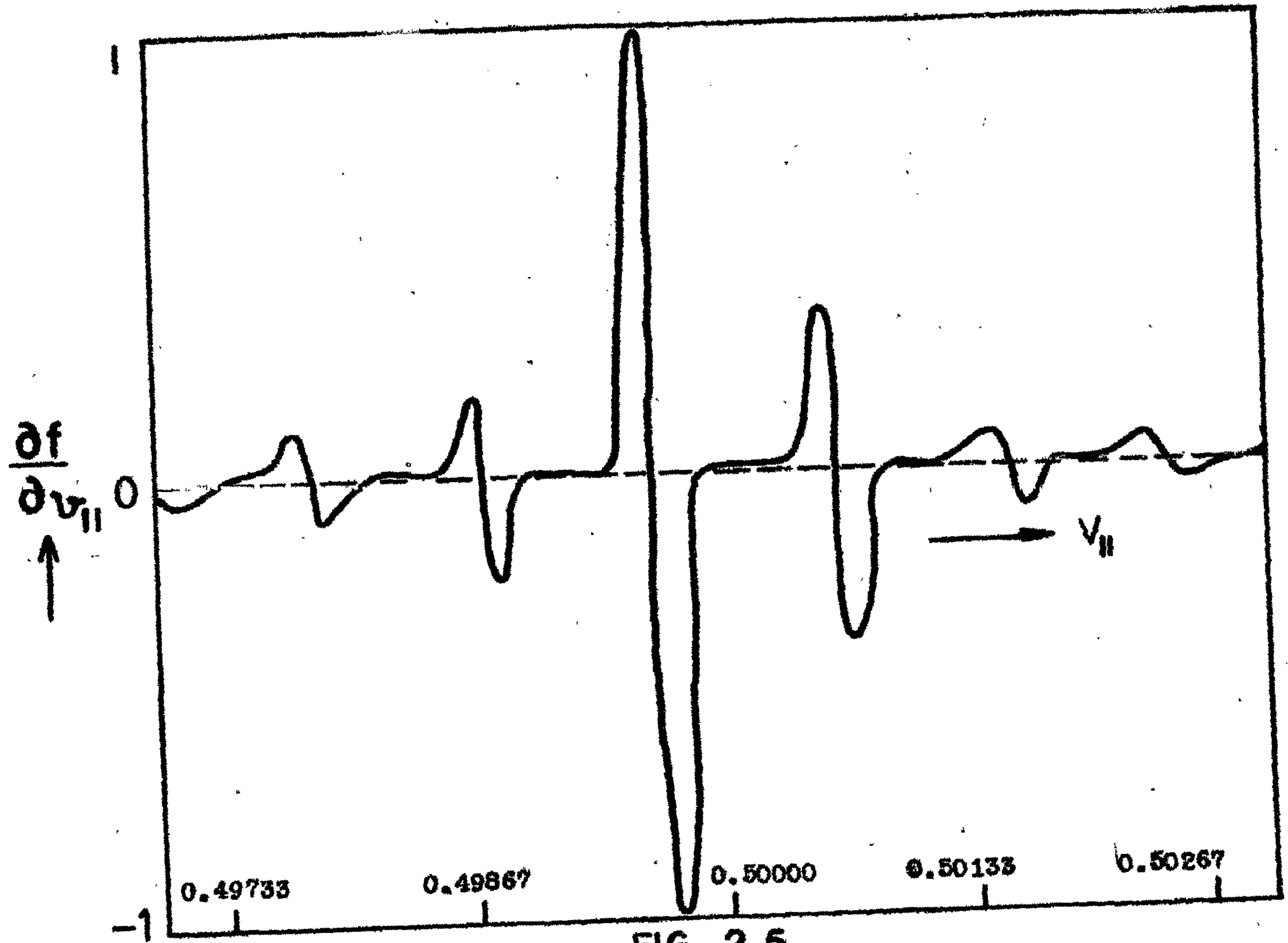

FIG 2.5

A PLOT OF $\partial f/\partial V_{\parallel}$ AGAINST $V_{\parallel}$ (UNITS ARBITRARY). BROKEN CURVE SHOWS INITIAL $\partial f/\partial V_{\parallel}$ WHICH IS SMALL AND DOES NOT SHOW UP. CONTINUOUS CURVE SHOWS $\partial f/\partial V_{\parallel}$ AFTER 1200 GP frequencies is expected if suitable perturbations exist in the system. It can be seen that there is no growth of waves corresponding to the central resonant velocity.

2.5 APPLICATIONS TO VLF EMISSIONS:

The system that we have studied so far closely resembles the equatorial magnetosphere where the VLF emissions are believed to be generated. The earth's magnetic field is approximately dipole in character and is roughly given by

$$B = (B_0/L^3)(1+3\sin^2\lambda)^{1/2}(\cos\lambda)^{-6} \qquad (2.6a)$$

where B is the field strength along field line at a geomagnetic latitude λ, B_0 is the equatorial field strength at the surface of the earth and L is the distance of the equatorial point of a field line from the centre of the earth measured by taking the radius of the earth as the unit of length.

In the vicinity of the equator, the equation (2.6b) can be closely approximated by

$$B = B_0 \left(1 + \frac{9}{2} \frac{z^2}{R_E^2 L^2} \right) \qquad \text{---(2.6b)}$$

Here R_E is the radius of the earth and z is the distance from the equator, of a point at which the field strength is B .

The most significant changes in the velocities of the particles take place in about a thousand gyroperiods' time as shown by the calculations mentioned in the last section. Considering the particle velocities to be roughly one tenth of the velocity of light and taking $L = 3$ at which a gyroperiod is approximately 3×10^{-5} sec., the distance travelled by the particles in 1000 gyro periods would be of the order of 1000 kilometers. The variation in the field strength over this distance is to the tune of 1% only and therefore the field may be considered to be fairly uniform over the duration and the spatial length of the effective resonance.

The plasma in the magnetosphere is fully ionised and at $L = 3$, and has a density merely of the order of 400 particles/c.c. It can be considered as collisionless because the collision frequencies are very small compared to the plasma frequency which is of the order of 10^6 radians/sec.

The plasma in radiation belts has two components very similar to those we have considered for our calculations: One is the low temperature back ground plasma and the other consists of magnetically trapped high energy particles. The distribution of the high energy particles gets modified by interaction with the waves. A loss cone has also been included in the previous calculations to make the resemblance complete.

We should consider the effects of the resonant interaction of the pulse with the particles all along the line of propagation. However, the resonances that take place away from the equator do not contribute much to the changes in $\nu_{\parallel}$ owing to their effectively shorter durations compared to those of the near equator resonances. This difference is caused by the nonuniformity of the static magneticfield in the regions away from the equator where k and $\nu_{\parallel}$ change relatively faster with z , being functions of B, the magnetic field. (see fig.2.6).

Now if the distribution function is considered to be already modified by an initial off angle VLF whistler mode pulse, it is easy to see that another pulse moving along a field line in the same mode will not be appreciably affected unless it reaches near the equator where the strong resonant effects come into picture and lead to a damping of the central frequencies and to the growth of its side bands. This modified pulse, when reaches again to the regions of highly nonuniform magnetic field on the other side of the equator, does not suffer any further significant changes in its spectrum and reaches the ground observation station in more or less the same shape.

A mechanism, like the one discussed above, leading to the growth of the side bands of a wave packet (Brinca 1972, Vyas 1974) may or may not directly trigger VLF emissions but the underlying physical process is very interesting and important and suggests that the investigation of similar processes might lead

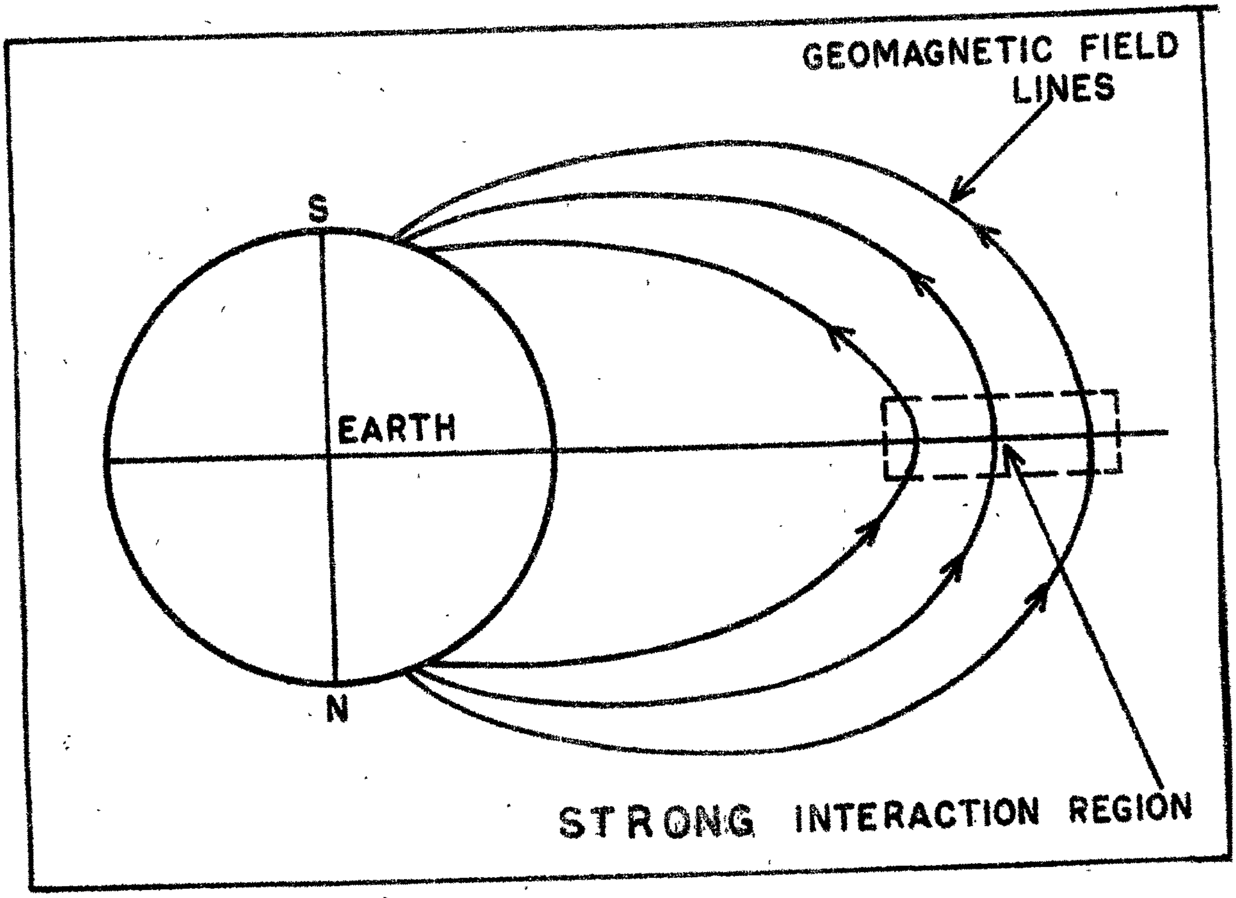

Fig. 2·6

THE REGION OF STRONG INTERACTION LIES CLOSE TO THE EQUATORIAL ZONE OF THE EARTHs MAGNETOSPHERE WHERE THE FIELD IS RELATIVELY UNIFORM us to a better understanding of the phenomenon of VLF emissions from the magnetosphere.

In the next chapter, we will take up the study of the gyroresonance of a wavepacket with the particles and discuss its relevance to VLF emissions.