2.1 Introduction

We have seen in chapter-1 that the general solution of the Liouville equation obtained by an iterative procedure, results in an infinite number of terms. Summation of this infinite series is equivalent to an exact solution of the N-body problem involved. Such a solution will be inconceivably difficult to obtain and also is not necessary since one considers only local properties in which only a few particles take part. The application of this general solution to a specific problem is done by selecting particular terms depending on the relevant time scales pertaining to the problem, by using a diagram technique. A diagram is graphical representation of the contribution of a given term in the series solution (1.1.23) for the problem concerned. There is a formal analogy with the quantum field theory but we deal here with the creation and annihilation of correlations instead of the creation and destruction of particles as in quantum field theory. Hence time and length scales are of paramount importance in this formulation.

A general dynamical system is characterized by writing a Hamiltonian of the form

\[ \mathcal{H} = \mathcal{H}_0 + \lambda V \]  

(2.1.1)
where $H_0$ - Non-interacting part of the Hamiltonian
$V$ - interaction potential
$\lambda$ - the coupling parameter.

The precise rules for constructing the diagrams for different contributions to the solution (1.1, 23) were explained in detail in a monograph by Prigogine (1962). Balescu (1963) in his monograph has also explained with a special reference to the electrostatic interactions amongst charged particles. In this chapter we explain the diagram technique given by Pratap (1967) with reference to his work on Microscopic Theory of Cerenkov Radiation and certain modifications in the general formulation.

2.2 Classical Theory of Cerenkov Radiation.

The phenomenon of Cerenkov Radiation is due to the passage of a very high energy charged particle through a dielectric medium, with a velocity greater than the phase velocity of light in that medium. This phenomenon was experimentally observed by Cerenkov and theoretically interpreted by Frank and Tamm in the context of classical electrodynamics. The state of the work in this field up to 1956 is detailed in the book by Jelley (1958). A good account of the calculation of the fields and the radiation of Cerenkov electron is also explained by Sommerfeld (1954).

The basic mechanism involved in the production of this radiation is the polarization of the atoms or the molecules constituting the medium by the electric field of the incoming high energy particle. In this state each molecule acts like an elementary dipole. Thus as the external particle passes through
the medium, each elemental region of the medium along its track will in turn receive a very brief electromagnetic pulse. It can be seen that when the velocity $\nu$ of the external particle is less than the phase velocity of light in that medium, polarization occurs in the medium symmetrically in all directions with reference to an instantaneous position of the particle. That is to say the interaction between the incoming particle and the molecules are confined within a sphere with the particle at the centre and this sphere will be moving along with the particle. But when $\nu$ exceeds the phase velocity of light in that medium the above symmetry in polarization is no more maintained and the interaction with the molecules is confined within a cone with the axis along the track of the external particle. This cone will be trailing behind the particle. This phenomenon is similar to shocks one observes in aerodynamics. In such a case it is possible for the wavelets from all portions of the track to be in phase with one another so that at a distant point of observation there is a resultant field. Following from the Huygens construction shown in Fig.1, this radiation is observed at a particular angle $\Theta$ with respect to the track of the external particle. This angle $\Theta$ satisfies a relation

$$\cos \Theta = \frac{1}{\eta \beta}, \quad \beta = \frac{\nu}{c}$$  \hspace{1cm} (2.2.1)

where $\eta$ - the refractive index of the medium
$c$ - the speed of light in vacuum.
Huygens construction to illustrate coherence.

Fig. 1
The above relation (2.2.1) is called the "Cerenkov relation" which is the condition for the coherence of the wavelets coming from the points $P_1$, $P_2$, $P_3$, etc. on the track $AB$ of the incoming particle (Fig. 1).

Thus the radiation originating from each element of the track of the Cerenkov particle is propagated along the surface of a cone, whose apex is at this element, whose axis coincides with the track and whose semi-vertical angle is angle $\theta$. As shown in Fig. 2, the polarization is such that the electric vector $E$ is everywhere perpendicular to the surface of the cone, and the magnetic field vector $H$ tangential to this surface.

In this chapter we will explain in brief the classical theory of Frank and Tamm. This is given here to have a better appreciation of the method we adopt to solve the problem microscopically.

Frank and Tamm in their original treatment of the problem of radiation of an electron moving uniformly in a dielectric medium have made the following simplifying assumptions:

1. The medium is considered as a continuum, so that the microscopic structure is ignored; the dielectric constant is then the only parameter used to describe the behaviour of the medium.
2. Dispersion is ignored, at least in the first approximation.
3. Radiation reaction is neglected.
4. The medium is assumed to be a perfect isotropic dielectric, so that the conductivity is zero, the magnetic permeability is unity and there is no absorption of radiation.
The formation of the Čerenkov cone, and the polarization vectors.

Fig. 2
5. The electron is assumed to move at constant velocity.
6. The medium is unbounded and the track length infinite.

Now the polarization of the medium due to the electric field \( E \) of the incoming particle is given by the polarization vector

\[
P = -c I e I S
\]  \hspace{1cm} (2.2.2)

where \( c I \) is the number of displaced electrons of charge \( e I \) and mass \( m I \) per unit volume and \( S \) is the displacement vector produced from the rest position.

Since the electrons in the system are bound "quasi-elastically" to their rest positions, they will seek to return to these positions. Thus \( S \) satisfies the following differential equation:

\[
m_I \ddot{S} + f_c S = -e_I E
\]

or

\[
\ddot{S} + \frac{f_c}{m_I} S = -\frac{e_I}{m_I} E
\]  \hspace{1cm} (2.2.3)

where \( f_c \) is the force constant. In terms of \( P \), the polarization vector, the equation (2.2.3) becomes

\[
\left( \frac{\partial^2}{\partial t^2} + \gamma^2 I \right) P = c I e I^2 \frac{E}{m_I}
\]  \hspace{1cm} (2.2.4)

where \( \gamma I \) is the characteristic frequency of the molecular oscillator. \( E \) and \( P \) may now be expanded in a Fourier series as

\[
E_I = \int_{-\infty}^{\infty} d\omega E_\omega \exp(i\omega t)
\]

\[
P_I = \int_{-\infty}^{\infty} d\omega P_\omega \exp(i\omega t)
\]  \hspace{1cm} (2.2.5)
and using it in (2.2.4) we get a relation between $P_\omega$ and $E_\omega$ as

$$P_\omega = (\eta^2 - 1) E_\omega$$  \hspace{1cm} (2.2.6)

where

$$\eta^2 = 1 + \left( \frac{c_0 e_0^2 / m_e}{\gamma_0} \right) / (\gamma_0^2 - \omega^2)$$  \hspace{1cm} (2.2.7)

is the refractive index of the medium at the frequency $\omega$. Expanding other field variables in a similar way we write the Maxwell's equations in the medium

$$\nabla \cdot H = 0, \quad \nabla \times H = \frac{4\pi j}{c} + \frac{1}{c} \frac{\partial D}{\partial t}$$

$$\nabla \cdot D = 4\pi j_c, \quad \nabla \times E = -\frac{1}{c} \frac{\partial H}{\partial t}$$  \hspace{1cm} (2.2.8)

as

$$\text{div} \ H_\omega = 0, \quad \text{curl} \ H_\omega = \frac{4\pi j_\omega}{c} + i\omega D_\omega$$

$$\text{div} \ D_\omega = 4\pi j_c, \quad \text{curl} \ E_\omega = -\frac{i\omega}{c} H_\omega$$  \hspace{1cm} (2.2.9)

where $j_c$ is the density of free charges and $j_\omega$ is the current density and the dielectric induction $D_\omega$ is connected to $E_\omega$ as

$$D_\omega = \eta^2 E_\omega$$  \hspace{1cm} (2.2.10)

The equation $\text{div} H_\omega = 0$ in (2.2.9) implies that

$$H_\omega = \text{curl} \ A_\omega$$  \hspace{1cm} (2.2.11)
and the equation for \( \nabla \times \mathbf{E} \) in (2.2.9) with (2.2.11) implies that
\[
\mathbf{E} = -i\omega \frac{\mathbf{A}}{c} - \nabla \Phi \quad (2.2.12)
\]
where \( \mathbf{A} \) and \( \Phi \) respectively are the vector and scalar potentials.

Making use of the general vector relation
\[
\nabla \times \nabla \times = \nabla \nabla \cdot - \nabla^2 \quad (2.2.13)
\]
the other two equations of the set (2.2.9) can be written as
\[
\nabla^2 \mathbf{A} + \frac{\gamma^2 \omega^2}{c^2} \mathbf{A} + \mathbf{j} = -\frac{1}{\gamma} \frac{4\pi i}{c} \mathbf{E} \quad (2.2.14a)
\]
and
\[
\nabla^2 \Phi + \frac{\gamma^2 \omega^2}{c^2} \Phi = -\frac{1}{\gamma^2} \frac{4\pi \rho}{c} \quad (2.2.14b)
\]
with the Lorentz condition
\[
\nabla \cdot \mathbf{A} + \frac{i\omega}{c} \gamma^2 \Phi = 0 \quad (2.2.15)
\]

If the Cerenkov particle is an electron moving through the medium along the \( z \)-axis with a constant velocity \( \gamma \), the corresponding current density will be given by
\[
j_z = e \gamma \delta(x) \delta(y) \delta(z - \gamma t) \quad (2.2.16)
\]
where \( \delta \) denotes the Dirac's delta function. When written in terms of Fourier components (2.2.16) becomes
Introducing the cylindrical co-ordinates, $\rho, \phi, z$; and assuming an axial symmetry, we get
\[ \delta(x) \delta(y) = \frac{1}{2\pi} \rho \delta(\rho) \tag{2.2.18} \]
and so
\[ j_z(\omega) = \frac{e^{\nu z^2}}{4\pi ^2 \rho} \exp(-i\omega z/\nu) \delta(\rho) \tag{2.2.19} \]
Substituting (2.2.19) in (2.2.14a) and putting $A_\rho = A_\phi = 0$ and $A_z(\omega) = U(\rho) \exp(-i\omega z/\nu)$ we obtain:
\[ \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \rho^2 U = -\frac{e}{\nu^2 \rho} \delta(\rho) \tag{2.2.21} \]
where
\[ \rho^2 = \frac{\omega^2}{\gamma^2} (\beta^2 - 1) = -\sigma^{-2} \tag{2.2.22} \]
Thus $U$ is a cylindrical function satisfying the Bessel equation:
\[ \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \rho^2 U = 0 \tag{2.2.23} \]
everywhere except at the pole $\rho = 0$. To find the condition to be satisfied by $U$ at $\rho = 0$, we replace the right hand side of (2.2.21) by $f$, such that
\[ \bar{f} = -\frac{2e}{\pi c S_0^2} \text{ if } S < S_0 \text{ and } \bar{f} = 0 \text{ if } S > S_0 \]

In Jelly's book, \( \bar{f} = -\frac{2e}{\pi c S_0} \) which is dimensionally not correct. The right way to consider this is to take
\[ \delta(S) = \frac{1}{S_0^2} \delta(S) \text{ with } S = \frac{S}{S_0} \]
Thus we can write (2.2.21) as
\[ \frac{i}{\bar{p}} \frac{\partial}{\partial S} \left( S \frac{\partial U}{\partial S} \right) + \bar{S}^2 U = -\frac{2e}{\pi c S_0^2} \]
Then integrating (2.2.24) over the surface of the disc or radius \( S_0 \) and lastly taking the limit \( S_0 \to 0 \), we obtain the condition
\[ \lim_{S \to 0} S \frac{\partial U}{\partial \bar{S}} = -\frac{e}{\pi c} \]

Now two cases of interest arise:
(a) When the electron is moving with a speed such that \( \beta \gamma < 1 \) (\( \beta = \gamma c \)), from (2.2.22) it is clear that \( \sigma^{-2} > 0 \) and \( \bar{S}^2 < 0 \).
Thus the solution of the equation (2.2.23) which satisfies the condition (2.2.25) and vanishes at infinity, is
\[ U = \frac{i e}{\pi c} H_0^{(1)}(i \sigma \bar{S}) \] (2.2.26)
where \( H_0^{(1)} \) is the Hankel function of the first kind. If \( \sigma \bar{S} \gg 1 \) one can use the following asymptotic expansion for \( H_0^{(1)}(Abramowitz and Stegun, 1970) \)
Thus with (2.2.20), (2.2.26) and (2.2.27), we write \( A_z(t) \) as

\[
A_z(t) = \frac{e}{c} \int_{-\infty}^{\infty} d\omega \exp[-\sigma \gamma + i\omega(t-z)]/\sqrt{2\pi\sigma} (2.2.28)
\]

The expression (2.2.28) for \( A_z(t) \) indicates that for \( \beta \eta < 1 \), the field of the electron decreases exponentially with \( \omega \). Thus at a point far away from the track of the electron, no radiation will be observed.

(b) When the velocity of the incoming particle (electron) is very high such that \( \beta \eta > 1, \sigma^2 > 0 \) and \( \sigma^2 < 0 \) and the general solution of equations (2.2.21) and (2.2.23) represents, at infinity, a cylindrical wave. Specifying \( U \) in this case to represent an outgoing and not an ingoing wave, the solution of (2.2.23) satisfying the condition (2.2.25) is

\[
U = -\frac{i\epsilon}{2c} H_0^{(2)}(\gamma \xi) \quad \text{if} \quad \omega > 0 \quad (2.2.29)
\]

and a complex conjugate expression if \( \omega < 0 \), \( \Lambda \) being assumed to be positive. Using the asymptotic expansion of \( H_0^{(2)}(x) \) for \( x(=\xi \gamma) \gg 1 \) which is

\[
H_0^{(2)}(x) = \sqrt{\frac{2}{\pi x}} \exp\left\{-i \left( x - \frac{1}{2} \gamma \pi^2 - \frac{\pi}{4} \right) \right\} \quad (2.2.30)
\]
A \text{C} \ toucan \ be; \ written \ as \ \textit{V}, \ (2.2.31)

when \ \omega > 0. \ From \ (2.2.22) \ \Delta^2 = \omega^2 + \frac{1}{\eta^2} (\beta^2 \eta^2 - 1), \ if \ we \ define \ an \ angle \ \theta \ such \ that \ \cos \theta = \frac{1}{\eta \beta}, \ then \ \Delta = \frac{\omega \sin \theta}{c/\eta} \ and \ \frac{1}{\nu} = \frac{\cos \theta}{c/\eta}.

Thus \ (2.2.31) \ can \ be \ written \ as

\[ A_z(\omega) = -\frac{ie}{c} \sqrt{\frac{2}{\pi}} \frac{1}{2\sqrt{\beta} \Delta} \times \exp \left\{ -i \left\{ \omega \left( \frac{z \cos \theta + \frac{\eta}{c} \sin \theta}{c/\eta} - \frac{\pi}{4} \right) \right\} \right\} \] \hspace{1cm} (2.2.32)

for \ \omega > 0. \ Considering \ the \ complex \ conjugate \ of \ (2.2.32) \ for \ \omega < 0 \ we \ can \ write \ \A_z(t) as

\[ A_z(t) = \frac{a}{\sqrt{\frac{\pi}{c}}} \int_0^\infty \frac{d\omega}{\sqrt{s}} \sin \chi \] \hspace{1cm} (2.2.33)

where

\[ a = \frac{e}{c} \sqrt{\frac{2}{\pi}} \] \hspace{1cm} \[ \chi = \omega \left( t - \frac{z \cos \theta + \frac{\eta}{c} \sin \theta}{c/\eta} \right) + \frac{\pi}{4} \] \hspace{1cm} (2.2.34)

Hence for the velocity of the Cerenkov particle for which \ \beta \eta > 1, \ a \ wave \ is \ propagated \ at \ infinity, \ along \ the \ direction \ \theta. \ The \ electric \ vector \ of \ the \ wave \ lies \ in \ the \ meridian \ plane \ (2, \ \eta).

When we calculate the field intensity in the wave zone with the Maxwell equations there are only three non-vanishing field vectors which are:

\[ H_\phi = -\frac{\partial A_z}{\partial \phi} = \frac{a}{\sqrt{\frac{\pi}{c}}} \int \sqrt{s} \cos \chi \ d\omega \]
\[ E_p = -\frac{1}{c} \frac{\partial A_p}{\partial t} - \frac{\partial \Phi}{\partial \rho} \]
\[ = \frac{a}{c\sqrt{\rho}} \int d\omega \omega \left( \frac{\beta^2 \eta^2 - 1}{\beta^2 \eta^2} \right)^{\frac{1}{2}} \frac{\cos \chi}{\sqrt{\beta}} \]  
(2.2.35)

\[ E_z = -\frac{1}{c} \frac{\partial A_z}{\partial t} - \frac{\partial \Phi}{\partial z} \]
\[ = -\frac{a}{c\sqrt{\rho}} \int d\omega \omega \left( 1 - \frac{1}{\beta^2 \eta^2} \right) \frac{\cos \chi}{\sqrt{\beta}} \]

The total energy radiated by the electron through the surface of a cylinder of length \( l \) whose axis coincides with the line of motion of the electron track is

\[ \tilde{E} = \int_{-\infty}^{\infty} dt \int \frac{c}{4\pi} \left( E_x H_y \right) \frac{dA_p}{\rho} \]

where \( dA_p \) - surface area element of the cylinder, or

\[ \tilde{E} = -2\pi p l \int_{-\infty}^{\infty} \frac{c}{4\pi} E_x H_\phi \frac{d\omega}{\rho} \frac{dt}{\omega} \]
\[ = - \frac{c \rho l}{2} \int_{-\infty}^{\infty} E_x H_\phi \frac{dt}{\omega} \]  
(2.2.36)

Thus the energy lost by the Cerenkov electron per unit path length is

\[ \frac{d\tilde{E}}{dl} = \frac{a^2}{2} \int_{-\infty}^{\infty} dt \int d\omega d\omega' \omega' \sqrt{\nu_{\nu'}} \cos \chi' \frac{\cos \chi \left( 1 - \frac{1}{\beta^2 \eta^2 (\omega)} \right)}{\nu} \]

(2.2.37)

With the aid of the formula

\[ \int_{-\infty}^{\infty} \cos (\omega t + \alpha) \cos (\omega' t + \beta) dt = \pi \delta (\omega - \omega') \cos (\alpha - \beta) \]  
(2.2.38)
we write \((2.2.37)\) as

\[
\frac{d\mathcal{E}}{dl} = \frac{e^2}{c^2} \int d\omega \, \omega \left( 1 - \frac{1}{\beta^2 \eta^2} \right)
\]

\((2.2.39)\)

which is the fundamental relation for the output of the radiation.

In the above, all the \(\omega\)-integrations have to be carried out only over the values of \(\omega\) for which \(\beta \eta(\omega) > 1\). In \((2.2.39)\) consideration of the dispersive nature of the medium and the finite size of the electron set an upper limit to the frequency spectrum and cause the radiation yield to be finite.

2.3 General Formulation of the Microscopic Theory

We shall give in this section the microscopic formulation of the above problem. We thereby relax the first assumption of Frank and Tamm given in the previous section.

In a many particle system, there will be two types of interactions: collective interactions and single particle collisions. In most of the radiation problems dealing with ionized media the collective interactions predominate over the single particle collisions. Collective interactions in general occur in three different modes: transverse \((\pi)\), longitudinal \((\sigma^-)\) and scalar \((\sigma^0 \sigma^-)\) modes. The transverse or electromagnetic interaction corresponds to the light waves and the longitudinal and the scalar waves are the consequences of the Coulomb interaction in the system.

We consider a general system of harmonic oscillators (constituting the medium) designated by the subscript \(l\), each with a
frequency \( \nu \), a relativistic charged (test) particle denoted by the subscript \( T \) and a radiation field. The interactions in the system are through this radiation field. In analogy with the field theory, we say that an interaction is complete if a particle emits a photon and the same particle absorbs the radiation after a finite time lapse.

We write the Hamiltonian of the system as

\[
\mathcal{H} = \mathcal{H}^j + \mathcal{H}^l + \mathcal{H}^T
\]

(2.3.1)

where \( \mathcal{H}^j, \mathcal{H}^l \) and \( \mathcal{H}^T \) are the Hamiltonians for the radiation field, medium and the test particle respectively. Even though the \( \mathcal{H} \) is written as the sum of three parts, the interaction between the particles in the system are contained through the interaction vector and scalar potentials \( \mathcal{A} \) and \( \Phi \) respectively appearing in each term. These \( \mathcal{A} \) and \( \Phi \) are functions of the position vectors of the particles and the variables representing the radiation field.

**Radiation Field:**

Following Heitler (1954) the radiation field is considered as the superposition of plane waves and hence we make the plane wave expansion of the interaction potentials \( \mathcal{A} \) and \( \Phi \) as follows. Since any vector field can be written as a sum of a divergence free part and a curl free part, we write \( \mathcal{A} \) as

\[
\mathcal{A} = \mathcal{A}^{\lambda} + \mathcal{A}^{\sigma}
\]

(2.3.2)
where $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = 0$.

Now making the planewave expansion we write the divergence free part i.e. the part of $\mathbf{A}$ which is transverse to the direction of wave propagation as

$$
\mathbf{A}^\lambda = \left( \frac{8 \pi c^2}{V} \right)^{1/2} \sum \hat{e}_\lambda \left( Q_\lambda \cos k_\lambda \cdot \mathbf{q} + Q_\lambda \sin k_\lambda \cdot \mathbf{q} \right) 
$$

(2.3.3a)

so that

$$
\hat{e}_\lambda \cdot k_\lambda = 0
$$

(2.3.3b)

and the curlfree part i.e. the longitudinal part in the direction of the wave propagation as

$$
\mathbf{A}^\sigma = \left( \frac{8 \pi c^2}{V} \right)^{1/2} \sum \hat{e}_\sigma \left( Q_\sigma \cos k_\sigma \cdot \mathbf{q} + Q_\sigma \sin k_\sigma \cdot \mathbf{q} \right) 
$$

(2.3.4a)

so that

$$
\hat{e}_\sigma \times k_\sigma = 0
$$

(2.3.4b)

Similarly the scalar potential is written as

$$
\Phi = \left( \frac{8 \pi c^2}{V} \right)^{1/2} \sum \left( Q_\sigma \sin k_\sigma \cdot \mathbf{q} - Q_\sigma \cos k_\sigma \cdot \mathbf{q} \right) 
$$

(2.3.5)

where $V$ is the volume of the system, $c$ the velocity of light, $Q$ the canonical variable for the photon (in general) with different modes, $\hat{e}$ and $k$ with the proper subscripts are the polarization vector and propagation vector respectively for different modes of interaction. In the above expressions for the potentials the summation index is over the wave vectors. But in (2.3.3a) the
summation index $\lambda$ also includes the summation over polarization vector $\hat{e}_\lambda$. It should be noted that $\hat{e}_\lambda$ and $\hat{e}_\sigma$ are mutually orthogonal so that

$$\hat{e}_\lambda \cdot \hat{e}_\sigma = 0$$  \hspace{1cm} (2.3.6)

With the above considerations, the wave equations for the fields when expressed in the canonical form (with canonically conjugate variables $Q_j$ and $P_j$, $j = \pm \lambda, \pm \sigma, \pm 0 \sigma$) reduce to the equations for forced vibration of an oscillator, the force being due to the presence of charged particles. These force terms are contained in the $\mathcal{H}_L^T$ and $\mathcal{H}_R^T$ (Heitler, 1954). Thus we consider our radiation field (in general) as a set of harmonic oscillators with frequency $\gamma_j$ and write $\mathcal{H}_j$ as

$$\mathcal{H}_j = \frac{1}{2} \sum_j \epsilon_j \left( P_j^2 + \gamma_j^2 Q_j^2 \right) + \frac{1}{2} \sum_j \tilde{\epsilon}_j \left( P_{-j}^2 + \gamma_j^2 Q_{-j}^2 \right) \hspace{1cm} (2.3.7)$$

where $\epsilon_j = +1$, when $j$ is $\lambda$ or $\sigma$, and $\epsilon_j = -1$ when $j$ is $0 \sigma$.

**Medium Oscillators:**

For the medium oscillators we write $\mathcal{H}_l$ as

$$\mathcal{H}_l^T = \sum_l \left\{ \frac{1}{2m_l} \left( \frac{p_l^2}{c} - \frac{e_l}{c} \vec{A}_l \right)^2 + \frac{m_l \gamma_l^2}{2} \left( q_l - q_{l0} \right)^2 + \epsilon_l \vec{\Phi}_l \right\} \hspace{1cm} (2.3.8)$$

$$= \sum_l \left\{ \frac{p_l^2}{2m_l} - \frac{e_l}{m_c} \frac{p_l \cdot A_l}{c} + \frac{m_l \gamma_l^2}{2} \left( q_l - q_{l0} \right)^2 + \epsilon_l \vec{\Phi}_l + \frac{e_l^2}{2m_c^2} \vec{A}_l^2 \right\} \hspace{1cm} (2.3.8)$$
where $\mathbf{q}_l$ and $\mathbf{p}_l$ are the canonically conjugate variables, $\mathbf{q}_{l0}$ is the vector representing the equilibrium position for the $l^{th}$ oscillator with frequency $\gamma_l$, $e$ and $m$ are the electronic charge and mass respectively of the particles of the system. $\mathbf{A}_l$ and $\Phi_l$, the vector and scalar potentials respectively, are functions of $\mathbf{q}_l$, $\mathbf{q}_j$. These can be obtained from the expressions (2.3.3) to (2.3.5) by taking into account the subscript $l$ for the position vector $\mathbf{q}_l$ of the oscillators. We can now go into an action-angle representation, action being $J_l$ and angle $\omega_l$ by the following transformation equations

$$
\mathbf{p}_l = \mathbf{a}_l \left(2 m_l \gamma_l J_l \right)^{\frac{1}{2}} \cos \omega_l \\
\mathbf{q}_l = \mathbf{q}_{l0} + \mathbf{a}_l \sqrt{2 J_l/m_l \gamma_l} \sin \omega_l
$$

(2.3.9)

where $\mathbf{a}_l$ is the oscillator strength giving the direction of the oscillation. Thus the expression (2.3.8) can be expressed as

$$
\mathbf{J}_l = \sum_l \left\{ \gamma_l J_l - \frac{e_l}{m_l c} (\mathbf{a}_l \cdot \mathbf{A}_l) (2 m_l \omega_l J_l)^{\frac{1}{2}} \cos \omega_l \\
+ \frac{e_l^2}{2 m_l c^2} \mathbf{a}_l^2 + e_l \Phi_l \right\}
$$

(2.3.10)

The term quadratic in $\mathbf{a}_l$ in (2.3.10) can be evaluated by using the expressions (2.3.3) and (2.3.4) for $\mathbf{a}_l$ as follows:

$$
\sum_l \frac{e_l^2}{2 m_l c^2} \mathbf{a}_l^2 = \sum_l \frac{e_l^2}{2 m_l c^2} \mathbf{a}_l \cdot \mathbf{a}_l \\
= \sum_l \frac{e_l^2}{2 m_l c^2} (\mathbf{A}_l^x + \mathbf{A}_l^\sigma) (\mathbf{A}_l^x + \mathbf{A}_l^\sigma^\prime)
$$
\[
\sum_{\ell} \frac{e_{\ell}^2}{2m_\ell c^2} A_{\ell}^2 = \sum_{\ell} \frac{e_{\ell}^2}{2m_\ell c^2} \left( A_{\ell}^\lambda \cdot A_{\ell}^{\lambda'} + A_{\ell}^\lambda \cdot A_{\ell}^{\sigma'} + A_{\ell}^{\sigma} \cdot A_{\ell}^{\lambda'} + A_{\ell}^{\sigma} \cdot A_{\ell}\right)
\]

(2.3.11)

The first term in (2.3.11) is
\[
\sum_{\ell} \frac{e_{\ell}^2}{2m_\ell c^2} A_{\ell}^\lambda \cdot A_{\ell}^{\lambda'} = \sum_{\ell} \frac{4\pi e_{\ell}^2}{2m_\ell c^2} \sum_{\lambda, \lambda'} \left( \hat{e}_{\lambda} \cdot \hat{e}_{\lambda'} \right) \left[ Q_{\lambda} Q_{\lambda'} \left\{ \cos (k_{\lambda} + k_{\lambda'}) \cdot q_{\ell} \right\} + \cos (k_{\lambda} - k_{\lambda'}) \cdot q_{\ell} \right] + (Q_{\lambda} Q_{\lambda'} + Q_{-\lambda} Q_{-\lambda'}) \times \sin (k_{\lambda} + k_{\lambda'}) \cdot q_{\ell} - (Q_{\lambda} Q_{-\lambda'} - Q_{-\lambda} Q_{\lambda'}) \times \sin (k_{\lambda} - k_{\lambda'}) \cdot q_{\ell} + Q_{-\lambda} Q_{-\lambda'} \times \left\{ \cos (k_{\lambda} - k_{\lambda'}) \cdot q_{\ell} - \cos (k_{\lambda} + k_{\lambda'}) \cdot q_{\ell} \right\}
\]

(2.3.12)

From (2.3.12) we have two types of contributions: the contribution for \( k_{\lambda} \neq k_{\lambda'} \) and the contribution for \( k_{\lambda} = k_{\lambda'} \), which can be written as (from (2.3.12))
\[
\sum_{\ell} \frac{e_{\ell}^2}{2m_\ell c^2} \left( A_{\ell}^\lambda \right)^2 = \frac{1}{2} \sum_{\lambda} \omega_{\ell \lambda} \left( Q^\lambda + Q^{-\lambda} \right) + \sum_{\ell} \frac{2\pi e_{\ell}^2}{V m_\ell} \sum_{\lambda} \left\{ (Q^\lambda - Q^{-\lambda}) \cos 2k_{\lambda} \cdot q_{\ell} \right\} + 2Q_{\lambda} Q_{-\lambda} \sin 2k_{\lambda} \cdot q_{\ell}
\]

(2.3.13)
is the square of the plasma frequency with \( N_L \) as the total number of oscillators constituting the medium. In (2.3.12) and (2.3.13) when we substitute the second equation of (2.3.9), we get totally two types of terms: the term \( \frac{1}{2} \omega_{pl}^2 \sum_{\lambda} \left( \frac{Q_{\lambda}^2}{N_L} + \frac{Q_{-\lambda}^2}{N_L} \right) \) from (2.3.13) which is independent of \( \omega_L \) and the other terms (from both \( k_{\lambda} \neq k_{\lambda}' \) and \( k_{\lambda} = k_{\lambda}' \) contributions) which are dependent on \( \omega_L \), the angle variable for the \( L \)-oscillators. We shall retain the \( \omega_L \) independent term from (2.3.12) and (2.3.13) and this is called the random phase approximation (Bohm and Pines, 1951). In this approximation, the angle dependent terms would vanish on averaging.

While considering the contributions from the second and the third terms in (2.3.11) by substituting the expressions (2.3.3) and (2.3.4) for \( A_{\lambda} \) and \( A_{\sigma} \), we get a term \( \hat{a}_{\lambda} \cdot \hat{a}_{\sigma} \) as a multiplication factor like \( \hat{E}_{\lambda}' \cdot \hat{E}_{\lambda} \) in (2.3.12). Thus with the orthogonality condition (2.3.6) we get vanishing contributions from these terms.

Similarly, with the above discussions we obtain a term \( \sum_{\sigma} \frac{1}{2} \omega_{pl}^2 \left( \frac{Q_{\sigma}^2}{N_L} + \frac{Q_{-\sigma}^2}{N_L} \right) \) from the fourth term of (2.3.11) from the contribution \( k_{\sigma} = k_{\sigma}' \).

Test Particle

We consider the test particle to be relativistic in connection with the phenomenon of Cerenkov radiation and thus \( \mathcal{H}_T \) can be written as
\[ H^T = \left[ m_T c^4 + c^2 \left( p_T - \frac{e_T}{c} A_T \right)^2 \right]^{\frac{1}{2}} + e_T \Phi_T (2.3.15) \]

where \( p_T \) and \( A_T \) are canonically conjugate variables for the test-particle, \( A_T \) and \( \Phi_T \) the vector and scalar potentials respectively which are functions of \( \eta_T \) and \( Q_j \). \( m_T c^2 \) is the rest energy of the test-particle. We define a dimensionless variable for the test-particle velocity (Henin, 1963) as

\[ u = \frac{1}{m_T c} \left( p_T - \frac{e_T}{c} A_T \right) \tag{2.3.16} \]

and write (2.3.15) as

\[ H^T = m_T c^2 \left( 1 + u^2 \right)^{\frac{1}{2}} + e_T \Phi_T (2.3.17) \]

Thus the Hamiltonian (2.3.1) for the total system can be written as

\[ H = \left\{ m_T c^2 \left( 1 + u^2 \right)^{\frac{1}{2}} + e_T \Phi_T \right\}^2 + \left[ \sum_i \left\{ \tilde{\gamma}_i \tilde{J}_i \right. \right. + \frac{e_i}{m_T c} (\omega_i \cdot A_i) (2 m_i \gamma_i \tilde{J}_i)^{\frac{1}{2}} \cos \omega_i + e_i \Phi_i \right. \left. \right\}^2 + \frac{1}{2} \omega_{pl} \left\{ \sum \lambda \left( Q_{\lambda}^2 + Q_{-\lambda}^2 \right) + \sum \sigma \left( Q_{\sigma}^2 + Q_{-\sigma}^2 \right) \right\}^2 \]

\[ + \frac{1}{2} \sum \epsilon_j \left( p_j^2 + \gamma_j^2 Q_j^2 \right) \]

\[ + \frac{1}{2} \sum \epsilon_j \left( p_{-j}^2 + \gamma_j^2 Q_{-j}^2 \right) \]

which can be rearranged as
\[ \mathcal{H} = \left\{ m_{\gamma}c^{2}\left(1 + u^{2}\right)^{\frac{1}{2}} + e_{\gamma} \Phi_{\gamma} \right\}_{\gamma} \right\} + \sum_{l} \left\{ \gamma_{l} J_{l} - \frac{e_{l}}{m_{\gamma}c} (\mathbf{a}_{l} \cdot \mathbf{A}) \right\}_{\gamma_{l}} \times \left( 2 m_{\gamma} \gamma_{l} J_{l} \right)^{\frac{1}{2}} \cos \omega_{l} + e_{\gamma} \Phi_{\gamma} \right\}_{\gamma} + \frac{1}{2} \sum_{j} \epsilon_{j} \left( p_{j}^{2} + \gamma_{j}^{2} q_{j}^{2} \right) + \frac{1}{2} \sum_{j} \epsilon_{j} \left( p_{j}^{2} + \gamma_{j}^{2} q_{j}^{2} \right) \]  

(2.3.18)

where \( \gamma_{j} = \gamma_{\lambda} \) or \( \gamma_{\sigma} \) when the subscript \( j \) is \( \lambda \) or \( \sigma \) respectively and \( \gamma_{j} = \gamma_{\sigma} \) when \( j \) stands for \( \sigma \) and

\[ \bar{\gamma}_{\lambda}^{2} = \gamma_{\lambda}^{2} + \omega_{\ell}^{2} \quad \text{and} \quad \bar{\gamma}_{\sigma}^{2} = \gamma_{\sigma}^{2} + \omega_{\ell}^{2} \]  

(2.3.19)

The modified frequencies \( \bar{\gamma}_{\lambda} \) and \( \bar{\gamma}_{\sigma} \) defined in (2.3.18) and (2.3.19) signify the superposition of the plasma waves on the radiation in \( \lambda \) and \( \sigma \) modes respectively. These modified frequencies do not appear in the work by Pratap (1967) and others (Pratap et al., 1972; Sridhar, 1972) because there the quadratic term in \( A_{2} \) was not taken into account at all. The part of the Hamiltonian in (2.3.18) corresponding to the radiation field can be expressed in terms of the action (\( J_{j} \)) and angle (\( \omega_{j} \)) representation by the following transformation equations:

\[ Q_{j} = \sqrt{\frac{2 J_{j}}{\bar{\gamma}_{j}}} \cos \omega_{j} \quad \text{and} \quad p_{j} = \sqrt{2 J_{j}} \bar{\gamma}_{j} \sin \omega_{j} \]  

(2.3.20)

Thus (2.3.18) can be written as

\[ \mathcal{H} = \left\{ m_{\gamma}c^{2}\left(1 + u^{2}\right)^{\frac{1}{2}} + e_{\gamma} \Phi_{\gamma} \right\}_{\gamma} \right\} + \sum_{l} \left\{ \gamma_{l} J_{l} - \frac{e_{l}}{m_{\gamma}c} (\mathbf{a}_{l} \cdot \mathbf{A}) \right\}_{\gamma_{l}} \times \left( 2 m_{\gamma} \gamma_{l} J_{l} \right)^{\frac{1}{2}} \cos \omega_{l} + e_{\gamma} \Phi_{\gamma} \right\}_{\gamma} + \frac{1}{2} \sum_{j} \epsilon_{j} \left( p_{j}^{2} + \gamma_{j}^{2} q_{j}^{2} \right) + \frac{1}{2} \sum_{j} \epsilon_{j} \left( p_{j}^{2} + \gamma_{j}^{2} q_{j}^{2} \right) \]  

(2.3.21)
where now:

\[ A^\lambda = \left( \frac{16 \pi^2 c^2}{V} \right)^{1/2} \sum \frac{\tilde{E}^\lambda}{\sqrt{\gamma^\lambda}} \left\{ \sqrt{J^\lambda} \cos \omega^\lambda \cos k^\lambda \cdot q^\lambda + \sqrt{J^\lambda} \cos \omega^\lambda \sin k^\lambda \cdot q^\lambda \right\} \]

\[ A^\sigma = \left( \frac{16 \pi^2 c^2}{V} \right)^{1/2} \sum \frac{\tilde{E}^\sigma}{\sqrt{\gamma^\sigma}} \left\{ \sqrt{J^\sigma} \cos \omega^\sigma \cos k^\sigma \cdot q^\sigma \right\} \]  

\[ \Phi = \left( \frac{16 \pi^2 c^2}{V} \right)^{1/2} \sum \frac{\tilde{E}^\sigma}{\sqrt{\gamma^\sigma}} \left\{ \sqrt{J^\sigma} \cos \omega^\sigma \sin k^\sigma \cdot q^\sigma \right\} \]

One can obtain \( \tilde{A}^\lambda (q^\lambda), \tilde{A}^\sigma (q^\sigma), \tilde{\Phi} (q^\sigma), \tilde{T}^\sigma (q^\sigma) \) by choosing the proper suffices in (2.3.22).

It may be seen that the significant difference between the present work and the previous papers is the fact, that we have taken into account the quadratic term in the Hamiltonian in the Random Phase Approximation. This results in a frequency shift from \( \gamma_j^2 \) to \( \gamma_j^2 + \omega_{\text{pl}}^2 \) and this present frequency now is dependent on the medium number density as well.

2.4 The Liouville Equation and its formal solution

The Liouville density in the present formulation is given by

\[ \mathcal{P} = \mathcal{P} (\mathbf{u}, q^\sigma; J^\lambda, \omega^\lambda; J^\sigma, \omega^\sigma; t) , \]

\[ j = \pm \lambda, \pm \sigma, \pm 0 \sigma \]  

(2.4.1)

and the evolution equation for \( \mathcal{P} \) which is obtained from the continuity equation in \( \Gamma \)-space is written as
\[
\frac{\partial \phi}{\partial t} + \left( \frac{d}{dt} \frac{\partial \mathcal{H}}{\partial \dot{\mathbf{\phi}}} + \frac{d\omega}{dt} \frac{\partial \mathcal{H}}{\partial \omega} \right) + \sum_j \left( \frac{d\mathcal{J}_j}{dt} \frac{\partial \mathcal{H}}{\partial \mathcal{J}_j} + \frac{d\omega_j}{dt} \frac{\partial \mathcal{H}}{\partial \omega_j} \right) = 0 \quad (2.4.2)
\]

The time derivatives appearing in the above can be obtained by using the Hamilton's canonical equations. We thus write

\[
\frac{d\omega}{dt} = \frac{\partial \mathcal{H}}{\partial \omega} = \gamma - e_T \mathbf{e} \cdot \frac{\partial \mathcal{A}^\lambda}{\partial \omega} - \frac{e}{m_e c} (2 m_i T^l J^l) \frac{1}{2} \cos \omega \times \frac{\partial}{\partial \omega} (\mathbf{a}_e \cdot \mathbf{A}^\lambda)
\]

\[
\frac{d\phi}{dt} = \frac{\partial \mathcal{H}}{\partial \phi} = - \phi - e_T \mathbf{e} \cdot \frac{\partial \mathcal{A}^\sigma}{\partial \phi} + \frac{e}{m_e c} (2 m_i T^l J^l) \frac{1}{2} \cos \omega \times \frac{\partial}{\partial \phi} (\mathbf{a}_e \cdot \mathbf{A}^\sigma)
\]

where

\[
\mathbf{e} = \frac{\mathbf{u}}{\sqrt{1 + \mathbf{u}^2}} = \gamma / c \quad (2.4.4)
\]

\[
\frac{d\mathcal{J}_j}{dt} = \frac{\partial \mathcal{H}}{\partial \omega_j} = e_T \mathbf{e} \cdot \frac{\partial \mathcal{A}^\sigma}{\partial \omega_j} + \frac{e}{m_e c} (2 m_i T^l J^l) \frac{1}{2} \cos \omega \times \frac{\partial}{\partial \omega_j} (\mathbf{a}_e \cdot \mathbf{A}^\sigma) \quad (2.4.5)
\]

\[
\frac{d\mathcal{J}}{dt} = \frac{\partial \mathcal{H}}{\partial \phi} = - e_T \mathbf{e} \cdot \frac{\partial \mathcal{A}^\sigma}{\partial \phi} - \frac{e}{m_e c} (2 m_i T^l J^l) \frac{1}{2} \cos \omega \times \frac{\partial}{\partial \phi} (\mathbf{a}_e \cdot \mathbf{A}^\sigma) - e_l \frac{\partial \phi}{\partial \mathcal{J}}
\]

\[
\frac{d\omega}{dt} = \frac{\partial \mathcal{H}}{\partial \omega} = \gamma - \frac{e}{m_e c} (2 m_i T^l J^l) \frac{1}{2} \frac{\partial}{\partial \mathcal{J}} \left\{ \sqrt{\mathcal{J}} \cos \omega \right\} \times (\mathbf{a}_e \cdot A^\lambda + A^\sigma_l) + e_l \frac{\partial \phi}{\partial \mathcal{J}}
\]
\[
\frac{dJ_l}{dt} = -\frac{\partial H}{\partial \omega_l} = \frac{e_l}{m_ic} (2m_i\gamma_l)^{\frac{1}{2}} \frac{\partial}{\partial \omega_l} \left[ \sqrt{J_l} \cos \omega_l \right.
\]
\[
\left. \times \left( A_{l}^\lambda + A_{l}^\sigma \right) \right] + e_l \frac{\partial \Phi_l}{\partial \omega_l} \tag{2.4.6}
\]

\[
\frac{dq_T}{dt} = \frac{\partial H}{\partial \gamma_T} = c \beta 
\tag{2.4.7a}
\]

\[
\frac{du}{dt} = \frac{1}{m_Tc} \left( \frac{d\gamma_T}{dt} - \frac{e_T}{c} \frac{dA_T^\lambda}{dt} - \frac{e_T}{c} \frac{dA_T^\sigma}{dt} \right)
\]
\[
= \frac{1}{m_Tc} \left[ \left[ -\frac{\partial H}{\partial \gamma_T} - \frac{e_T}{c} \{ \frac{\partial A_T^\lambda}{\partial J_T} \frac{dJ_T}{dt} + \frac{\partial A_T^\lambda}{\partial \omega_T} \frac{d\omega_T}{dt} \right.ight.
\]
\[
\left. \left. \left. + \frac{\partial A_T^\sigma}{\partial \omega_T} \frac{d\omega_T}{dt} \right] + \frac{d\gamma_T}{dt} \frac{\partial}{\partial \gamma_T} \right) \right] \left( A_{T}^\lambda + A_{T}^\sigma \right) \right] \tag{2.4.7b}
\]

or

\[
\frac{du}{dt} = \frac{e_T}{m_Tc} \beta \times \left\{ \nabla_T \times \left( A_T^\lambda + A_T^\sigma \right) \right\} - \frac{e_T}{m_Tc^2} \left( 2 \left( \frac{\gamma_T}{m_Tc} \frac{\partial A_T^\lambda}{\partial \omega_T} \right.ight.
\]
\[
\left. \left. + \frac{\gamma_T}{m_Tc} \frac{\partial A_T^\sigma}{\partial \omega_T} \right) \right) - \left( \frac{e_T}{m_Tc} \right) \nabla_T \Phi_T \tag{2.4.7c}
\]

where \( \nabla_T = \frac{\partial}{\partial \gamma_T} \)

Substituting the time derivatives obtained above in (2.4.2) and separating the terms containing the interaction parameter 'e', we can write the Liouville equation (2.4.2) as
\[
\frac{\partial \Phi}{\partial t} + \left( C_B \cdot \nabla_T + \sum \gamma_\ell \frac{\partial}{\partial \omega_\ell} + \sum \epsilon_j \frac{\partial}{\partial \omega_j} \right) \Phi \\
- \epsilon [i \delta \mathcal{L}] \psi = 0
\]

which can be written as

\[
\frac{\partial \Phi}{\partial t} + i \mathcal{L}_0 \Phi = e \left( i \delta \mathcal{L} \right) \Phi \tag{2.4.8}
\]

where

\[
i \mathcal{L}_0 = \mathcal{L}_0^T + i \mathcal{L}_0^l + i \mathcal{L}_0^j
\]

with

\[
i \mathcal{L}_0^T = C_B \cdot \nabla_T
\]

\[
i \mathcal{L}_0^l = \sum \gamma_\ell \frac{\partial}{\partial \omega_\ell}
\]

\[
i \mathcal{L}_0^j = \sum \epsilon_j \frac{\partial}{\partial \omega_j}
\]

and

\[
i \delta \mathcal{L} = A_j^T + B_j^T + A_j^l + B_j^l \tag{2.4.10}
\]

The explicit expressions for the above operators \( A \) and \( B \) can easily shown to be

\[
A_\lambda^T = - \frac{m_T}{m_T} \sqrt{\frac{16 \pi}{V}} \sum \lambda \frac{1}{\sqrt{\gamma_\lambda}} \frac{\partial}{\partial u_\lambda} \left( \frac{e^{\gamma_\lambda}}{c} \sqrt{f_\lambda} \sin \omega_\lambda \right.
\]

\[
\times \cos k_\lambda \cdot q_T - \sqrt{f_\lambda} \beta \left( \frac{k_\lambda \times e^{\gamma_\lambda}}{c} \right)
\]

\[
\times \cos \omega_\lambda \sin k_\lambda \cdot q_T \right)
\]

\[
A_\sigma^T = - \frac{m_T}{m_T} \sqrt{\frac{16 \pi}{V}} \sum \sigma \frac{1}{\sqrt{\gamma_\sigma}} \frac{\partial}{\partial u_\sigma} \left( \frac{e^{\gamma_\sigma}}{c} \sqrt{f_\sigma} \right.
\]

\[
\times \sin \omega_\sigma \cos k_\sigma \cdot q_T \left.
\right)
\]
\[
\begin{align*}
A\text{\textsuperscript{T}}_{\alpha\sigma} &= \frac{n}{m} \sqrt{\frac{16\pi}{V}} \sum_{\sigma} \frac{1}{\sqrt{\mathcal{V}}_\sigma} \frac{2}{\omega_{\sigma}} \left\{ k_{\sigma} \sqrt{J_{\sigma\sigma}} \cos \omega_{\sigma}\sin \left( k_{\sigma} \cdot q_{\sigma\sigma} \right) \right\} \\
B\text{\textsuperscript{T}}_{\alpha} &= -4nC \sqrt{\frac{\pi}{V}} \sum_{\alpha} \frac{\beta}{\sqrt{\mathcal{V}}_{\alpha}} \cos \left( k_{\alpha} \cdot q_{\alpha\alpha} \right) \left\{ \frac{2}{\omega_{\alpha}} \sqrt{J_{\alpha}} \cos \omega_{\alpha}\sin \left( k_{\alpha} \cdot q_{\alpha\alpha} \right) \right\} \\
B\text{\textsuperscript{T}}_{\sigma\sigma} &= 4nC \sqrt{\frac{\pi}{V}} \sum_{\sigma} \frac{1}{\sqrt{\mathcal{V}}_{\sigma}} \sin \left( k_{\sigma} \cdot q_{\sigma\sigma} \right) \left\{ \frac{2}{\omega_{\sigma}} \sqrt{J_{\sigma\sigma}} \cos \omega_{\sigma}\sin \left( k_{\sigma} \cdot q_{\sigma\sigma} \right) \right\} \\
A_{\lambda\alpha} &= -\left( \frac{32\pi k}{m_{l}V} \right)^{\frac{1}{2}} n \sum_{\alpha} \frac{\alpha_{l}}{\sqrt{\mathcal{V}}_{\lambda}} \sqrt{J_{\alpha}} \cos \omega_{\alpha} \left( \frac{2}{\omega_{l}} \sqrt{J_{l}} \right) \\
&\quad \times \cos \omega_{l} \cos \left( k_{\alpha} \cdot q_{\alpha\alpha} \right) \left( \frac{2}{J_{l}} \right) - \left( \frac{4}{\omega_{l}} \right) \cos \left( k_{\alpha} \cdot q_{\alpha\alpha} \right) \\
&\quad \times \cos \left( k_{\alpha} \cdot q_{\alpha\alpha} \right) \left( \frac{2}{\omega_{l}} \right) \\
A_{\lambda\sigma} &= -\left( \frac{32\pi k}{m_{l}V} \right)^{\frac{1}{2}} n \sum_{\sigma} \frac{a_{l}}{\sqrt{\mathcal{V}}_{\lambda}} \sqrt{J_{\sigma}} \cos \omega_{\sigma} \left( \frac{2}{\omega_{l}} \sqrt{J_{l}} \right) \\
&\quad \times \cos \omega_{l} \cos \left( k_{\sigma} \cdot q_{\sigma\sigma} \right) \left( \frac{2}{J_{l}} \right) - \left( \frac{4}{\omega_{l}} \right) \cos \left( k_{\sigma} \cdot q_{\sigma\sigma} \right) \\
&\quad \times \cos \left( k_{\sigma} \cdot q_{\sigma\sigma} \right) \left( \frac{2}{\omega_{l}} \right)
\end{align*}
\]
\[
A_{00}^L = 4 n_1 C \sqrt{\frac{\alpha}{V}} \sum_\sigma \sqrt{J_{0\sigma}} \cos \omega_{0\sigma} \left( \frac{\partial}{\partial \omega_l} \sin (k_x q_l) \frac{\partial}{\partial J_l} \right)
\]

\[- \frac{3}{2} \frac{\partial}{\partial J_l} \sin (k_x q_l) \frac{\partial}{\partial \omega_l} \right) \tag{2.4.13c}
\]

\[
B_{0}^l = - \left( \frac{32 \pi n}{m_l v} \right)^{\frac{1}{2}} n_1 \sum_\sigma \frac{\alpha_k \hat{e}_x}{\sqrt{J_l}} \cos \omega_l \cos (k_x q_l) \]

\[\times \left( \frac{\partial}{\partial \omega_x} \sqrt{J_{0\sigma}} \cos \omega_{0\sigma} \frac{\partial}{\partial J_{0\sigma}} - \frac{3}{2} \frac{\partial}{\partial J_{0\sigma}} \cos \omega_{0\sigma} \frac{\partial}{\partial \omega_{0\sigma}} \right) \tag{2.4.14a}
\]

\[
\hat{B}_{0}^l = \left( \frac{32 \pi n}{m_l v} \right)^{\frac{1}{2}} n_1 \sum_\sigma \frac{\alpha_k \hat{e}_x}{\sqrt{J_l}} \cos \omega_l \cos (k_x q_l) \]

\[\times \left( \frac{\partial}{\partial \omega_x} \sqrt{J_{0\sigma}} \cos \omega_{0\sigma} \frac{\partial}{\partial J_{0\sigma}} - \frac{3}{2} \frac{\partial}{\partial J_{0\sigma}} \cos \omega_{0\sigma} \frac{\partial}{\partial \omega_{0\sigma}} \right) \tag{2.4.14b}
\]

In the above expressions, \( n \) with proper subscripts are the number of charges on the test particle and the medium oscillator respectively. Expressions for the operators \( A_{-j}, B_{-j}, A_{-j}, B_{-j} \) can be obtained by replacing \( \cos k_j q \) by \( \sin k_j q \) and \( \sin k_j q \) by \( -\cos k_j q \) in the above expressions.
The formal solution of the equation (2.4.8) which is given in chapter I is

\[ P(t) = \exp(-i \mathcal{L}_0 t) \left[ 1 + \sum_{n=1}^{\infty} \mathcal{E}_n \int_0^t dt_1 \int_0^{t_2} \cdots \int_0^{t_{n-1}} dt_n \mathcal{O}_n \right] P(0) \]  

(2.4.15)

where the operator

\[ \mathcal{O}_n = \exp \left( i \mathcal{L}_0 t_1 \right) (i \delta \mathcal{L}) \exp \left( -i \mathcal{L}_0 t_2 \right) (i \delta \mathcal{L}) \times \exp \left( -i \mathcal{L}_0 t_{23} \right) (i \delta \mathcal{L}) \times \cdots \times \exp \left( -i \mathcal{L}_0 t_{n-1,n} \right) (i \delta \mathcal{L}) \exp \left( -i \mathcal{L}_0 t_n \right) \]  

(2.4.16)

with

\[ t_{ij} = t_i - t_j \]  

(2.4.17)

It can be noted that the operators \( \mathcal{A} \) in the expressions (2.4.11)-(2.4.14) contain derivatives with respect to the test particle or the oscillator variables while the operators \( \mathcal{B} \) contain derivatives with respect to the photon field variables. In the operators \( \mathcal{A}_j^T \) the differentiation with respect to \( \mathcal{U} \) commutes with the quantities on its right. The complete system of operators would consist of operators corresponding to \(-j\) also.

2.5 **Diagrams and Test-Particle Distribution Function.**

In any real physical problem, one studies only the local properties such as the local temperature, velocity etc. Hence when we insert a probe into a system, the particles in its neighbourhood interact with this probe and hence we measure the
result of these interactions with the probe. It is precisely this that we do with the test particle which acts like a probe in our formulation. Hence we need to find out the one particle distribution function from the N-particle distribution function given by (2.4.15). In particular the one particle distribution function can be obtained from the Liouville density by using the equation (1.1.11) i.e. by integrating \( \mathcal{J} \) over all the particles and the field variables except that of the test particle. In general, any medium particle can act as a test particle (In this case one will get a factor \( N!/ (N-S)! \) as the multiplying factor in r.h.s. of (1.1.11). But there we have written for S-specific particles). But in the present case, since we are investigating the response of the system on the relativistic particle, we shall integrate \( \mathcal{J} \) over all the variables except the relativistic test particle variables \( \mu \) and \( q \). Thus we define the one particle distribution function \( f(\mu, q, t) \) as

\[
 f(\mu, q, t) = \int \mathcal{P} \prod_{\ell} d\mathcal{T}_\ell d\omega_\ell \prod_{j} d\mathcal{J}_j d\omega_j \quad (2.5.1)
\]

The evaluation of (2.5.1) from (2.4.15), requires the knowledge of the initial state of the system due to the presence of the term \( \mathcal{J}(0) = \mathcal{J}(t=0) \) in (2.4.15). We shall write this as

\[
 \mathcal{J}(0) = f(\mu, q, 0) \prod_{\ell} g_\ell(\mathcal{T}_\ell, \omega_\ell; 0) \prod_{j} g_j(\mathcal{J}_j, \omega_j; 0) \quad (2.5.2)
\]

This above expression for \( \mathcal{J}(0) \) signifies that at \( t=0 \), the system was uncorrelated. We shall further specify that the medium
particle oscillators are in equilibrium and hence $\varphi_{\ell}(J_{\ell}, \omega_{\ell}; t=0)$ can be specified by a normalised equilibrium distributed functions as

$$\varphi_{\ell}(J_{\ell}, \omega_{\ell}; 0) = \varphi_{\ell}(J_{\ell}) = \frac{\gamma_{\ell}^{2}}{2\pi K_{B} T} \exp \left( -\frac{\gamma_{\ell} J_{\ell}}{K_{B} T} \right) \quad (2.5.3)$$

where $T$ is the kinetic temperature of the medium, $K_{B}$ the Boltzmann constant. We consider the electromagnetic vacuum condition for the radiation field which can be represented by

$$\varphi_{j}(J_{j}, \omega_{j}; 0) = \varphi_{j}(J_{j}) = \frac{1}{\sqrt{2\pi}} \delta(J_{j}) \quad (2.5.4)$$

Various other forms of the above initial conditions will be discussed in the chapter V. In the expression for (2.5.2) for $\mathcal{J}$ at $t=0$, the correlation between the particles are assumed to be zero. But in time the correlations are built up and destroyed due to the interactions in the system. The choice of the initial state is important in the non-Markoffian limit. However in a Markoffian limit, as described above, this choice of the initial state is not very crucial, since the correlations are created and destroyed within the system and hence after a long time, the initial state is forgotten by the system.

Also while evaluating (2.5.1) one may note that in the definition of $\mathcal{J}$ all the operators except $A_{j}$ are in the form of Poisson brackets. Hence we write the expression for $\varphi_{n}$ in (2.4.16) as
in (2.4.16) as

\[ O_n = \exp(i \mathcal{L}_0 t_1) \left[ \mathcal{A}_j^T + B_j^T + A_j^l \right] O_{n-1}' \]

(2.5.5)

with

\[ O_{n-1}' = \exp(-i \mathcal{L}_0 t_2) (i \delta \mathcal{L}) \exp(-i \mathcal{L}_0 t_3) (i \delta \mathcal{L}) \]

(2.5.6)

When we integrate over all the variables except the test-particle variables, using the well-known property of the Poisson brackets, viz

\[ \int dx \, dy \, \{ U, V \} = 0 \]

(2.5.7)

if \( U \) and \( V \) are functions vanishing at the boundary, it can be seen that in (2.5.5), except the first term i.e. \( \mathcal{A}_j^T \), all the other terms would vanish. This implies that while evaluating (2.5.1), the contribution of each term of the series (2.4.15) with \( O_n \) in (2.4.16) will have an operator \( \mathcal{A}_j^T \) at the extreme left. One can also see that a \( \mathcal{B}_j \) operator (vertex) always succeeds a vertex and never precedes the same.

We shall consider a few contributions of the series (2.4.15) to (2.5.1) with the operators defined by the expressions (2.4.9) - (2.4.14) and subsequently we will explain the diagram technique we use in the later chapters.
1. **First Order in e:**

The term corresponding to the first order in $e$ ($n=1$) in the series (2.5.1) and (2.4.15) is

$$e \int_0^t dt_1 \int d\mathbf{f}_j \, d\mathbf{\omega}_j \, d\mathbf{\omega}_l \, \exp \left\{ -i \mathcal{L}_0 (t - t_1) \right\} \times A_j^T \exp (-i \mathcal{L}_0 t_1) \mathcal{P}(0)$$

(2.5.8)

But this will give zero contribution on integration over the field angle variables $\omega_j$ since $A_j^T$ contains a harmonic function in $\omega_j$. This is also true for all odd powers of $e$ i.e. $e^{2n+1}$ ($n=0,1,2,...$). Hence the series will have only even powers of $e$.

2. **Second Order in e:**

The contribution from the series (2.4.15) to (2.5.1) in this order is

$$e^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \int d\mathbf{f}_j \, d\mathbf{\omega}_j \, d\mathbf{\omega}_l \, \exp \left\{ -i \mathcal{L}_0 (t - t_1) \right\} A_j^T \exp (-i \mathcal{L}_0 t_1)$$

$$\times \left[ A_j^T + B_j^T + A_j^L + B_j^L \right] \exp (-i \mathcal{L}_0 t_2) \mathcal{P}(0)$$

(2.5.9)

Because of the above results, in (2.5.9) only the contributions of the form $A_j^T A_j^T$ and $A_j^T B_j^T$ will survive. But due to the integration over $\mathbf{f}_j$ and $\omega_j$ with (2.5.2)-(2.5.4) for $\mathcal{P}(0)$, the combination $A_j^T A_j^T$ would give a vanishing contribution and
the term that is left is
\[
e^{2} \int_{0}^{t} dt_{1} \int_{0}^{t_{1}} dt_{2} \prod_{j} dJ_{j} dw_{j} \prod_{i} dJ_{i} dw_{i} \exp \left\{ iL_{0}(t-t_{1}) \right\} 
\times A_{j}^{T} \exp \left\{ -iL_{0}t_{12} \right\} (B_{j}^{T} \exp \left\{ -iL_{0}t_{2} \right\} f(0)
\]
\[(2.5.10)\]

But with the definition for \( L_{0} \) in mind we make use of the following relations:
\[
\int dJ_{j} dw_{j} \exp \left\{ -iL_{0}(t-t_{1}) f(\omega_{j}, J_{j}) \right\} = \int dJ_{j} dw_{j} f(\omega_{j}, J_{j})
\]
\[(2.5.11)\]
\[
\exp \left\{ -iL_{0}(t-t_{1}) \right\} f(\omega_{j}, J_{j}) = f(\omega_{j}, J_{j})
\]
\[(2.5.12)\]

and
\[
\exp \left\{ -iL_{0}t_{2} \right\} f(0) = f(0)
\]
\[(2.5.13)\]

and write (2.5.10) as
\[
e^{2} \int_{0}^{t} dt_{1} \int_{0}^{t_{1}} dt_{2} \prod_{j} dJ_{j} dw_{j} \exp \left\{ -iL_{0}(t-t_{1}) \right\} A_{j}^{T}
\times \exp \left\{ -iL_{0}t_{12} - iL_{0}t_{12} \right\} (B_{j}^{T} \exp \left\{ -iL_{0}t_{2} \right\}
\times \int f(0, q_{T}, 0) \right) G(J_{j})
\]
\[(2.5.14)\]

The relation (2.5.11) can be shown to be true by using the condition that the function and its derivatives vanish on the boundary and the relation (2.5.13) is due to the angle independence of \( f(0) \).
Now the expression (2.5,14) can be represented diagrammatically as in the figure 3.

Conventions

(i) Time increases from right to left as shown by an horizontal arrow in Fig. 3.

(ii) Since \( \exp(i\mathcal{L}_0 \tau)f(t) = f(t+\tau) \) (2.5.15) the operator \( \exp(i\mathcal{L}_0 \tau) \) is called a propagator in time. In the diagram we represent the particle propagation by a solid line and the photon propagation by a dashed line.

(iii) A square represents a \( B \)-type vertex. Since this operator contains the derivatives with respect to the photon variables (field variables), this on integration would give a non-zero contribution. Hence we shall call this a creation vertex.

(iv) A circle represents a \( A \)-type vertex. Since the first order term in \( e (2.5.8) \) vanishes due to the integration over \( J_j \) and \( \omega_j \), we call this a destruction vertex.

Thus the contribution to the one particle distribution function for the test-particle given by (2.5.1) from the series (2.4.15) for \( n=2 \) can be pictured as the particle at \( t_2 \) creating a correlation with the photon field and then the correlation is annihilated at \( t_4 \). During this time interval \( t_1-t_2 \) (the duration of field-particle interaction) the particle propagates along the upper line and the photon along the lower line. We then say that the particle and field interaction starts at \( t_2 \) and the interaction is complete at \( t_1 \).
3. Fourth order term in $e$ ($n=4$):

The contribution in this order can be written as

$$
\mathcal{E}^t \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \prod_j dJ_j \, d\omega_j \prod_l dJ_l \, d\omega_l 
\times \exp \{ -i \mathcal{L}_0 (t - t_1) \} \mathcal{A}_j^T \exp (-i \mathcal{L}_0 t_{12}) \mathcal{A}_j^T + \mathcal{B}_j^T + \mathcal{A}_{j'}^T + \mathcal{B}_{j'}^T \} \exp (-i \mathcal{L}_0 t_{23}) \left[ \mathcal{A}_{j''}^T + \mathcal{B}_{j''}^T + \mathcal{A}_{j'''}^T + \mathcal{B}_{j'''}^T \right] 
\times \exp (-i \mathcal{L}_0 t_{34}) \left[ \mathcal{A}_{j'''}^T + \mathcal{B}_{j'''}^T + \mathcal{A}_{j''''}^T + \mathcal{B}_{j''''}^T \right] 
\times \exp (-i \mathcal{L}_0 t_4) \mathcal{P}(0) 
\tag{2.5.16}
$$

With the discussions for $n=2$ and with the property of Poisson bracket (2.5.4) we write the following non-zero contributions in this order.

(i) \[ \exp \{ -i \mathcal{L}_0 (t - t_1) \} \mathcal{A}_j^T \exp (-i \mathcal{L}_0 t_{12}) \mathcal{A}_j^T \times \exp (-i \mathcal{L}_0 t_{23}) \mathcal{B}_j^T \exp (-i \mathcal{L}_0 t_{34}) \mathcal{B}_j^T \tag{2.5.17} \]

and the corresponding diagram is given in figure 4. If the fourth $\mathcal{B}$ operator has the index $j'$ instead of the third, the corresponding diagram will be as given in figure 5.

(ii) \[ \exp \{ -i \mathcal{L}_0 (t - t_1) \} \mathcal{A}_j^T \exp (-i \mathcal{L}_0 t_{12}) \mathcal{B}_{j'}^T \times \exp (-i \mathcal{L}_0 t_{23}) \mathcal{A}_{j'}^T \exp (-i \mathcal{L}_0 t_{34}) \mathcal{B}_{j'}^T \tag{2.5.18} \]
The corresponding diagram (Fig. 6) gives two connected cycles as in second order term. In all the above cases only the test-particle interacting with the field is involved. Each vertex gives \((v)\cdot l\), \(v\) being the volume element, and there is no medium particle involved in the diagram. Hence \(V \to \infty\), this gives a vanishing contribution.

\(\text{(iii) } \exp \left\{ -i \mathcal{L}_0 (t - t_1) \right\}^2 \mathcal{A}_j^T \exp (-i \mathcal{L}_0 t_{12}) B_j^l \times \exp (-i \mathcal{L}_0 t_{23}) \mathcal{A}_{j'}^T \exp (-i \mathcal{L}_0 t_{34}) B_{j'}^T\)

\((2.5.19)\)

This involves an interaction of the test particle with a medium oscillator with a corresponding diagram as given in figure 7. The interaction between the test particle and the medium particle (oscillator) is through the creation and annihilation of the correlation through the photons and the interaction is over at time \(t_1\). The diagram gives \(\frac{1}{V^2}\) as against \(N_e\) which is obtained by the summation over the medium particle index \(l\). Hence we will have \(\frac{1}{V} \left( \frac{N_l}{V} \right)\) and \(\frac{N_l}{V} \to C\) the concentration as \(N_e, V\) both go to \(\infty\) in the thermodynamic limit. The extra \(\frac{1}{V}\) term is absorbed in the conversion of the summation to integration as according to the relation

\[ \frac{8 \pi^3}{V} \sum_{k_{j'}} \longrightarrow \int dk_{j'} \]

as \(k_{j'}\) goes to infinity.

There are still various other forms of this above diagram depending on the position of the oscillator. These are (Figs. 8, 9, 10
Fig. 6
These are the only possible contributions one gets for the case when \( n=4 \). For further simplification we use the following approximations:

(a) **The Thermodynamic Approximation**

\[ N_l \to \infty, \quad V \to \infty, \quad \frac{N_l}{V} = C_l \text{ (finite concentration)} \]  
(2.5.23)

This is the assumption of the large extent of the system. With this assumption we neglect the boundary effects of the system.

(b) **Ring Approximation**:

The presence of a pair of medium oscillator vertices and the summation over all medium particles in each diagram provides
a factor like $e_2^2 N_x / \sqrt{y}$ which in the thermodynamic limit goes over to $e_2^2 C_L$. It may be seen that each vertex gives a factor "e". 

Thus in general, in a diagram of $2n$ vertices and $m$ medium particles, we have $e^{2n} C_L^{m'} (n > m)$ which can be written as $(e^2)^{n-m'} (e^2 C_L)^{m'}$.

Following Balescu (1963) we consider the case $n-m=1$ and consider only the terms of the order $e^2 (e^2 C_L)^m$ ($m=0,1,2,...$) and neglect all other unsaturated terms in 'e'. This is called the ring approximation. It is precisely for this reason that we have neglected the $\omega_L$ dependent terms in considering the $A^2_{L}$ term in $H^L$ (see the expressions (2.3.8)-(2.3.14)). This procedure for neglecting these angle dependent terms is called the random phase approximation (Bohm and Pines, 1951). If we had included these terms as well in the beginning, we should have had in the diagrams having $B_j^L \bar{A}_j^L$ vertex terms with co-efficients $e^2 (e^2 C_L)$ which remains unsaturated. These terms are important beyond the ring approximation.

With the above criteria for $n=4$, the terms (i) and (ii) can be neglected, since they deal with only the field and the test particle interaction and so one gets a term $1/\sqrt{y}$ which tends to zero in the thermodynamic approximation. The only terms with factors of order $e^2 (e^2 C_L)^m$ for $n=4$ in (2.4.15) are (with the relations (2.5.11)-(2.5.13) in mind)

\begin{align}
(i) \exp \xi - i L_0^T (t-t^\prime) \frac{3}{2} A_j^T \exp (-i L_0^T t^\prime_{I4} - i L_0^T t_{I2}) B_j^L \\
\times \exp (-i L_0^L t_{23}) A_j^L \exp (-i L_0^L t_{34}) B_j^L \\
(ii) \exp \xi - i L_0^T (t-t^\prime) \frac{3}{2} A_j^T \exp (-i L_0^T t_{I2}) A_j^T \\
\times \exp (-i L_0^L t_{24} - i L_0^L t_{I3}) B_j^L \exp (-i L_0^L t_{34}) B_j^L
\end{align}

(2.5.24a, 2.5.24b)
From the previous discussions a general rule for writing out the matrix element for each diagram can be made. For this, one should first write the propagator $ \exp \left( -i \mathcal{L}_0^T (t-t_r) \right)^J \overline{A}_j^T$ and then the vertex $\overline{A}_j^T$. Following this the next procedure is to proceed along the upper line and write the propagator or the vertex in the sequence as they appear till the last vertex. Before writing the last vertex, proceed along the lower line excluding the first vertex $\overline{A}_j^T$ appearing at $t_1$ and after covering the lower line, write the last vertex together with the last propagator for the test-particle i.e. $\exp(-i \mathcal{L}_0^T t_{14}) B_j^l$ where $n$ is the order of the term under consideration and then the initial distribution function $f(0)$. In the very mode of writing the matrix elements, it is evident that the two sets of particles appearing on the upper and lower lines of the diagram do not interact between themselves and hence the time ordering involving the particles appearing on the two lines becomes irrelevant. This in essence is the factorization theorem due to Resibois (1963) which will be used extensively in the evaluation of the matrix elements. However
there are certain limitations in this procedure which are detailed in Balescu's (1963) monograph.

We can get the other contributions to (2.5.1) from the series (2.4.15) for \( n > 4 \) in a similar manner. With all these considerations when we sum all the matrix elements, (2.5.1) can be represented pictorially as in figure 11 where the wavy line represents the sum

\[
\begin{align*}
\sum & = - \hspace{1em} + \hspace{1em} \square \quad \circ \quad - \hspace{1em} + \hspace{1em} \square \quad \circ \quad \ldots \quad \square \quad \circ \quad \ldots \\
& \quad \begin{array}{c}
\ j \ j' \\
\ j \ j' \\
\ j \ j' \\
\ j \ j' \\
\ j \ j' \\
\ j \ j' \\
\ j \ j' \\
\ j \ j'
\end{array} + \hspace{1em} (2.5.26)
\end{align*}
\]

In the above, the intermediate vertices are provided by the oscillators. In fact, only in second and third diagrams of (2.5.25) we have an oscillator with both \( \Theta_3 \)-type vertices.

We shall apply this general formulation in later chapters. In later chapters (Chapters III & IV) we use Born-approximation which allows us to consider only the first diagram in the expression (2.5.25). The other two diagrams correspond to contributions beyond Born approximation and contribute to the collisions between the test particle and the other particles (Henin, 1963).