CHAPTER III

NONLINEAR TWO-STREAM INSTABILITY MECHANISM ASSOCIATED
WITH TYPE I IREGULARITIES

3.1 Introduction:

In recent years, the type I irregularities in the equatorial electrojet have been extensively studied observationally by HF and VHF backscatter radar and in-situ rocket measurements (see secs. 1.31(a) and (c) for the detailed observations). These were first observed by Bowles and Cohen (1957) and Bowles et al. (1960, 1963). Although a substantial theoretical research in both linear and nonlinear regimes has been advanced in accounting their production and observed features discussed in sec. 1.31, there remain a few other important basic characteristics whose explanation is still uncertain and the detailed agreement between the theories and these observations is not wholly established. Because of highly complex nature of the ionospheric plasma and its parameters, it is most likely that there are many different instability mechanisms which may be operating at a time individually or simultaneously, so as to give rise to the occurrence of type I irregularities over a wide range of wavelengths observed (from 1 meter or less to a few tens of meters). Of these mechanisms, the one, so-called two-stream instability, which is believed to be the most successful physical process as the source of the origin of the electrojet type I irregularities, will be the topic of our present investigation to be done below in this chapter.

It is now widely recognized that if the plasma embedded
in electric and magnetic fields, contains two groups of particles drifting with respect to each other with directed energy due, for example, to the electron drift velocity (giving rise to the relative electron streaming) of the current carrying electrons, a high frequency \((\omega \gg \Omega_i)\) electrostatic instability, termed as the 'two-stream instability' can be excited. When the current (or relative drift velocity) reaches a certain critical threshold level, the plasma becomes unstable and the growing electrostatic type modes with phase-fronts aligned with the magnetic field are generated by the excitation of the two-stream instability. These growing modes which derive their energy from the kinetic energy associated with the relative drift velocity of the particles (relative electron streaming), have their phase velocity, \(V_p\) equal to ion acoustic speed, \(C_s\) and are favoured at short wavelengths. Such a plasma situation and threshold condition on the electron drift velocity are easily visualized in the equatorial electrojet, at least during daytime, and such ion acoustic electrostatic plasma waves can, therefore, be excited in the electrojet due to streaming electrons by the two-stream instability. This instability was first developed independently by Farley (1963a,b) and Buneman (1963) for the electrojet conditions. These authors showed that these waves travelling normal to the magnetic field, would grow when \(k \cdot V_e/|k|\), the component of streaming electron drift velocity in the wave direction exceeds a critical threshold value, the ion acoustic velocity, \(C_s\). If the orthogonal condition for \(k \times B_0\) is violated by more than a degree or so, the threshold velocity resistive as the collisions (electron-neutral collisions, in
is increased substantially.

As has been discussed in the Chapter I, the equatorial electrojet is characterized by a weakly ionized plasma with magnetized electrons ($\nu_e \ll \Omega_e$), collisional ions ($\nu_i \gg \Omega_i$) and the vertical density gradients. The crossed ambient vertical polarization electric field and the horizontal northward earth's magnetic field, then leads to the streaming of electrons with respect to the ions in east-west direction due to the disparity in the electron-neutral ($\nu_e$) and ion-neutral ($\nu_i$) collision frequencies (see secs. 1.4(b) and 2.3). It is now, generally, agreed that this relative electron-ion streaming drift velocity when it exceeds the above discussed threshold (the ion acoustic velocity, $C_s$), excites the two-stream instability giving rise to a form of spectrum of longitudinal electrostatic waves, known as ion acoustic waves, with the wave-fronts parallel to the ambient magnetic field. These ion acoustic waves have been associated, in the literature, with the type I irregularities (Farley, 1963a,b; Buneman, 1963; Kato and Hirata, 1967; Balsley, 1969b; Register and D'Angelo, 1970; etc.). On the other hand, the combined influence of the vertical electron density gradient and the polarization electric field (i.e., east-west electron drift) excites the gradient-drift (or $E \times B$) instability generating low-frequency, long-wavelength modes (Type II irregularities) at very low threshold (Register and D'Angelo, 1970; Jain and Das, 1971), which has been discussed by the author in the previous chapter.

The physical nature of this instability is purely resistive as the collisions (electron-neutral collisions, in
particular, as is obvious from its linear dispersion relation) which provide resistivity to the medium, extract energy from the relative electron drift (electron streaming current) and transfer to the waves which then grow. That is, the plasma electrons in the perturbation given to the medium, exhibit oscillations as a result of their being excited by electron streaming, and these oscillating plasma electrons, in turn, interact back on the streaming due to their collisions. Thus, the amplitude of the waves increases exponentially along the streaming. The physical mechanism of this resistive type instability has already been demonstrated in details in sec. 1.4(b). It has very recently been shown, however, that the physical nature of this instability changes from resistive to reactive for off angle propagation of the waves \( k_{\parallel} \neq 0, k_{\perp}/k_{\parallel} \ll 1 \), being probably resistive-inductive rather than purely resistive or purely reactive, for a large range of parameters applicable to the electrojet (Ossakow et al., 1975). There is general agreement that the so-called two-stream instability favours the excitation of high-frequency shorter-wavelength modes, while the gradient-drift instability favours the excitation of low-frequency long-wavelength modes.

The linear analysis of the two-stream instability applicable to the equatorial ionospheric conditions was first performed independently by Farley (1963a,b) considering the kinetic equations of the plasma and by Runesman (1963) in the fluid approximations. These authors predicted that the unstable waves are highly elongated along the field lines, as
the observational features of the type I irregularities show. It has now become apparent that many of the observed features of type I irregularities discussed in sec. 1.3(a), are more or less consistent with the linear Farley-Buneman instability theory. Since their prediction, a number of authors have studied this instability using linear theory (Waldteufel, 1965; Kato and Hirata, 1967; Register and D'Angelo, 1970; Lee et al., 1971; Kaw, 1972; Lee and Kennel, 1973a,b; Sudan et al., 1973; Ossakow et al., 1975; Fejer et al., 1975b; Farley and Fejer, 1975).

All these linear analyses except a few (for example, Lee and Kennel, 1973a,b; Ossakow et al., 1975, etc.) as well as a host of linear analyses predict mostly the highly field aligned nature of the growing waves, consistent with the observations of the equatorial electrojet type I irregularities. Also, since $k_{\parallel} \neq 0$ modes have large group velocity $|v|$ to $B_\circ$, they should be weakly convectively amplified, and if they are at all important, they would appear with largest amplitudes at northern and southern extremities of the electrojet, where the assumption of complete suppression of vertical currents (i.e., complete vertical polarization) made here in the investigation fails. Until good convective amplification studies (see below for such studies) of the $k_{\parallel} \neq 0$ modes in a more realistic electrojet model are completed, it is difficult to evaluate their significance, as pointed out by Lee et al. (1974). Henceforth, with these facts in mind, we will assume the waves to propagate in the plane orthogonal to $B_\circ$ in the present investigation.
Rogister and D'Angelo (1970) and Sudan et al. (1973) gave a unified interpretation of both the type I and type II irregularities and derived a unified linear dispersion relation for the instabilities associated with them. These authors obtained a unified linear growth rate expression for the unstable waves, which contains two main driving terms: one involving ion inertia and referred to as the two-stream term and the other involving the electron density gradient identified as the gradient-drift term. In the literature, these two terms have been attributed respectively to the two-stream instability giving rise to the type I irregularities and the gradient drift instability associated with the type II irregularities. The type II irregularities, which are directly generated at long wavelengths even when the mean electron drift velocity is much smaller than $C_s$ (i.e., the two-stream term is weak) and can not be produced directly at shorter wavelengths, but indirectly through some nonlinear processes (Farley and Balsley, 1973; Sudan et al., 1973; Sato, 1968, 1973; Rognlien and Weinstock, 1973, 1975b; Schmidt and Gary, 1973; McDonald et al., 1974; Ott and Farley, 1974; Rognlien and Weinstock, 1974, etc.), have been discussed in the Chapter II. Here the author devotes to the discussions of the type I irregularities associated with the two-stream term which is strong enough, at least during daytime, to produce direct generation of shorter wavelengths irregularities (the gradient-drift term being negligible for short wavelength modes) at the threshold velocity, approximately equal to $C_s$. 
In Farley's (1963a,b) kinetic description which neglected finite Debye length effects \( k_D \rightarrow 0 \), important for shorter wavelengths (comparable with ion or electron gyroradii), it was shown that most unstable modes were more or less exactly perpendicular to the magnetic field. He inferred that if the orthogonality condition for \( \mathbf{k} \times \mathbf{B}_0 \) is violated by more than \( 1^\circ \) or so, the threshold drift velocity for very small but nonzero \( k_n \) modes is substantially enhanced which may not be observable in the electrojet and hence the waves would not grow. Lee et al. (1971) have extended Farley's (1963a,b) analysis by including finite Debye length effects and found higher-frequency shorter-wavelength instabilities to be excited. Their calculation was restricted to modes propagating exactly perpendicular to \( \mathbf{B}_0 \) \( (k_n = 0) \). Recently, Lee and Kennel (1973b) have performed a simple parallel propagation effect \( (k_\parallel \neq 0) \) analysis on type I instabilities in the fluid limit, and noted that even within the fluid theory, these modes may be more unstable and have larger growth rates than those with \( k_n = 0 \), when

\[
\frac{\mathcal{V}_i}{\Omega_i} \frac{\mathcal{V}_e}{\Omega_e} \text{ is sufficiently small, as it is at the top of the electrojet.}
\]

Very recently, Ossakow et al. (1975) reexamined the linear theory of the resistive-type Farley-Buneman instability by considering finite Debye length effects (ion gyroradius is assumed to be large) for higher-frequency shorter wavelength modes having small but finite \( k_\parallel \) \( (k_\parallel /k_n \sim (m_i/m_i)^{1/2}) \). They showed that when the current is driven sufficiently hard (drift speeds in the range 2–3 times the ion thermal velocity), these modes with \( \mathcal{V}_p \) closer to \( \mathcal{V}_S \) than those propagating purely
perpendicular to $B_0$, grow more unstable as these are found to have maximum growth slightly away from the perpendicular to $B_0$, showing the importance of $k_{||} \neq 0$. That is, the off angle ($k_{||} \neq 0$) modes might actually dominate the linear stage in the higher drift velocity regimes. These authors suggested that the physical nature of the instability changes for off angle ($k_{||} \neq 0$) modes from 'resistive' to 'reactive' type, usually called a 'modified two-stream instability' strongly excited (strong growth) even in a collisionless plasma with $T_e \sim T_i$ (Krall and Lieber, 1971; McBride et al., 1972), and that for a large range of the electrojet parameters, the instability is probably 'resistive-inductive' rather than purely resistive or purely reactive. They inferred, further, that $V_F$ is rapidly decreasing function of angle for such modes as one moves away from pure perpendicular direction. Although this higher-frequency shorter-wavelength resistive-inductive type two-stream instability, could be applied to explain some of radar backscatter observations, it is felt by us that the ion viscosity effects play an important role in the development of unstable shorter wavelength modes. As these effects are significantly noticeable for the wavelengths comparable to ion mean free path or ion or electrons gyroradius, they are likely to contribute towards the stabilization of such modes. It is, therefore, important to examine the instability both linearly and nonlinearly by including the ion viscosity and this is what has been done in the investigation made in this chapter.
An idea was proposed by Kaw (1972) that since all waves have the vertical group velocity component $| \omega/\partial k_z |$ being upward during the day and downward at night, the electro-jet Farley-Buneman instability is undeniably convective. He argued that the linear convective amplification and refraction of plasma waves propagating through the horizontally stratified electrojet medium, might account for several of the puzzling features of type I irregularities and for their marginal stable spectra when the waves reach their largest amplitudes (hence spectral peak) near the end of their propagation path. Kaw, however, based his arguments on linear rather than nonlinear theory. Although this refraction certainly takes place, it cannot consistently be included in the linear theory, as pointed out by Lee and Kennel (1973a) who subjected Kaw's suggestion to quantitative tests. They concluded that there are several difficulties with his idea and consequently, it fails on several counts and that only a nonlinear refraction and convective amplification theory could produce a constant Doppler spectrum and account for the type I observations in much better way. The major difficulty is that typical growth rates are so large that nonlinear effects must become important before the wave has travelled far enough to make refraction, significant. Furthermore, the refraction rate is weakest where the growth rate is strongest (i.e., at the spectral peak near the altitude of 102.5 km where $\nu / \nu_e / \Omega_e = 1/3$ (Lee and Kennel, 1973a)) and refraction cannot account for the vertically and very oblique propagating waves that are observed. Several other
difficulties have been enumerated in the works of Lee and Kennel (1973a) and Lee et al. (1974) and in a short review article by Farley (1974).

Kaw et al. (1974), recently, stressed upon the possible importance of the electromagnetic (EM) effects in the linear electrostatic (ES) streaming instability theory and examined the excitation of mixed EM-ES linear (two-stream) instability (especially in the long wavelength range) by the cross-field electron current. It was inferred by these authors that the retention of the finite EM corrections \(\frac{ck}{\omega_{pe}} < 1\), where \(c/\omega_{pe} = 20\) m is the collisionless skin depth) for both the cases of \(k_{\parallel} = 0\) and \(k_{\parallel}/k_{\perp} \ll 1\) modes, contributes the additional (rather small) destabilizing terms that could excite the new cross-field streaming instability of mixed EM-ES waves, even when the electron streaming velocity \((\sim 200\) ms\(^{-1}\)) is below the two-stream threshold and the electron density gradient is absent. But it is not yet clear, however, whether or not this new instability is of much importance in the equatorial electrojet, as the present measurements do not show the existence of such mixed type irregularities. Also it apparently operates at long wavelengths \((> 500\) meters\) and has a slow growth rate (typical growth time \(\sim 1\) min).

Very recently, Farley and Pejer (1975) studied in details, the effect on the linear type I instability of the gradient drift term, the importance of which was first pointed out by Lee et al. (1974). It was shown by Lee et al. (1974) that the vertical density gradient corrections to the linear two-stream
instability theory are significant only when $v_0/v_c \approx 1$ and the density gradient, scale length, $L << 1$ km. But the parameter $L$ has been observed in the unstable electrojet plasma to range from 3 km to about 6 km or more (Prakash et al., 1969, 1972, 1973). Hence they concluded that for short wavelength type I modes ($kC_A/v_1 \approx 1$), the density gradient corrections are a negligible correction to the linear theory. This problem was reexamined by Farley and Fejer (1975) and hypothesized on the basis of strong observations that the phase velocity of the unstable type I modes always remain very close to that which corresponds to the marginal excitation (i.e., to instability threshold condition), even when the destabilizing terms (electron drift and plasma density gradient) exceed the thresholds. It was clearly shown by these authors that this constant threshold velocity of unstable waves can be affected appreciably by gradient drift term even at short wavelengths, particularly at night when the density gradient are much sharper and highly variable than they are during the day. At night, this phase velocity may differ significantly and vary over a considerable range from the usual constant threshold acoustic velocity, in consistent with the existing nighttime equatorial radar observations of considerable variability in the type I velocity. However, this hypothesis does not have strong convincing footing, as adequate measurements of type I irregularities at nighttime are, unfortunately, not yet available.

The linear theory of the instability can merely decide when a wave will become unstable and in which mode of
propagation it will grow more. The ultimate (saturation) level of the linearly growing modes, energy spectrum, phase velocity and many other characteristics can only be found from a satisfactory nonlinear theory. The nonlinear evolution of the instability is used most frequently to unravel its long time behaviour and several detailed features which are otherwise not possible from the linear theory. It is well conceived fact that all the observational features of the concerned irregularities occurring in the electrojet arise from a final saturated or marginally stable nonlinear state. The biggest question is thus a nonlinear saturation mechanism for the two-stream instability so as to know about the detailed properties of the instability and have a closer agreement between the theory and the observations. Therefore, in this chapter a nonlinear analysis of two-stream instability associated with type I modes is presented which, including ion viscosity and some quasilinear stabilization effects, discusses a special class of coherent nonlinear harmonic stationary type I modes observed to exist in such a nonlinear state in the equatorial electrojet (Balsley, 1973a,b; Farley and Balsley, 1973).

Several nonlinear theories of the two-stream instability have been attempted by a number of authors so as to decide a proper nonlinear mechanism to explain the marginally stable saturation spectrum of unstable type I modes and their observed features (Skadron and Weinstock, 1969; Kamenetskaya, 1971; Rogister, 1971; Weinstock and Williams, 1971; Sato, 1972, 1973; Sleeper and Weinstock, 1972, 1973; Weinstock and
Sleeper, 1972; Farley and Balsley, 1973; Sudan et al., 1973; Weinstock, 1973; Lee et al., 1974; Weinstock and Rognlien, 1975; Rogister and Jamin, 1975).

All of these studies of nonlinear saturation mechanisms of type I modes can, in general, be viewed from four different angles. In one approach, Skadron and Weinstock (1969) and Weinstock and Sleeper (1972, 1973) using the Kinetic theory description (developed by Weinstock, 1968, 1969, 1970) and Weinstock and Williams (1971) and Sleeper and Weinstock (1972, 1973) based on the fluid theory description, followed the Dupree-Weinstock nonlinear perturbed electron orbit diffusion theory (Dupree, 1966, 1967, 1968; Weinstock, 1968, 1969, 1970). These authors have suggested various nonlinear modifications for the form of the growth rate, when a turbulent wave spectrum is excited. The saturation is referred to perturbed or diffusing orbits, since the transfer of energy from waves to electrons (at the rate at which the unstable waves gain from the electrojet) is associated with the perturbation (heating) of the electron orbits by the waves. In other words, the balance of energy takes place when the configuration and velocity orbits of electrons supporting the waves, diffuse away from their unperturbed orbits as a result of heating. However, these authors have estimated only the integrated saturated amplitude, ignoring the actual detailed structure of the saturation spectrum of the instability.

A second approach (quasi-linear) was proposed by Kamenetskaya (1971), Rogister (1971) and Sato (1972, 1973) who showed the importance of quasilinear reduction of the vertical
polarization electric field in obtaining marginally stable spectrum of type I modes. They concluded that the form of the growth rate remains unchanged, but enhanced electron scattering (flux) across the magnetic field lines by irregularities could increase the vertical component of the average electron Pedersen current and thereby short out the polarization field. In turn, this reduces down the electron streaming drift driving the instability to its marginal stable value or below, thereby quenching the instability, and thus provides a marginal stable state for the type I modes.

The third approach to the study of marginal stable spectrum of the type I modes is based on the Kaw's (1972) idea, but as has been discussed earlier, he, however, based his arguments on linear rather than nonlinear theory, which was subjected to quantitative tests by Lee and Kennel (1973a) and Lee et al. (1974). These authors noted that there are several difficulties with the Kaw's idea and inferred that it fails to explain the observations on several counts. In fourth approach, Farley and Balsley (1973), Sundan et al. (1973) and Sato (1973) put forward the turbulent mode coupling mechanism in which they suggested that gradient drift instability first generates large scale \( (> 10^2 \text{ m}) \) irregularities propagating primarily in horizontal direction. When these grow to sufficiently large amplitudes \( (n/n_0 \sim \text{a few percent}) \), the first-order gradient and velocity perturbations associated with the wave are large enough (comparable to zero-order equilibrium values) to generate short wavelength type I irregularities which can easily travel vertically or at highly oblique angles to the horizontal.
However, this theory can perhaps be thought as the combination of the two linear theories rather than a nonlinear theory, since the mechanisms that limit the growth and thus account for the constant spectral shape, are not considered.

Although all the above approaches are very interesting and physically sound, it has been made clear by Lee et al. (1974) that none of them is individually sufficient enough in explaining all aspects of observations. They showed, however, that a combination of all of them seems to be capable of doing so. These authors argued that since both quasi-linear polarization reduction and nonlinear perturbed electron orbits diffusion damping arise from spatial electron diffusion, they cannot be considered separately. They, further, argued that since many modes are required for diffusion approximations to be valid, these nonlinear mechanisms are incomplete and do not give the detailed structure of the saturation spectrum, without the inclusion of other nonlinear mechanisms, such as wave-wave coupling (mode coupling). They stressed clearly upon the importance of Ka"{w}'s (1972) idea that it not only provides a natural way to produce a marginal stable spectrum, but also forces reformulation of the above nonlinear theories which were carried out for a homogeneous plasma. Therefore, Lee et al. (1974) investigated into all of the above-discussed nonlinear limiting mechanisms by incorporating the Ka"{w}'s idea of convective effects and then formulated a rather elaborate self-consistent convective nonlinear theory of type I irregularities in the electrojet. They concluded that this self-consistent nonlinear theory, combining wave refraction and convective amplification, quasi-linear polarization field
reduction and nonlinear perturbed electron orbit diffusion damping, could account for the radar backscatter observations of ubiquitous marginally stable or 'constant ion acoustic Doppler shift' - saturation spectrum better than any of these mechanisms separately. But it can not explain the presence of vertical type I echoes, however, and appears to be very hard put to account for the very oblique echoes and east-west symmetry.

Very recently, Weinstock and Rognlien (1975) developed a nonlinear theory and derived a unified nonlinear dispersion relation for the electrojet two-stream and gradient drift instabilities, by generalizing the unified linear dispersion relation (Rogister and D'Angelo, 1970; Sudan et al. 1973; Sato, 1973) and using the perturbed electron orbit diffusion damping theory (Weinstock and Williams 1971; Sleeper and Weinstock, 1972).Basically, the saturation mechanism can be viewed as arising due to slowing down of electrons by turbulent unstable waves to their marginal velocity, i.e., essentially it is equivalent to quasilinear theory of Rogister (1971), since the electrons experience the enhanced friction (viscosity) as they travel through the linearly growing waves approaching turbulent level. This friction is caused by the scattering of electrons by the turbulent unstable waves. Though, Weinstock and Rognlien's theory could account only for the observations of saturation level, frequency shift and asymmetrical angular spectrum about the horizontal plane of the vertically propagating waves, it can not explain the detailed structure of
marginally stable saturation spectrum of type I modes propagating both horizontally and vertically.

Simultaneously, Register and Jamin (1975) also studied the two-dimensional nonlinear theory of the turbulence in a uniform electrojet, based on the strong turbulence theory of Weinstock and Sleeper (1972). They invoked in their analysis both the one-dimensional quasi-linear modification of the vertical polarization field (Register, 1971; Sato, 1972) resulting into quasi-linear enhancement of vertical Pedersen conductivity and the two-dimensional nonlinear wave coupling processes which transfer energy from the linearly growing short wavelength modes to linearly damped long wavelength ones. It was concluded by these authors that their theory could provide a plausibly better interpretation of some of the yet puzzling observations and account for the detailed stabilization over the entire spectrum of type I irregularities. However, the definiteness of their conclusions and of the agreement of their predictions with observations are not yet fully established, which can, as they remarked, only be achieved by numerical calculations.

One of the basic features of VHF radar backscatter observations of type I irregularities in the equatorial electrojet, which has not yet been explained, is that short wavelength (1-3 m) irregularities travel in fairly discrete packets, i.e., in the shape of fairly sharp isolated pulses (Balsley, 1969a,b; 1973a,b; Parley and Balsley, 1973). This feature of the irregularities certainly suggests that in the electrojet, they exist in marginally stable state having some particular structures. This stable state of irregularity
particular structures must be caused by some nonlinear mechanism.

One of the stabilization mechanisms suggested so far is, a quasilinear theory (referred above as second approach of attempts in developing nonlinear saturation theories) proposed by Kamenetskaya (1971), Rogister (1971) and Sato (1972). These authors indicated, as has been discussed above, that quasilinear effects undeniably play a significant role in stabilization of the unstable modes growing linearly to large amplitudes. Though a few limitations of this quasilinear saturation theory were pointed out by Lee and Kennel (1973a) and Lee et al. (1974), even then its importance and reality cannot be denied, as they confessed that quasi-linear effects are remarkably efficient and are most favoured at least within the electrojet region where $\nu_1 \nu_e/\Omega_1 \Omega_e \lesssim 1/3$ (low altitude regions < 105 km), but with some modifications (i.e., including convective effects) within $\nu_1 \nu_e/\Omega_1 \Omega_e > 1/3$ regions where the efficiency of quasilinear effects decreases with increasing altitudes.

In this chapter, the author discusses a special class of coherent weakly nonlinear type I modes excited by two-stream instability in the equatorial electrojet. Having admitted the importance of quasilinear effects (discussed above and in sec. 3.2 here), we have included them in our nonlinear treatment of the two-stream instability. Introducing $\xi = (\omega t - k \cdot \mathbf{r})$ as the only independent variable, the nonlinear partial differential equations describing one-dimensional type I modes have been converted into the nonlinear ordinary differential
wave equation (sec. 3.3). Having retained the quasilinear modification of the vertical polarization field (quasilinear effects), the author has shown that in the weakly nonlinear limit, almost harmonic nonlinear stationary waves exist as special solutions. An amplitude dependent dispersion relation for these nonlinear modes is obtained in sec. 3.4. To get the nonlinear stationary modes, we have to retain dispersion of type I irregularities. For wavelengths comparable to ion mean free path or ion gyroradius (a few meters), this dispersion arises due to kinetic effects (Lee et al., 1971). A simplified way of taking into account of the dispersion, which produces qualitatively identical effects, is to introduce ion viscosity in the fluid equations: this is what we have done here.

A study of stability of the nonlinear stationary modes made in sec. 3.5, shows that they are stable against long wavelength slowly varying amplitude perturbations. However, as shown by Karpman and Kruskal (1969), generally for such waves, a large amplitude perturbation has a tendency to breakup the nonlinear stationary modes into envelope solitons. Thus we suspect that nonlinear type I modes might exist in the form of wave-packets with soliton shaped envelopes (fairly sharp isolated pulses). There seems to be some evidence from the VHF radar backscatter experiments for such a nonlinear state that the short wavelength irregularities do in fact travel in fairly discrete packets (Balsley, 1969a,b; 1973a,b; Farley and Balsley, 1973). Finally, as the complexity of full nonlinear equations demands, the author has carried out a series of numerical
calculations (in sec. 3.6) to verify some of the conclusions of the present theory, and to establish their definiteness.

3.2 Basic Equations and Quasilinear Modification of Polarization Electric Field:

For wavelengths of a few meters or more (longer than either iongyroradius or ion-mean free path), the fluid equations are adequate to describe the E-region plasma. In the frame work of a two fluid theory, the E-region plasma, in the presence of a steady electric field $\mathbf{E}_0$ and a uniform and steady magnetic field $\mathbf{B}_0$, is considered to be governed by continuity and momentum transfer equations for electrons and ions, where the collisions of the charged particles with a neutral background are dominant. The following assumptions are made to simplify these equations.

1. The ions are considered to be unmagnetized since their gyrofrequency $\Omega_i (= eB_0/m_i)$ is much smaller than the ion-neutral collision frequency $\nu_i$.

2. The electron inertia is neglected, because the electron-neutral collision frequency $\nu_e$ is much larger than the wave frequency Doppler-shifted to the electrojet drift frame.

3. The wave electric field is essentially electrostatic $\mathbf{E} = -\nabla \phi$, since the plasma pressure is much smaller than the ambient magnetic energy density and $ck/\omega_{pe} \gg 1$ for wavelengths of interest ($\omega_{pe}$ is the electron plasma frequency).

4. The waves are assumed to propagate exactly perpendicular to the magnetic field, since these waves experience least damping due to diffusion.
5. The ions and electrons are assumed to be isothermal.

In the nonlinear regime, higher harmonics are produced and steepening occurs for waves of short wavelengths (Sudan et al., 1973) for which case, the ion viscosity effects become significantly important, which in turn, lead to dispersion of the waves (sec. 3.1-9th and 22nd para). Therefore, the ion viscosity term must be included into the ion momentum transfer equation. This has been ignored by most of the previous authors, although these effects might compete with the nonlinear effects and then lead to a steady state solution. However, the ion viscosity term ($\propto k^2\mu$, where $\mu$ is the coefficient of viscosity and $k$ is wave number) becomes more significant when $k^2\mu/\nu_1 \approx 1$, i.e., when wavelength is comparable to ion-mean free path and for such cases, one must use the kinetic theory for ions (Lee et al., 1971).

With the inclusion of ion viscosity to take account of dispersive effects, the governing equations for electrons (3.1 and 3.2) and singly charged ions (3.3 and 3.4) are given by

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) &= 0 \quad \text{(3.1)}
- \frac{eE}{m} - \mathbf{v} \times \mathbf{B} - \frac{Z}{n} \mathbf{E} \mathbf{v} - \nu_e \mathbf{v} &= 0 \quad \text{(3.2)}
\frac{\partial N}{\partial t} + \nabla \cdot (N \mathbf{V}) &= 0 \quad \text{(3.3)}
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} &= \frac{eE}{M} - \frac{Z}{N} \mathbf{E} \mathbf{V} - \nu_i \mathbf{V} + \frac{\mu}{N} \nabla \mathbf{V} \quad \text{(3.4)}
\end{align*}
\]

The Poisson equation is

\[
\mathbf{V} \cdot \mathbf{E} = \mu \frac{1}{e} (N-n) \quad \text{(3.5)}
\]
Where \( \vec{E} = - \vec{V} \phi \) is the electric field, \( n \) and \( N \) are the number densities of electrons and ions respectively, \( M \) and \( m \) denote the ion and electron masses, \( \vec{V} \) and \( \vec{V} \) represent the electron and ion velocities, \( \vec{\omega}_e \) is the gyrofrequency of the electrons \( (\vec{\omega}_e = e\vec{B}/m_e) \) and \( \mu_0 \) is the coefficient of ion viscosity. \( S_e \) and \( S_i \) are respectively electron and ion thermal velocities. All other symbols have their conventional meaning.

We choose a rectangular coordinate system (Fig. 3.1) with x-axis pointing (magnetic) southward, y-axis eastward and z-axis vertically upward. Before solving the equations (3.1) - (3.5) describing the type I modes for the nonlinear analytical solutions, let us briefly describe how the equilibrium conditions in the equatorial electrojet get modified in nonlinear state due to quasi-linear effects (Kamenetskaya, 1971; Register, 1971; Sato, 1972). In the simplified model of equatorial electrojet (Sugiura and Cain, 1966; Lee and Kennel, 1973), the primary eastward electric field \( \vec{E}_p \) and the vertical polarization field \( \vec{E}_{oz} \) are such as to cause complete suppression of vertical currents, i.e.,

\[
\vec{J}_z = e n_o (\nu_{oz} - \nu_{oz}') = 0 \tag{3.6}
\]

which yields the equation (Baker and Martyn, 1953)

\[
\vec{E}_{oz} = \left( \frac{\sigma_2}{\sigma_1} \right) \vec{E}_p \tag{3.7}
\]

where \( \sigma_1 \) and \( \sigma_2 \) are respectively the Pedersen and Hall conductivities (Chapman, 1956). The horizontal electron drift velocity is then given by

\[
\nu_o \sim - \frac{e}{B_o} \vec{E}_{oz} = - \frac{e}{B_o} \frac{S_2}{\sigma_1} \vec{E}_p \tag{3.8}
\]

Register (1971) and Sato (1972) have argued that in the
Fig. 3.1. The coordinate system used.
presence of type I instability, the polarization electric field $E_{oz}$ is reduced. Physically, the fully developed instability, however, produces a net negative nonlinear vertical electron flux $\langle \vec{n} \cdot \vec{v}_z \rangle$ in the direction opposite to that of the primary flux $n_0 v_{oz}$ so that equation (3.6) must be modified as

$$\nabla \times E = \epsilon n_0 \left( v_{oz} - v_{oz} \right) + (\epsilon) \langle \vec{n} \cdot \vec{v}_z \rangle = 0 \quad \ldots(3.9)$$

The nonlinear ion flux is not included since it is negligible. Equation (3.9) leads to a modified polarization field

$$E_{oz}' = E_{o} \left[ 1 + \left( \frac{v_{oz}}{v_{oi}} \right)^2 \langle \vec{n} \cdot \vec{v}_z \rangle \right]^{-1} \quad \ldots(3.10)$$

and a modified electron drift velocity

$$v_{om}' = v_{o} \left[ 1 + \left( \frac{v_{oz}}{v_{oi}} \right)^2 \langle \vec{n} \cdot \vec{v}_z \rangle \right]^{-1} \quad \ldots(3.11)$$

This produces the simplest quasi-linear saturation mechanism for type I modes. As the waves attain large amplitude, the electron drift $v_{om}$ decreases reducing the growth rate of the wave until marginal stability is reached. In other words, the turbulence level inhibits the formation of large current and thereby maintains the plasma in a state relatively close to linear marginal instability. In fact, irregularities at turbulence level scatter the electrons across the magnetic fields, and thereby lead to an enhancement of the vertical Pedersen conductivity of the medium, increasing the vertical component of the average electron Pedersen current. This increase, in turn, tends to short out the polarization electric field and the electrojet current which drive the instability, and provide a mechanism whereby the electrons are slowed...
down by the unstable wave to their marginally stable value of
the drift velocity, i.e., to the threshold value (Rogister,
1971; Sato, 1972; Lee et al., 1974; Weinstock and Rognlien,
1975; Rogister and Jamin, 1975). We shall now use above
results in deriving an equation describing coherent nonlinear
type I modes.

3.3 Wave Equation for Nonlinear Type I Modes:

Because we are interested in wavelengths much larger
than the Debye length, we shall assume quasi-neutrality, i.e.,
\[ \gamma \approx N \]  \hfill (3.12)

We are interested to look for nonlinear stationary waves by
which we mean those waves in which all the oscillating quanti­
ties are functions of only \( s-ut \), where \( s \) is the coordinate along
the direction of wave propagation and \( u = \omega/k \) is the phase
velocity of the wave. In case of periodic stationary waves, it
is therefore, convenient to introduce the phase of oscillation
as

\[ \xi = \omega t - \frac{\omega}{k} \lambda \]  \hfill (3.13)

where \( k = 2\pi/\lambda \) and \( \lambda \) is the wavelength, then all the oscillat­
ing quantities can be represented as periodic functions of \( \xi \)
(with period \( 2\pi \)). We define each variable \( q \) as

\[ q = q_o + \widetilde{q} \]  \hfill (3.14)

where \( q_o \) is the equilibrium value and \( \widetilde{q} \) is the perturbation.
Including the modified quasilinear electron drift \( v_{e0m} \) instead
of linear electron drift \( v_e \), the equations (3.1) to (3.4) may
be combined to give
for ions and

\[
\left(1 - \frac{n_0}{n}\right) \omega = -\frac{\nu_e k}{\Omega_e} \left[\frac{e}{m} \frac{d\Phi}{d\xi} - \frac{S_2^2}{\hbar} \frac{d\eta}{d\xi}\right]
\]  

(3.16)

for electrons, where \(n = n_0 + \tilde{n}\) and \(\Omega_e (= |\Omega_e|) \gg \nu_e\) has been considered. Eliminating \(d\Phi/d\xi\) from (3.15) and (3.16), we obtain the wave equation for nonlinear type I modes as

\[
\frac{d^2\eta}{d\xi^2} - \frac{2}{\nu_e k^2} \left(\frac{d\eta}{d\xi}\right)^2 - \frac{\nu_e k}{\mu} \left[1 - \frac{C_s^2}{\omega^2} \left(1 + \eta^2\right)^{\frac{1}{2}} \right] \frac{1}{\nu_e k^2} \frac{\nu_e k}{\mu} \left[\frac{\nu_e k}{\mu} \Omega_{om} - 1\right] - 1 \frac{\nu_e k}{\mu} \left(1 + \eta^2\right) \eta = 0
\]

(3.17)

where \(\eta = \tilde{n}/n_0\), \(\psi = \nu_1 \nu_e/n_1 n_e\) and \(C_s^2 = S_1^2 + (\Omega_1/\Omega_e)S_e^2\), where \(C_s\) is the acoustic velocity. The first two terms in (3.7) arise because of the ion viscosity.

3.4 Solutions of the Nonlinear Wave Equation and Nonlinear Dispersion Relation:

3.41 Linear Type I Modes:

In the linear limit, we may assume that \(\eta = \tilde{n}/n_0 \rightarrow 0\) so that the wave equation (3.17) can be written as

\[
\frac{d^2\eta}{d\xi^2} - \frac{\nu_e k}{\mu} \left(1 - \frac{C_s^2}{\omega^2}\right) \frac{d\eta}{d\xi} + \frac{\nu_e^2}{\mu} \left(1 - \frac{\nu_e k \Omega_{om}}{\omega} - 1\right) \eta = 0
\]

(3.18)

where \(\Omega_{om}\), in the linear limit, has been replaced by \(\nu_1\) (eq.3.11). The stationary solution of the form

\[
\eta = A \cos (\omega t - \mathbf{k} \cdot \mathbf{x} + \phi)
\]

(3.19)

satisfies the equation (3.18) if
The condition (3.21) is obtained by equating the coefficient of first derivative of \( \gamma \) in (3.18) to zero, which gives the condition of marginal instability (no growth). The critical electron drift velocity for instability to occur can thus be determined by combining (3.20) and (3.21) and is given by the magnitude of \( v_o \), say \( v_o^* \):

\[
\nu_o^* = C_5 \left[ 1 + \psi \left( 1 + \frac{k^2 \mu}{\nu_i} \right) \right] \tag{3.22}
\]

These results are essentially the same as obtained by earlier workers in the linear case (Parley, 1963a,b; Buneman, 1963; Register and D'Angelo, 1970). The following differences arise because of inclusion of ion viscosity effects: (a) it effectively changes the ion collision frequency from \( \nu_i \) to \( \nu_i + k^2 \mu \) and thus introduces dispersion of the waves; (b) the threshold \( v_o \) is increased. Note that the ion viscosity term becomes important when \( k^2 \mu / \nu_i \approx 1 \), i.e., when the wavelength is comparable to ion-mean free path. For such cases, a kinetic theory for ions (Lee et al., 1971) would be more appropriate. However, since inclusion of ion viscosity gives qualitatively similar results, we have chosen that for simplicity.

An important result to be noted here, which differs from earlier results obtained from fluid theory, is that in the linear limit, our analysis of the instability leads to the wavelength dependent dispersion relation (3.20) and the
wavelength dependent threshold electron drift velocity, \( v_c \) (3.22) for marginally stable waves (described by the condition (3.21)). This agrees with the result predicted by marginal stability calculations from kinetic theory (Parley, 1963a,b) that the phase velocity of the marginally stable waves are wavelength dependent. The phase velocities of type I irregularities are, in fact, wavelength dependent; yet they still correspond to marginally stable waves (Cohen and Bowles, 1967; Balsley and Parley, 1971).

3.42 Nonlinear Type I Modes:

It is obvious that the analytical treatment of the nonlinear wave equation (3.17) is too difficult to handle mathematically and to obtain the physical real solutions. Our attention will, therefore, be devoted mainly to the case of weak nonlinearities, in which case we assume sufficiently small amplitude perturbation (\( \gamma \ll 1 \)), in consistent with the observations. Thus, in the case of weakly nonlinear modes for which \( \gamma \ll 1 \), the wave equation (3.17) can be written as

\[
\frac{d^2 \eta}{ds^2} + K_0 (1 + \eta) \eta = \frac{\nu}{k} \left[ \frac{1 - \eta + \eta^2}{(1 + \eta)^2} \right] \frac{d \eta}{ds}
\]

\[
+ \frac{2}{(1 - \eta)} \left( \frac{d \eta}{ds} \right)^2
\]

where

\[
K_0 = \frac{\gamma_2^2}{k^2 \mu} \left[ \frac{1}{\nu} \left( \frac{\nu}{\omega} \frac{1}{1 + (\sigma^2/\sigma_1)^2} \right) - 1 \right] \quad \text{(3.23a)}
\]

In the weakly nonlinear regime, the amplitude and the phase of the wave vary slowly with \( s \). We use the method of
averaging (Bogoliubov and Mitropolsky, 1961) to solve equation (3.27). Let the solution be

$$\eta = A \cos (K_0 \xi + \Phi) \quad \ldots (3.24)$$

where the amplitude $A$ and the phase $\Phi$ are slowly varying functions of $\xi$. Following Bogoliubov and Mitropolsky (1961), we obtain the dependence of $A$ and $\Phi$ on $\xi$ as

$$\frac{dA}{d\xi} = \frac{\mu}{k \mu} \left(1 - \frac{C_s^2}{u^2}\right) \frac{A}{2} - \frac{C_s^2}{4k \mu} \frac{A^3}{2} \quad \ldots (3.25)$$

$$\frac{d\Phi}{d\xi} = 0 \quad \ldots (3.26)$$

The equation (3.26) shows that the waves travel with constant phase in the wave frame.

In nonlinear stationary regime, the amplitude gets saturated and does not vary with $\xi$, i.e., the saturated amplitude is given by

$$\frac{dA}{d\xi} = 0 \quad \ldots (3.27)$$

This condition and the wave-equation (3.23) are satisfied by the solution with stationary amplitude ($A = \text{constant}$, say $A_0$) of the type

$$\eta = A_0 \cos \left(\omega t - \frac{k}{\mu} \xi + \Phi\right) \quad \ldots (3.28)$$

if

$$\omega = \frac{k}{\mu} \sqrt{\frac{1}{1 + \psi \left(1 + \frac{k^2 \mu}{\psi \nu}\right)}} \cdot \frac{1}{1 + \psi \left(1 + \frac{k^2 \mu}{\psi \nu}\right)^2 \frac{A^2}{2}} \quad \ldots (3.29)$$

and

$$\omega^2 = C_s^2 \left(1 + \frac{A^2}{2}\right) \quad \ldots (3.30)$$

where in equation (3.29) the value of $\langle \eta \eta \rangle$ has been put equal to $A_0^2/2$ over a period.
Equation (3.29) shows that the phase velocity of the nonlinear wave is a function of the amplitude $A_0$. This dependence arises because of the quasilinear reduction of vertical polarization field. From equation (3.30), we note that the critical condition for marginal stability is also modified. The amplitude dependent term here can be traced back to $(\nabla, \nabla V)$ term in the ion equation of motion (i.e., like an ion ponderomotive force).

It is, therefore, concluded that nonlinear stationary sinusoidal solutions exist for coherent type I modes in equatorial electrojet. Now the question arises whether these nonlinear stationary wave propagating in the electrojet are stable to long wavelength slowly varying large amplitude perturbations or not. This is discussed in the next section.

3.5 Stability Criteria of Nonlinear Stationary Waves:

It is obvious from the nonlinear dispersion relation that the frequency of the wave $\omega$ depends not only on the wave number $k$, but also on the nonlinear parameter, the amplitude $A$ which is assumed small in linear approximation. The dependence of $\omega$ on $k$ and $A$ is called nonlinear dispersion relation and is given by (3.29). In general, the nonlinear dispersion relation can be expressed as

$$\omega = \omega(k, A)$$

(3.31)

Considering the waves of sufficiently small (but finite) amplitude, we can restrict ourselves to the first two terms of the expansion of the dispersion equation (3.31) in powers of $A^2$, i.e.,
\[ \omega = \omega^0(k^2) + \left( \frac{\partial \omega}{\partial \alpha} \right)_0 \alpha^2 \quad \ldots (3.32) \]

where \( \omega^0(k^2) \) determines the dispersion law in the linear approximation.

For studying the stability of these nonlinear periodic stationary solutions obtained in earlier section, we consider a nonlinear non-stationary wave where the amplitude, the wavelength, etc. change sufficiently slowly at a distance of the order of wavelength and over a time of the order of the period of oscillations and which differs from the stationary wave (the latter has the parameter \( k_0, \omega_0, A_0 \)) satisfying the dispersion equation (3.31), the phase of the wave can be represented in the form

\[ \xi(x, t) = \omega_0 t - k_0 \phi + \bar{\phi}(x, t) \quad \ldots (3.33) \]

where \( \bar{\phi}(x, t) \) is the contribution to the phase made by the non-stationarity. Karpman and Kruskal (1969) have investigated the modulation of nonlinear stationary waves in nonlinear dispersive media. By comparing the set of equations with the hydrodynamic equations they found that the 'adiabatic compressibility' is determined by

\[ \chi = - \frac{\rho_0^2}{u'_0} \left( \frac{\partial \omega}{\partial \alpha} \right)_0 \quad \ldots (3.34) \]

where \( u'_0 = \left( \frac{\partial \omega}{\partial k^2} \right)_{\alpha=0, \omega^0} \) and the square of the velocity of sound is given by

\[ c_s^2 = -\chi \quad \ldots (3.35) \]

It follows from this that for a plane wave to be stable against the changes in phase and amplitude, it is necessary that the quantity \( c_s^2 \) must be positive, i.e.,
For $\lambda > 0$, the plane waves considered will be unstable. This result was first obtained by Lighthill (1967). To apply this criteria, we have to know only the dispersion equation in the form (3.32). Karpman and Kruskal (1969) found the following dispersion equation for the perturbations

$$\omega = \left( \frac{\gamma^2 - \lambda^2}{2} \right)^{1/2} \gamma$$

where $\omega$ is the frequency and $\gamma$ is the wave number of the amplitude and phase perturbations. Using the linear and nonlinear dispersion relations (eqs. (3.20) and (3.29)), we obtain the expression for $\alpha$ as

$$\alpha = \frac{\mu_0^2}{2} \frac{1}{\psi \mu} \frac{\left[ 1 + \psi(1 + k, \mu, \gamma) \right]^3}{\left[ 1 + \psi(1 - k, \mu, \gamma) \right]^2 (1 + \psi(1 + k, \mu, \gamma))}$$

where $\mu_0 = k_0^2 \mu / \gamma_1$. For the parameters appropriate to the equatorial electrojet and for the wavelength of interest, it can be shown that the quantity $\mu_0 \ll 1$ and hence $\alpha < 0$. This means that the frequency $\omega$ is real leading to nongrowing waves. For not too large $A - A_0$, the set of equations obtained by Karpman and Kruskal (1969) reduces to the Korteweg-de Vries equation (an equation of hydrodynamics with dispersion, in which the role of density is played by $A^2$). As is well known the profile of the solution of the Korteweg-de Vries equation decays for large time into solitary waves (solitons) and small oscillations. Thus, modulational envelopes of nonstationary waves distort into solitons or soliton-like shapes (fairly sharp isolated pulses) instead of shock waves when $\alpha < 0$. 

\[ \lambda < 0 \]
It may thus be concluded that the nonlinear and dispersive properties of type I modes are such that nonlinear stationary solutions are modulationally stable but at the same time large perturbations in the amplitude tend to develop into envelop solitons. Balsley (1973b) has pointed out that some backscatter observations at Jicamarca are consistent with the propagation of fairly isolated wave-packets of short wavelength (e.g., 3 m) type I modes (Balsley, 1969a,b, 1973a; Farley and Balsley, 1973), an experimental observation, which may be of relevance to this result.

3.6 Numerical Calculations and Results:

The above analytical results have been derived for weak nonlinearities by using Bogoliubov-Mitropolsky method. The approach followed above is approximate one, as this method is valid only for sufficiently small (but finite) amplitude perturbation and the terms of order higher than the cube of amplitude were neglected. In order to establish the definiteness of these results obtained in secs. 3.4 and 3.5, they need to be verified by some other method. An accurate and simplest method of verification of the results is one which involves solving either the full nonlinear wave equation (3.17) or the exact set of the basic equations (3.1)-(3.4) or both, numerically on computer, and then comparing the numerical solutions so obtained with the analytical ones. And this is done below in this section. The parameters appropriate to equatorial electrojet were chosen as; \( \gamma_c = 5 \times 10^6 \) cps; \( \gamma_e = 8 \times 10^4 \) cps; \( \gamma_i = 1.3 \times 10^2 \) cps; \( \gamma_i = 5 \times 10^3 \) cps; \( B_p = 1 \text{ mV/m} \) and \( C_g = 330 \text{ m/sec} \).
a) **Numerical Solutions of the Exact (Full) Nonlinear Wave Equation (3.17):**

As a check on our analytical results, for the reasons mentioned above, we have therefore solved the exact nonlinear wave equation (3.17) on the computer. The numerical procedure adopted is briefly laid out as follows: The wave equation (3.17) is solved as a system of two first order differential equations by the unconditionally stable trapezoidal rule. The initial values of the amplitude $A$ of the perturbation $\eta$ and $d\eta/d\xi$ at $\xi = 0$, are arbitrarily chosen. For taking $\eta$ forward with $\xi$, one needs the values of $\nu_{om}$ and $u$, both of which depend upon $<\eta^2>$. Since $\eta$ is a periodic function of $\xi$, for each cycle $<\eta^2>$ is calculated from the one preceding cycle. Fig. 3.2 shows an example of such a numerical calculation.

In the Fig. 3.2 is depicted the numerical solution of the exact nonlinear wave equation (3.17) for $A(\xi = 0)$ equal to 0.005 (which is 0.5/ of the equilibrium density, $n_o$), wherein the wavelength chosen is of the order of a meter. One finds from this figure that for large $\xi$, the amplitude does not seem to grow any more and a periodic stationary solution with a constant amplitude, say $A_o$, which is nearly equal to 1.38%, is finally achieved, whereas initially the perturbation is found to grow with linear growth. This initial linear growth arises because of the fact that at the initial stage, the amplitude is not sufficiently large so as to make quasilinear effects (quasilinear reduction of $\nu_{om}$) play an efficient role. When the amplitude increases to sufficiently high value, i.e., when it reaches the turbulent level, electron drift velocity $\nu_{om}$
Fig. 3.2. A plot of $\eta$ versus $\xi$ showing nonlinear stationary state followed by linear growth and later characterized by steady amplitude of the perturbation $\eta$. 
reduces down to its threshold value for marginal instability due to quasilinear effects and consequently, saturates the nonlinear perturbation at some finite level, $A_0$. In this nonlinear saturation regime, the electron drift velocity $v_{om}$ and the phase velocity $u$ (computed by 3.11 and 3.29), are found to satisfy respectively the relation (obtained by using 3.11, 3.29 and 3.30)

$$v_{os}^2 = -\zeta_0 \left[ 1 + \psi \left( 1 + \frac{L^2}{k^2} \right) \right] \left[ 1 + A_0^2 / 2 \right]^{1/2}$$

and the relation

$$u_o^2 = C_s^2 \left( 1 + A_0^2 / 2 \right)$$

where $v_{os}$ and $u_o$ are the respective values of $v_{om}$ and $u$ in nonlinear stationary (saturation) state. This clearly indicates that the numerical calculations show that the nonlinear stabilization of nonlinear stationary wave will occur when both the expressions (3.29) and (3.30) for the phase velocity are satisfied, a fact which was found analytically in earlier sections.

However, all these above results have been derived by solving the nonlinear wave equation (3.17) which resulted in after the assumption of a single independent variable $\xi$ was made, and hence, do not describe the nonlinear state of type I modes with respect to time and space variations separately. This kind of information can be achieved by solving further the set of basic equations numerically which is done below.
b) Space and Time Dependences of Solutions of the Exact Set of Basic Equations:

The above numerical computation was made on the nonlinear ordinary differential equation which resulted after the assumption of a single independent variable $\xi = (\omega t - \vec{k} \cdot \vec{x})$ had already been made. As a further check of the above theory in the general case where dependences on $\vec{x}$ and $t$ are independent of each other we, therefore, carried out a numerical computation using the exact set of partial differential equations (3.42) to (3.46) which are derived as follows.

Substitution of $\vec{x}$ from (3.2) into (3.4) and taking the components of resulting equation in $y$- and $z$-directions give us two equations. The $z$-component of equation (3.2) gives the third equation. The assumption of quasineutrality (equation 3.12) and combination of electron and ion continuity equations (3.1) and (3.3) lead to fourth equation which is, in fact, $\vec{v}_j = 0$. The $5^{th}$ equation is the ion continuity equation (3.3) itself. Thus these are the five basic equations to be solved for five unknowns $\gamma$, $\vec{v}_y$, $\vec{v}_z$, $\gamma'$, $\vec{v}'$. We define the following dimensionless parameters

\[
\tau = \omega_0 t \quad \kappa y' = \lambda \quad \vec{v} = U_0 \vec{v}' \quad \vec{v}' = U_0 \vec{v}'
\]

where $\omega_0$ is the wave frequency at time $t = 0$ (calculated by 3.20) and $U_0 = \omega_0 / k$. Thus the set of basic equations to be solved numerically for five unknowns $\gamma$, $\vec{v}_y$, $\vec{v}_z$, $\gamma'$ and $\vec{v}'$ is written in the dimensionless parameters as

\[
\frac{d}{dt} + (V_{by} + V_{y}') \frac{d}{dx} \frac{\vec{v}_y'}{2\lambda} \vec{v}_y' = \frac{\omega_0}{\omega_{0z}} \left( \vec{v}_z' - \vec{v}_y' - \frac{\rho_0}{\delta n_0} \vec{v}_y' - \frac{\rho_i}{\delta n_i} \vec{v}_y' \right)
\]

\[
\frac{\omega_0}{\omega_{0z}} \left( \vec{v}_z' - \vec{v}_y' - \frac{\rho_0}{\delta n_0} \vec{v}_y' - \frac{\rho_i}{\delta n_i} \vec{v}_y' \right)
\]

\[
- \frac{\rho_0}{\delta n_0} \frac{\partial \gamma}{\partial \lambda} + \frac{k_0}{U_0} \frac{\partial ^2 \vec{v}_y'}{\partial \lambda^2} \quad (3.42)
\]
\[
\left( \frac{\partial}{\partial t} + (v_0 y + v' y) \frac{\partial}{\partial \xi} \right) \vec{v}'_z = -\frac{\partial v}{\partial \xi} \vec{v}'_z + \frac{k \mu}{u_0} \cdot \frac{\partial^2 \vec{v}'}{\partial \xi^2} 
\] ..(3.43)
\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \xi} \left[ (1 + \eta) \vec{v}'_y + \eta \vec{v}'_y \right] = 0 
\] ..(3.44)
\[
\vec{e}_e \cdot \vec{v}'_y + \eta \vec{v}'_z = 0 
\] ..(3.45)
\[
\frac{\partial}{\partial \xi} \left[ (1 + \eta) (\vec{v}'_y - \vec{v}'_y) + \eta \vec{v}'_{om} \right] = 0 
\] ..(3.46)

Where, we have made use of the quasilinear modification \( v_{om} \) instead of \( v_0 \) and \( V_{om}' \) is dimensionless equilibrium (zero order) ion drift velocity in \( y \)-direction given by

\[
\vec{v}'_{om} = v_{om}' = \frac{e}{m v_e} E_p 
\] ..(3.47)

The equation (3.46) can easily be integrated analytically and others are integrated by the numerical procedure given below.

At \( \xi = 0 \), we start with an initial density perturbation, with the space dependence

\[
\eta = A \cos k y = A \cos \lambda 
\] ..(3.48)

Since \( A \) is assumed small, the initial values of variables \( \vec{v}'_y, \vec{v}'_y, \vec{v}'_z \) and \( \vec{v}'_z \) are calculated respectively from the linearized set of equations (3.1), (3.3) and (3.45) where \( \vec{v}'_z \) is obtained by \( z \)-component of linearized equation (3.4) and are given by,

\[
\vec{v}'_y = (\omega / k u_0 - v'_0) \eta ; \quad \vec{v}'_y = (\omega / k u_0) \eta 
\] ..(3.49)
\[
\vec{v}'_z = -\left( \frac{\partial v}{\partial \xi} \right) \vec{v}'_y ; \quad \vec{v}'_z = 0 
\] ..(3.50)

For simplicity, we will consider the monochromatic disturbance (defined by 3.48) to be of one fundamental wavelength long with wave number \( k \) and period \( 2\pi \), i.e., \( ky = 2\pi \) is
dimensionless fundamental wavelength $\lambda$. The range of $\lambda$ from 0 to $2\pi$ is divided into $M$ equal parts with $M + 1$ points. All the spatial derivatives are replaced by central differences of second order at all the $(M + 1)$ points. The finite difference replacements at the boundary points involve the points outside the range. These outside points are eliminated using the periodicity conditions, $p_{M+2} = p_2$ and $p_M = p_0$ where $p$ is any of the dependent variables. This system of $3(M + 1)$ ordinary ordinary differential equations in time are solved by the Trapezoidal rule. Thus, the finite difference method adopted is Crank-Nicolson scheme.

For each time step, $\gamma$, $\bar{v}_y$ and $\bar{v}_z$ are obtained first by the simple explicit scheme and these are iteratively corrected using the semi-implicit scheme. Whenever new values for $\gamma$, $\bar{v}_y$ and $\bar{v}_z$ are calculated while advancing the time by $\Delta \tau$ or while iteratively correcting for the same time, $\bar{v}_y$ and $\bar{v}_z$ are calculated using equations (3.45) and (3.46) followed by the evaluation of $v_{0m}$ using (3.11). For the calculation $v_{0m}$, $\langle \gamma \rangle$ is needed and is calculated by evaluating $\int_0^{2\pi} \gamma^2 \, d\lambda$ by Trapezoidal rule. In the computation presented $M = 24$ and $\Delta \tau = 1.5 \times 10^{-6}$; $\Delta \lambda = .2518 \approx 2\pi/24$. These values of $\Delta \tau$ and $\Delta \lambda$ are found satisfactory considering the accuracy requirements.

Our aim is to verify the analytical results obtained in sections (3.3) and (3.4) by exactly solving the set of nonlinear basic equations (3.42 through 3.46) numerically. We want to show that the use of nonlinear dispersion relation quasilinear modification of polarization field due to turbulent
vertical electron flux make the perturbation stabilized at some steady amplitude $A_0$ and which might have, in this nonlinear stationary state, the phase velocity given by (3.30). As there are $3M + 3 (= 75)$ ordinary differential equations (3.42, 3.43 and 3.44) in time and $2M + 2 (= 50)$ algebraic equations (3.45 and 3.46) to be solved and the time step $\Delta \tau$ is so low to satisfy the accuracy requirement, the computational time taken in IBM 360/44 computer becomes fantastically high to establish the solutions of these equations even up to the linear growth time. Therefore, we are compelled to follow an alternative approach given below to verify the results satisfactorily by restricting the solutions of the equations up to $\tau = 1.13 \times 10^{-1} \text{ (real time } \approx 0.5 \times 10^{-5} \text{ sec)}$.

In this approach, we study the time evolution of an initial density perturbation with amplitude $A = 1$ and fundamental wavelength $\lambda = 2\pi$, both linearly and nonlinearly. The results are shown in Fig. 3.3. The linear evolution of $\eta$ is shown by the thin solid curve and the nonlinear evolution by the dashed curve. For the latter case, the nonlinear modification of the electron drift $v_\text{om}$ (eq. 3.11) is taken into account. The thick solid curve represents the initial shape of $\eta$ at $\tau = 0$. Comparing these curves, it is inferred that in the linear case, the perturbation grows at the linear growth rate, whereas in the nonlinear case, the growth of the perturbation is negligible. This clearly indicates that if we use the nonlinear modification to electron drift and the nonlinear dispersion relation, then a nearly stabilized nonlinear solution is finally obtained. Furthermore, we have also performed the computation for higher
Fig. 3.3. Time evolution of an initial density perturbation \( \eta \) with amplitude \( A = 0.1 \) and fundamental wavelength \( \lambda = 2\pi \). The linear evolution of \( \eta \) is shown by thin solid curve and the nonlinear evolution by dashed curve, both until \( t = 1.13 \times 10^{-1} \). The thick solid curve represents the initial shape of \( \eta \) at \( t = 0 \).
initial values of $A$. In this case it is found that the dashed curve in Fig. 3.3 becomes indistinguishable from the initial values (thick curve). This shows that an initial nonlinear perturbation, satisfying the appropriate dispersion relation, persists in its original form. We take this as evidence for the existence of nonlinear stationary solutions of the kind discussed above. Unfortunately, limitations of computational time did not permit us to verify numerically the stability of such nonlinear solutions against slowly varying large amplitude modulational perturbations, although it is highly desirable to do so.

3.7 Discussions and Conclusions:

It is shown that the coherent type I modes observed in the equatorial electrojet can be described by a nonlinear ordinary differential wave equation (3.17). This equation has been solved both analytically and numerically, and harmonic weakly nonlinear stationary waves as its special solutions are obtained, under certain conditions prevailing in the equatorial electrojet. We, therefore, infer that such nonlinear periodic stationary waves might exist in the electrojet and might be responsible for the generation of coherent short wavelength ($>1\text{ m}$) type I irregularities giving rise to type I echoes. The theory predicts that the conditions under which such waves exist, are those for marginally stable state, implying that the type I irregularities are the result of marginally stable nonlinear state, in agreement with the inference drawn from the observations. The results of the theory can account satisfact-
orily for some of the observed features of type I irregularities. These features have already been mentioned in sec. 1.31 (a) and are now discussed here below in relevance to the theory presented.

In the linear limit, the earlier results obtained both from kinetic theory (Farley, 1963a,b) and the fluid approach (Buneman, 1963; Rogister and D'Angelo, 1970) are recovered. However, an important difference from earlier linear theories developed within fluid limit, can certainly be noted that the present analysis leads to the wavelength dependent linear dispersion relation (3.20) and the wavelength dependent linear threshold $v_c$ (3.22) for marginally stable waves described by the condition (3.21). This result agrees well with the observations (Cohen and Bowles, 1967; Balsley and Farley, 1971) which indicated that the phase velocity of type I irregularities is, in fact, wavelength dependent and corresponds to marginally stable waves. It is also remarkable to note that these results obtained here within fluid approximation, are similar to those predicted by marginally stability calculations from the kinetic theory (Farley, 1963a,b). It can easily be seen from eq. (3.22) that the threshold $v_c$ is higher for smaller wavelength modes, implying that the strong electrojet current (large electron drift velocity) is required for the generation of shorter wavelength type irregularities, an observational fact.

The present analysis leads to an amplitude dependent nonlinear dispersion relation (3.29) for short wavelength
type I modes existing in the electrojet. The nonlinear phase velocity thus depends upon both the wavelength (the dependence arising because of the dispersive effects introduced by ion viscosity) and the nonlinear amplitude (the dependence arising due to quasilinear effects). A requirement for steepening of the wave is that its phase velocity must depend in some manner upon local quantities, such as the density or temperature. The amplitude dependence of the phase velocity found here suggests that the waves might steepen in the nonlinear stage, similar to the case of sound waves. The phase velocity of sound wave depends upon the local temperature. In the compressional part of the sound wave, the temperature is higher, and since any local disturbance in this region propagates at a higher phase velocity, the sound wave profile gradually steepens. The waves discussed here are also dispersive and the dispersion resists the steepening of the waves. The amount of steepening will thus depend upon how important the dispersion is. That is, those waves for which the viscosity term $k^2/\mu$ is small compared to $\gamma_i$, might have appreciable steepening due to nonlinear interactions. As has already been discussed, this term is quite important at least for smaller wavelength, though its effect may be small compared to quasilinear effects. It is thus inferred that this steepening of the waves due to quasilinear effects limits the growth of the initially unstable waves, making it stabilized and in turn, leads to marginally stable state described by (3.27) in nonlinear regime.
We have shown that in this marginally stable nonlinear state, the nonlinear harmonic stationary type I modes get stabilized at some constant finite nonlinear amplitude $A_0$ nearly equal to 1.38% and travel with the nonlinear phase velocity $u_0$ defined by (3.30). This implies that type I irregularities occurring in the electrojet propagate with a constant nonlinear amplitude and a nonlinear phase velocity. This is in fair agreement with the observations reported by Prakash et al. (1970, 1971a,b, 1972, 1973) that the short wavelength irregularities which they called as type Ss and identified with type I, are observed having the amplitudes of the order of a few percent (ranging from 0.21% to 3.44% with a value of 1.55% around 104 km). This result is also seen to agree with the magnitude of the amplitudes ($\lesssim 3\%$) deduced from the backscatter radar measurements (Weinstock, 1968).

We have also shown analytically and later verified numerically that the nonlinear stabilization of nonlinear stationary waves will occur when both the expressions (3.29) and (3.30) for the phase velocity are satisfied. This result suggests two things. Firstly, the quasilinear effects tend to reduce its linear phase velocity (3.20) to a constant nonlinear phase velocity $u_0$ in nonlinear state. This reduction occurs because of the fact that the electron drift velocity $v_{om}$ on which linear phase velocity depends ($v_{om} = v_0$ in linear regime), gets decreased due to increase in turbulent negative vertical electron flux in nonlinear state (Register, 1971; Sato, 1972), as discussed in sec. 3.2. The nonlinear phase velocity of
marginally stable modes is slightly different from the linear threshold value by an amplitude dependent factor. Looking at the ion equation of motion, it can easily be realized that this effect arises due to the presence of a nonlinear term $(\mathbf{V} \cdot \nabla)\mathbf{v}$, i.e., through a ponderomotive force acting on ions due to quasilinear interactions. As the amplitude of nonlinear modes in marginally stable state remains constant ($\approx A_o$), the equation (3.30) expresses the remarkable fact that within fluid theory, the marginally stable modes have the threshold phase velocity which is constant and is slightly greater than the ion acoustic velocity, in contrast to the linear result that it is equal to ion acoustic velocity. This conclusion, therefore, appears to account for the observed constant phase velocity of type I irregularities, slightly differing from ion acoustic speed. Secondly, the electron drift velocity $v_{om}$ in the marginal stable state assumes a constant threshold value, $|v_{os}|$ which seems to be appreciably exceeding the ion acoustic velocity, as it is obvious from (3.39). This supports an observational result demonstrated by Cohen (1973) that the electron drift velocity may appreciably exceed the ion sound velocity, when type I irregularities appear. From equation (3.39), one easily observes the wavelength and amplitude dependences of the threshold electron drift velocity, $v_{os}$ for marginally stable nonlinear waves. Thus, the theory predicts that the nonlinear threshold $|v_{os}|$ differs slightly from its linear value by the amplitude dependent term, the effect (as discussed above) arising through ponderomotive force on ions.
The numerical solutions of the full nonlinear wave equation (3.17) and the exact set of basic equations (3.1) - (3.4) are found to support many of the analytical results of this theory. We must remark here, however, that the numerical verification of the above results in support of them, could not be done completely and exactly because of lack of availability of computer time. It is, rather, unfortunate that we were compelled to make use of the nonlinear dispersion relation and the quasilinear modification $v_{om}$ which are important only when the nonlinearities start playing role after the linear growth, in finding out the solutions of the exact nonlinear set of basic equations. Furthermore, we could integrate these equations only up to the (real) time $\sim 0.5 \times 10^{-5}$ sec which is short compared to all time scales of interest (e.g., even much shorter than the linear growth time). Nevertheless, the inferences made by extrapolating the numerical results in Fig. 3.3 for larger time, reasonably support the analytical results of our theory. We point out, further that if it were feasible to solve numerically the exact nonlinear set for sufficiently large time, it would have been possible to show clearly first the linear growth and later on, the final nonlinear stable state of the initial small amplitude perturbation. Consequently, this would have provided us with the nonlinear saturation value of the amplitude, $A_0$, the marginally stable threshold electron drift velocity, $v_{os}$, the nonlinear phase velocity, $u_0$ and other nonlinear characteristics.

The amplitude dependence of wave frequency and the wave dispersion is such that the coherent nonlinear stationary
type I modes discussed above are found to be stable against slowly varying large-amplitude modulational perturbations. We have shown, though not by rigorous treatment, that the large amplitude modulational perturbations to such modes are, however, expected to develop adiabatically into modulational envelope solitons. As mentioned in sec. 1.31 (a), there may already be some evidence to this effect from backscatter observations. (Balsley, 1969a,b, 1973a,b; Farley and Balsley, 1973) that the type I irregularities in the electrojet might travel in the form of fairly discrete wave packets with soliton shaped envelopes.

Unfortunately, limitations of computational time did not permit us to numerically investigate the stability property of nonlinear solutions discussed above, although such a study would be highly desirable. As a result, it could also not be possible to check numerically one of our conclusions, e.g., the conversion of the type I modes into modulational envelope solitons in adiabatically nonstationary case.

In conclusion, the principal results of the theory presented herein, may now be summarized as follows. The author has demonstrated the existence of stationary nonlinear solutions to the equations describing one-dimensional type I irregularities in the equatorial electrojet. The nonlinear phase velocity and threshold electron drift velocity for the marginally stable mode are slightly different from their linear values, the effect arising through ponderomotive force on ions. The nonlinear solutions are also found to be stable
against slowly varying large amplitude modulational perturbations. However, in the adiabatically nonstationary case, one expects the formation of modulational envelope solitons. Some of the observed features of type I backscatter echoes as well as type $S_2$ irregularities are consistent with the predictions of the theory presented.