In this Chapter we study mainly, invariant subspace problem and conditions under which an operator turns out to be unitary. In Section 1, we study basic properties of a maximal vector subspace of an operator. Moreover, we study conditions under which a maximal vector subspace is non-trivial. In Section 2, we extend the result of J.G. Stampfli [73] for operator of class (N, k). Also, conditions under which an invertible operator turns out to be a scalar multiple of unitary, are studied. Section 3 is devoted to a study of principal points of an operator.

1. MAXIMAL VECTOR SUBSPACE

The idea of maximal vector subspace of an operator \( T \) defined on a Hilbert space \( H \) is connected with the invariant subspace problem. J.G. Stampfli [73] has proved that if a hyponormal operator has a maximal vector, then it has a non-
trivial invariant subspace. We extend this result for operator of class \((N, k)\). Also, we show that this result can not be extended further for normaloid operators.

A non-zero vector \(x \in H\) is called a maximal vector for an operator \(T\) if \(||Tx|| = ||T|| \cdot ||x||\). Let

\[ M(T) = \left\{ x \in H : ||Tx|| = ||T|| \cdot ||x|| \right\}. \]

Then clearly \(0 \in M(T)\). Since \(x \in M(T)\) if and only if

\[ (||T||^2 I - T^*T)x, x) = 0. \]

As \(||T||^2 I - T^*T \geq 0\), it follows that \(M(T)\) is a closed subspace. We call \(M(T)\) the maximal vector subspace of \(T\). \(M(T)\) has some nice properties found in the following theorem.

**Theorem 6.1** \([16, 63]\) : For any operator \(T\),

(i) \(M(T) = M(\alpha T)\), \(\alpha \neq 0\) scalar.

(ii) \(M(T) = M(T^*T)\) and \(M(T^*) = M(TT^*)\).

(iii) \(T[M(T)] \subseteq M(T^*)\).

**Proof** : (i) As the proof is simple, we omit it.

(ii) Let \(x \in M(T)\). Then
\[ \| T^*Tx \| \leq \| T \| \cdot \| x \| = \frac{\| Tx \|^2}{\| x \|} \leq \| T^*Tx \|. \]

i.e. \( M(T) \subseteq M(T^*T) \).

Now, if \( x \in M(T^*T) \), then
\[ \| Tx \| \leq \| T \| \cdot \| x \| = \frac{\| T^*Tx \|}{\| T \|} \leq \| T \|. \]

i.e. \( M(T^*T) \subseteq M(T) \).

Thus \( M(T) = M(T^*T) \). We can prove similarly,
\[ M(T^*) = M(TT^*). \]

(iii) For \( x \in M(T) \), \( x \in M(T^*T) \) and
\[ \| T^*Tx \| = \| T^*T \| \cdot \| x \| = \| T^* \| \cdot \| Tx \|. \]

i.e. \( T \cdot x \in M(T^*) \).

**Corollary 5.1**: If \( M(T) = M(T^*) \neq \{ 0 \} \), then \( M(T) \) reduces \( T \) and \( T/M(T) \) is a scalar multiple of unitary.

**Proof**: It follows from (iii) of this theorem that \( M(T) \) is invariant under both \( T \) and \( T^* \), i.e. \( M(T) \) reduces \( T \).
If \( S = T/M(T) \), then \( \| sx \| = \| S \| \cdot \| x \| = \| S^*x \| \) for
x \in M(T). Hence S/||S|| is unitary, i.e. T/M(T) is a scalar multiple of unitary, as desired.

Now consider the operators $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then T and S are unitarily equivalent but $M(T) \neq M(S)$. This shows that if the operators are unitarily equivalent, their maximal vector subspaces are not necessarily equal. However, their dimensions are equal. Happily, if the operators are metrically equivalent, their maximal vector subspaces are equal. For commuting operators, we prove the following theorem.

**Theorem 5.2**: If T commutes with an unitary operator U and $M(T) \subseteq \{ 0 \}$, then $M(T) = M(TU)$.

**Proof**: Since U is unitary, $||T|| = ||TU||$. For $x \neq 0 \in M(T)$,

$$||TUx|| = ||UTx|| = ||Tx|| = ||T||.||x||$$

$$= ||TU||.||x||$$

i.e. $M(T) \subseteq M(TU)$.

The reverse inclusion follows similarly. This proves the theorem.

We consider the following examples.
Example 1: Let $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1 & 0 \end{pmatrix}$ be an operator defined on a three dimensional unitary space with respect to the usual orthonormal basis $\{e_1, e_2, e_3\}$. Then

(i) $T$ is normaloid. (ii) $T^{-1}$ is not normaloid.

(iii) $M(T) = [e_1, e_2]$. (iv) $M(T^*) = [e_1, e_3]$.

(v) $M(T^{-1}) = [e_2]$, (vi) $M(T^2) = [e_1]$.

Example 2: Let $T : L^2[0, 1] \to L^2[0, 1]$ be defined as $(Tf)(t) = tf(t)$ for $f(t) \in L^2[0, 1]$, $0 \leq t \leq 1$.

Then, $\sigma(T) = \pi(T) = [0, 1]$ and $M(T) = \{\emptyset\}$, because if $x \neq \emptyset \in M(T)$ then $T^*Tx = T^2x = ||T||^2x$. i.e. $\emptyset \subset ||T|| \subset \pi_o(T)$, a contradiction.

From these examples it follows that, $M(T) \neq \{\emptyset\}$ for a non-normaloid operator $T$, and $M(T) = \{\emptyset\}$ even if $T$ is self-adjoint. A natural question to be raised is under what conditions $M(T) \neq \{\emptyset\}$? In the following theorem, an attempt is made in this direction.

**Theorem 5.3** \[66\]: For an operator $T$, $M(T) \neq \{\emptyset\}$ if any one of the following conditions is satisfied.
(i) $T$ is compact;
(ii) $\dim H < \infty$;
and (iii) $H(T) \cap W(T) \neq \emptyset$.

**PROOF:** (i) The conclusion follows from the fact that a compact operator attains its norm. (ii) Since $T^*T$ is self-adjoint, there is $z \in \pi_0(T^*T)$ such that $|z| = ||T^*T||^2 = ||T||$, i.e. there is a non-zero vector $x$ in $H$ such that $T^*Tx = ||T||^2 x$. i.e. $x \in M(T)$. (iii) $H(T) \cap W(T) \subseteq \pi_0(T) \subseteq \mathcal{L}(H)$. Hence there is $z \in \pi_0(T)$ such that $|z| = ||T||$, i.e. there is a non-zero vector $x$ in $H$ such that $||Tx|| = ||T|| \cdot ||x||$, i.e. $x \in M(T)$.

**COROLLARY 5.2:** If $T$ is normaloid and $W(T)$ is closed then $M(T) \neq \{0\}$.

**PROOF:** As the proof is simple, we omit.

2. INVARIANT SUBSPACE.

In [73], J.G. Stampfli has proved that if a hyponormal operator has a maximal vector, it has a non-trivial invariant subspace. We extend this result in the following theorem.

**THEOREM 5.4:** If an operator of class $(N, k)$ has a maximal vector, it has a non-trivial invariant subspace.
Let $T$ be an operator of class $(N, k)$. Then $T$ is normaloid. Now $x \notin 0 \in M(T)$,

$$||T(Tx)|| \leq ||T|| \cdot ||Tx|| = ||T||^2 \cdot ||x||$$

$$\leq \frac{||Tx||^k}{||T^{k-2}||} \cdot ||x||^{k-1}$$

$$\leq \frac{||Tk_x||}{||T^{k-2}||} \leq ||T^2x||$$

i.e. $Tx \in M(T)$.

This completes the proof of the theorem.

**Remark**: Example 1 shows that this result cannot be extended further for normaloid operators.

**Theorem 5.5**: If $T$ is normaloid and $T^n$ is unitary for some positive integer $n$, then $T^n$ is unitary.

**Proof**: Since $T$ is normaloid, for $x \in M(T^n)$,

$$||T^{n-1}x|| \leq ||T||^{n-1} \cdot ||x|| = \frac{||T^n||}{||T||} \cdot ||x|| \leq ||T^{n-1}x||$$

i.e. $M(T^n) \subseteq M(T^{n-1})$. 
Thus, \( H = M(T^n) \subseteq M(T^{n-1}) \subseteq H \). i.e. \( M(T) = H \).

Hence \( T \) is a scalar multiple of isometry. \( T \) being invertible, \( T \) is unitary.

**REMARKS**:

(i) From the proof of this theorem it follows that \( M(T^n) \subseteq M(T^{n-1}) \) whenever \( T \) is normaloid. However, the converse is not true. i.e. \( M(T^n) \subseteq M(T^{n-1}) \) does not imply that \( T \) is normaloid because if \( T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) then \( M(T) = M(T^n) \) for all \( n \) without \( T \) being normaloid.

(ii) Since an operator of class \((M)\) is normaloid, Theorem 3.12 follows immediately from this theorem.

We now study the situation in which an invertible operator turns out to be a scalar multiple of a unitary operator. We prove the following theorems.

**THEOREM 5.6**: For an invertible operator \( T \) with \( M(T) \neq \{0\} \), \( T \mid M(T) \mid = M(T^{-1}) \) if and only if \( \| T \| \cdot \| T^{-1} \| = 1 \).

**PROOF**: Let \( T \mid M(T) \mid = M(T^{-1}) \). Then, for \( x \neq 0 \in M(T) \), \( T x \in M(T^{-1}) \) and
\[ ||x|| = ||T^{-1}Tx|| = ||T^{-1}|| \cdot ||Tx|| = ||T^{-1}|| \cdot ||T|| \cdot ||x|| \]

i.e. \[ ||T|| \cdot ||T^{-1}|| = 1 \]

Now we assume \[ ||T|| \cdot ||T^{-1}|| = 1 \]. Then for \( x \in M(T) \)

\[ ||x|| = ||T^{-1}Tx|| \leq ||T^{-1}|| \cdot ||Tx|| \]

\[ \leq ||T^{-1}|| \cdot ||T|| \cdot ||x|| = ||x|| \]

i.e. \( Tx \in M(T^{-1}) \) and hence \( T \left[ M(T) \right] \subseteq M(T^{-1}) \). In order to prove the reverse inclusion, let \( x \in M(T^{-1}) \). If \( T^{-1}x = y \) then \( x = Ty \) and \( ||y|| = ||T^{-1}|| \cdot ||Ty|| \) Thus

\[ ||Ty|| = \frac{||y||}{||T^{-1}||} = ||T|| \cdot ||y|| \]

i.e. \( y \in M(T) \) i.e. \( x \in T \left[ M(T) \right] \). Hence \( M(T^{-1}) \subseteq T \left[ M(T) \right] \). This completes the proof of the theorem.

In [38], W. Malk has proved the following result.

**Theorem Q**: An invertible operator \( T \) is a scalar multiple of a unitary if and only if \[ ||T|| \cdot ||T^{-1}|| = 1 \].
Theorem 5.7: An invertible operator $T$ is a scalar multiple of a unitary if and only if $M(T^*) = M(T^{-1}) \oplus \{ \theta \}$

Proof: By definition, $M(T^*) = \left\{ x : TT^*x = ||T||^2 x \right\}$ and $M(T^{-1}) = \left\{ x : TT^*x = x / ||T^{-1}||^2 \right\}$. Hence $M(T^*) = M(T^{-1})$ if and only if $||T|| = 1 / ||T^{-1}||$ if and only if $||T|| \cdot ||T^{-1}|| = 1$ if and only if $T$ is a scalar multiple of a unitary (by Theorem Q).

Theorem 5.8: An invertible operator $T$ is a scalar multiple of a unitary if and only if $T \subseteq M(T) \subseteq M(T^{-1}) \oplus \{ \theta \}$

Proof: Suppose $T \subseteq M(T) \subseteq M(T^{-1}) \oplus \{ \theta \}$. Then for $x \in M(T)$, $Tx \in M(T^{-1})$ and

$$||x|| = ||T^{-1}Tx|| = ||T^{-1}|| \cdot ||Tx|| = ||T^{-1}|| \cdot ||T|| \cdot ||x||$$

i.e. $||T|| \cdot ||T^{-1}|| = 1$.

Now the conclusion follows from Theorem Q. The converse part is an immediate consequence of Theorem 5.6.

Corollary 5.3: An invertible operator $T$ is a scalar multiple of a unitary if $T$ is an operator of class $(N, k)$ and $M(T) = M(T^{-1}) \oplus \{ \theta \}$. 
PROOF: Since $T$ is an operator of class $(N, k)$, $M(T)$ is invariant under $T$ by Theorem 5.4. i.e.

$$T \left[ M(T) \right] \subseteq M(T) = M(T^{-1}) \neq \left\{ \theta \right\}.$$ 

Now, the conclusion follows from the theorem.

REMARK: In this corollary, the condition $M(T) = M(T^{-1}) \neq \left\{ \theta \right\}$ is not sufficient to ensure unitariness of $T$, as

$$T = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

then $M(T) = M(T^{-1}) \neq \left\{ \theta \right\}$, without $T$ being unitary.

3. PRINCIPAL POINTS OF AN OPERATOR.

Stefen Hildebrandt [32] has defined principal points of an operator as follows.

If $T$ is an operator defined on a Hilbert space $H$, then the points of the set

$$H(T) = \overline{W(T)} \cap \left\{ z : ||z|| = ||T|| \right\}$$

are called the principal points of $T$. In [32], it is shown that
(i) \( H(T) \) is non-empty if and only if \( T \) is normaloid;
(ii) \( H(T) \subseteq \pi(T) \);
(iii) \( H(T) \cap W(T) = H(T) \cap \pi_0(T) \).

Throughout the discussion, we will consider \( T \) to be normaloid in order to ensure \( H(T) \neq \{ \emptyset \} \). The set \( H(T) \) possesses the following properties.

**Theorem 5.9:** If \( T \) is a normaloid operator, then

(i) \( H(T) \) is a closed set;
(ii) \( H(\alpha T) = \alpha H(T) \), \( \alpha \) scalar;
(iii) \( H(T^n) = \bigcap_{i=1}^{n} H(T) \), \( n = 1, 2, 3, \ldots \);
(iv) \( H(T) = H(S) \) whenever \( T \) is unitarily equivalent to \( S \);
(v) \( H(T^*) = \{ \bar{z} : z \in H(T) \} \).

**Proof:** (i), (ii) and (v) are easy to prove. The proof (iii) can be furnished by using spectral mapping theorem and the fact that \( \sigma(T) \subseteq \overline{W(T)} \). The proof of (iv) follows from the relations \( \sigma(S) = \sigma(T) \); \( W(S) = W(T) \) and \( ||S|| = ||T|| \).

We know that if \( T \) is normaloid then \( T+zI \) is not necessarily normaloid. (otherwise, if \( T \) is quasi-hyponormal
then it is transloid and hence convexoid, a contradiction. Therefore, it may happen that \( H(T+zI) = \emptyset \), for some normaloid \( T \). Thus, generally the following relation does not hold.

\[
(\ast_1) \quad H(T+zI) = H(T) + zI
\]

There are self-adjoint and unitary operators for which \( (\ast_1) \) does not hold. It will be interesting to study conditions which imply \( (\ast_1) \). We prove the following theorem in this direction.

**THEOREM 5.10**: For a normaloid operator, \( (\ast_1) \) holds if and only if \( T \) is a scalar multiple of identity.

**PROOF**: If \( T \) is a scalar multiple of identity, the result is immediate. Suppose \( T \) is normaloid and \( (\ast_1) \) holds. \( T \) being normaloid, \( H(T) = \emptyset \). At first, we prove that \( H(T) \) is a singleton set. If possible, let \( z_1 \neq z_2 \) and \( z_1', z_2' \in H(T) \). Then for any scalar \( z \), \( z_1+z \) and \( z_2+z \) are in \( H(T+zI) \), i.e.

\[
|z_1+z| = |z_2+z| = ||T+zI|| \quad (i)
\]

Now, if \( z_0 \) is not on the perpendicular bisector of the
line joining $z_1$ and $z_2$, then $|z_1 - z_0| + |z_2 - z_0|$, which
contradicts (i) for $z = -z_0$. Thus $H(T) = \{ u_0 \}$, say. We
assert that $\sigma(T) = \{ u_0 \}$. In the case contrary to this,
$u \neq u_0 \in \sigma(T)$. As $u \notin H(T)$, $|u| < ||T||$. Select $\alpha$
such that $|u-\alpha| > |u_0-\alpha|$. Now $u_0 - \alpha \in H(T-\alpha I)$, which
implies $|u_0 - \alpha| = ||T-\alpha I||$. Also $u-\alpha \in \sigma(T-\alpha I)$,
implies $|u-\alpha| \leq ||T-\alpha I||$. Thus $|u-\alpha| \leq |u_0-\alpha|$, a
contradiction to the choice of $\alpha$. Hence $\sigma(T) = \{ u_0 \}$.
Since $H(T) \neq \emptyset$, $H(T+zI) \neq \emptyset$. i.e. $T+zI$ is normaloid
and consequently $T$ is convexoid. Therefore, $\sigma(T) = W(T) = \{ u_0 \}$
Hence $(T-u_0 I)x, x = 0$ for all $x$ in $H$. Therefore, $T-u_0 I$.

**COROLLARY 5.4** : If $T \in R$ and $\sigma(T)$ is finite, then
$(*_1)$ holds.

**PROOF** : From $[37]$, it follows that $T$ is a scalar
multiple of identity. Now, the result follows from this theorem.

**COROLLARY 5.5** : If $T \in R$ and $\dim H < \infty$ then $(*_1)$ holds.

**PROOF** : Easy to prove.

If $T$ is an invertible normaloid operator then $T^{-1}$ is
not necessarily normaloid, (see example 1). Hence, for an invertible normaloid operator, the following relation does not hold.

\[(*)_2 \quad H(T^{-1}) = \left[ H(T) \right]^{-1}\]

However, if \( T \) is unitary or a non-zero scalar multiple of a unitary, then \((*)_2\) holds. Moreover, the relation

\[(*)_3 \quad H(T) = \sigma(T)\]

holds. Now the following questions arise naturally.

1. For an invertible operator \( T \) if \((*)_2\) holds, does it follow that \( T \) is a scalar multiple of a unitary?

2. For a non-zero operator \( T \) if \((*)_3\) holds, does it follow that \( T \) is a scalar multiple of a unitary?

By taking \( T/\|T\| \), we can reformulate the second question as follows:

2'. For an operator \( T \) if \((*)_3\) holds and \( \sigma(T) \) lies on the unit circle, does it follow that \( T \) is unitary?
The answer of 2' is negative. Consider the following example [52] .

Let $H$ be the set of sequences $a = \left\{ a_n \right\}_{-\infty}^{\infty}$ of complex numbers such that $\sum |a_n|^2$ is finite. Then $H$ is a Hilbert space with the usual inner product $(a, b) = \sum_{-\infty}^{\infty} a_n \overline{b_n}$ .

Let $\left\{ x_n \right\}_{-\infty}^{\infty} \subseteq H$ be the canonical orthonormal basis for $H$ . i.e. $x^n$ is the 'characteristic function on $n'$. Let $0 < t < 1$ and let $G$ be the operator defined by

$$G x^n = \begin{cases} t x^{(n)} & \text{if } n = 0 \\ x^{(n)} & \text{if } n \neq 0 . \end{cases}$$

Let $U$ denote the bilateral shift operator $U x^{(n)} = x^{(n+1)}$ for all $n$ . Then $T = UG$ is a contraction which is not unitary and $\sigma(T)$ is the entire unit circle.

We are thankful to Prof. T. Saitô for having pointed out to us this example of [52] . We now answer 1 and 2' in the following theorems .
**THEOREM 5.11**: For an invertible normaloid operator $T$ if $(\star_2)$ holds then $T$ is a scalar multiple of a unitary.

**PROOF**: For $z \in H(T)$, $z^{-1} \in H(T^{-1})$. i.e. $||T|| = |z|$ and $||T^{-1}|| = |z^{-1}|$. i.e. $||T|| \cdot ||T^{-1}|| = 1$. Hence $T$ is a scalar multiple of a unitary by Theorem Q.

**THEOREM 5.12**: Let $T$ be an operator such that $(\star_3)$ holds and $\sigma(T)$ lies on the unit circle. If

(i) $\dim H < \infty$ or

(ii) $T$ satisfies the condition $G_1$ then $T$ is unitary.

**PROOF**: (i) From the hypothesis, $H(T) = \sigma(T) = \pi_o(T)$. Now, the result follows from the facts that $z, u \in \pi_o(T)$, $z \neq u$ and $|z| = |u| = ||T||$ then $N_T(u) = N_{T^*}(u^*)$ and $N_T(u) \perp N_T(z)$.

(ii) Since $T$ is invertible and satisfies the condition $G_1$, $||T^{-1}|| \leq 1$. As $\sigma(T^{-1})$ lies on the unit circle, $||T^{-1}|| = 1$. Thus $||T|| \cdot ||T^{-1}|| = 1$ and hence $T$ is unitary by Theorem Q.