The purpose of this Chapter is to characterize operators of class \((N)\) and generalize some known results of normaloid operators. In Section 1, we give essentially the same characterization for operators of class \((N)\) and recover two theorems of [20, 35]. Moreover, we extend the result of [29]; a partial isometry is subnormal if and only if it is hyponormal. In Section 2, we generalize some known results for normaloid operators. Section 3 is devoted to new proof of the theorem due to T. Saitô and derive it again as a corollary from the different generalizations.

1. OPERATOR OF CLASS \((N)\)

T. Andô [2] has characterized operators of class \((N)\) as follows:

**Theorem M**: An operator \(T\) is paranormal if and only if
\[
T^* T^2 - 2 z T^* T + z^2 I \geq 0, \quad (z > 0).
\]
We shall give below an essentially same characterization for operators of class (N) with simple proof, by which we recover two theorems of [20, 35]. In the proof, we use an elementary property of real quadratic forms: If \( a > 0 \), \( b \) and \( c \) are real numbers, then \( at^2 + bt + c \geq 0 \) for any real \( t \) if and only if \( b^2 - 4ac \leq 0 \).

**Theorem 4.1 [59]:** An operator \( T \) is an operator of class (N) if and only if \( T^* T^2 + 2zT^*T + z^2I \geq 0 \) for all real \( z \).

**Proof:**

\[
T^* T^2 + 2zT^*T + z^2I \geq 0
\]

\[\iff ( (T^* T^2 + 2zT^*T + z^2I)x, x) \geq 0\]

\[\iff \| T^2x \|^2 + 2z \| Tx \|^2 + z^2 \| x \|^2 \geq 0\]

\[\iff \| Tx \|^2 \leq \| T^2x \| \text{ (by the remark made above)}\]

\[\iff T \text{ is an operator of class (N)}\]

**Corollary 4.1 [59]:** If \( T \) is an invertible operator of class (N) then \( T^{-1} \) is an operator of class (N).

**Proof:** By the theorem we have
for every real $z$. Putting $u = 1/z$, for every real $u \neq 0$, we have

$$T^{* -2}T^{-2} + 2uT^{* -1}T^{-1} + u^2I \geq 0$$

which is also trivially true when $u = 0$, so that $T^{-1}$ is an operator of class $(N)$.

**Corollary 4.2** : If $T$ is an operator of class $(N)$ then $T^k$ is an operator of class $(N)$ for every positive integer $k$.

**Proof** : We give the proof by induction. At first, we prove that $T^2$ is an operator of class $(N)$. By the theorem, for any real $z$ we have,

$$T^{*3}T^3 + 2zT^{*2}T^2 + z^2T^{*}T = T^*(T^{*2}T^2 + 2zT^{*}T + z^2I)T \geq 0$$

which is equivalent to

$$\|T^3x\|^2 + 2z\|T^2x\|^2 + z^2\|Tx\|^2 \geq 0$$
for every unit vector $x$. As $||T^3x||^2 = ||TT^2x||^2$
\[\leq ||T^4x||.||T^2x||, \text{ for every real } z \text{ we have },\]
\[||T^4x||.||T^2x|| + 2z||T^2x||^2 + z^2 . ||Tx||^2 > 0\]
Hence,
\[||T^2x||^4 \leq ||T^4x||.||T^2x||.||T^2x||\]
i.e. $||T^2x||^2 \leq ||T^4x||$.

Thus $T^2$ is an operator of class (N).

Now assuming $T^k$ an operator of class (N), we show that $T^{k+1}$ is also an operator of class (N). By the theorem, for any real $z$ we have
\[T^{*2k+1}T^2k+1 + 2zT^{*k+1}T^k+1 + z^2T^2T^*T \geq 0\]
This implies
\[||T^{2k+1}x||^2 + 2z||T^{k+1}x||^2 + z^2||Tx||^2 \geq 0\]
for every unit vector $x$. Hence, $||T_{k+1}x||^4 < ||T_{2k+1}x||^2$.

Now, $T$, being an operator of class (N), we have $||T_{n+1}x||^2 \geq ||T_nx||^2 ||T_2x||$ for any positive integer $n \geq 20$. Thus $||T_{k+1}x||^4 < ||T_{2(k+1)}x||^2$ and $T_{k+1}$ is an operator of class (N).

We know that the product of two doubly commuting operators of class (N) is not necessarily an operator of class (N). However, T. Saitô [54] has proved the following theorem using the spectral integral.

**THEOREM N**: If an operator of class (N) is doubly commutative with a hyponormal operator $S$ (i.e. $TS = ST$ and $TS^* = S^*T$) then $TS$ is an operator of class (N).

Using Theorem 4.1, we prove the following theorem:

**THEOREM 4.2** [58]: If $T$ is an operator of class (N) and commutes with an isometric operator $S$ then $TS$ is an operator of class (N).

**PROOF**: If $A = TS$, then we have for any real $z$,

$$A^2 = A^2 + 2zA^* + z^2I = S^*T^*S^*T^*TST + 2zS^*T^*TS + z^2I.$$
Using \( TS = ST \), \( T^*S^* = S^*T^* \) and \( S^*S = I \), we get

\[
A^2 + 2zA^2 + z^2I = T^2 + 2zT^*T + z^2I \geq 0
\]

so that \( A \) is an operator of class (N) by Theorem 4.1.

**Remark:** In this theorem \( ST = TS \) does not imply \( ST^* = T^*S \).

We know that a partial isometry is an operator restriction of which to the orthogonal complement of its null-space is an isometry \([66]\). Equivalently, an operator \( T \) is a partial isometry if and only if \( T = TT^* \) \([29]\). In \([59]\), we have shown that a partial isometry of class (N) is quasi-hyponormal. Here, we extend this result and prove that a partial isometry of class (N, k) is quasi-hyponormal. This work was motivated by an interesting consequence of the proof of \([29, \text{Problem 161}]\). There, it is proved that a partial isometry is subnormal if and only if it is hyponormal.

**Theorem 4.3** \([57]\): For a partial isometry \( T \), the following conditions are equivalent,

(i) \( T \) is an operator of class \((N, k)\), \( k \geq 2 \).

(ii) \( T \) is quasi-normal.

(iii) \( N(T) \subseteq N(T^*) \).
PROOF: (i) $\Rightarrow$ (ii): If $T$ is an operator of class $(N, k)$, then

$$
\|Tx\|^k = \|TT^*Tx\|^k
$$

$$
\leq \|T^kT^*Tx\|.\|T^*Tx\|^{k-1}
$$

$$
= \|T^kx\|.\|Tx\|^{k-1}
$$

i.e. $\|Tx\|^k \leq \|T^kx\| \leq \|T^2x\| \leq \|Tx\|$. 

i.e. $\|Tx\| = \|T^2x\|$. 

Now,

$$
\|T^*TTx - Tx\|^2 = \|T^*TTx\|^2 + \|Tx\|^2
$$

$$
- (T^*TTx, Tx) - (Tx, T^*TTx)
$$

$$
= \|T^2x\|^2 + \|T^2x\|^2 - 2 \|T^2x\|^2.
$$

$$
= 0
$$

Thus $T^*TTx = Tx$ for all $x$ in $H$. i.e. $T^*TT = T = TT^*T$ 
i.e. $T$ is quasi-normal.

(ii) $\Rightarrow$ (iii): Obvious.
(iii) $\implies$ (i) : $\mathcal{N}(T) \subseteq \mathcal{N}(T^*) \implies \mathcal{N}(T^*)^\perp \subseteq \mathcal{N}(T)^\perp$

$\implies$ Initial space of $T^*$ $\subseteq$ Initial space of $T$ $\implies$

$TT^* \leq T^*T \implies T$ is hyponormal and hence operator of class $(N, k)$

REMARKS : (i) In condition (i) of this theorem, we cannot take $T$ to be normaloid, as $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a normaloid partial isometry but not quasi-normal. (ii) The referee has kindly pointed us out the stronger result than this one \cite{30}. (iii) T. Furuta has generalized this result in \cite{21}. (iv) Theorem 3 of \cite{40} is a direct consequence of this theorem.

COROLLARY 4.3 \cite{57} : If $T$ is a partial isometry of class $(N, k)$ such that $\mathcal{N}(T^*) = \{ \emptyset \}$, then $T$ is unitary.

PROOF : From the theorem, $\mathcal{N}(T) \subseteq \mathcal{N}(T^*) = \{ \emptyset \}$. Hence $\mathcal{N}(T) = \mathcal{N}(T^*) = \{ \emptyset \}$. Since $\mathcal{N}(T)^\perp = \emptyset$, $T$ is isometry. i.e. $T^*T = I$. This implies $T^*$ is both one-one and onto. Hence $T^*$ and consequently $T$ is invertible. Therefore, $T$ is unitary.
2. SOME GENERALIZATIONS

We know that a hyponormal operator with pure point spectrum is normal. But $\sigma(T) = \pi_0(T)$ does not imply normality of a hyponormal operator $T$. However, if the underlying Hilbert space is separable, $T$ becomes normal. G.H. Constantin [17] has proved the following theorem:

**THEOREM 0**: If $T$ is an operator of class $(N, k)$, $\sigma^{-}(T) = \pi_0(T)$ and has zero as the single limit point, then $T$ is normal.

In the following theorem, we generalize this result for restriction - normaloid operator (T is restriction - normaloid if restriction to every invariant subspace is normaloid).

**THEOREM 4.4**: If $T$ is restriction - normaloid, $\sigma^{-}(T) = \pi_0(T)$ and has zero as the single limit point, then $T$ is normal.

The proof of this theorem can be constructed just similar to that of Theorem 2.4 [17]. For the sake of completeness, we give the proof.

**PROOF**: Since $T$ is normaloid, there exists $z_0 \in \sigma^{-}(T)$ such that $|z_0| = ||T||$ and therefore $z \in \pi_0(T) \cap W(T)$. For $x \in N_T(z_0)$,
\[ ||T^*x - z_0x||^2 = ||T^*x||^2 - (T^*x, z_0x) - (z_0x, T^*x) + |z_0|^2 \leq 0 \]

Hence \( N_T(z_0) \) reduces \( T \) and \( T/NT(z_0) \) is normal. From the fact that \( T/NT(z_0) \) is normaloid we may continue this process and we conclude that \( T \) is normal and the underlying Hilbert space is the direct sum of the eigenspaces of \( T \).

It is known that every Riesz operator \( T \) on a Hilbert space can be decomposed into

\[ (*) \quad T = C + Q \]

where \( C \) is a compact operator and \( Q \) is a quasinilpotent operator \([75]\). Now, Bonsall's problem is as under \([76]\).

Is every hyponormal operator of the form \((*)\) normal? We know that this question has an affirmative answer \([54]\). In this thesis we have studied two different generalizations of hyponormal operators. Naturally, we would like to raise the same question for these operators, i.e. for quasi-hyponormal operators and operators of class \((M)\). For quasi-hyponormal operators, the question has an affirmative answer \([54]\). We answer this question for restriction - normaloid operators, that include both, quasi-hyponormal operators and operators of class \((M)\).
THEOREM 4.5 : Let $T$ be a restriction - normaloid operator of the form (*). Then $T$ is a normal operator.

PROOF : Since $T$ is normaloid, there exists $z \in \sigma(T)$ such that $|z| = \|T\|$. We can assume that $z \neq 0$. Now $z \in \nu(T)$ and $z \notin \nu(T)$. Hence $z \in \nu(T)$. Consequently,

$$\overline{z} \in \nu(T^*) \text{ and } N_T(z) \cap N_{T^*}(\overline{z}) = \{0\}.$$ Let

$$H = \sum N_T(z) \cap N_{T^*}(\overline{z}) \text{ where } z \in \nu(T) .$$ Then $H$ reduces $T$ and $T_1 = T/H$ is normal. We assert that $T_2 = T/H = 0$,

whence $T = T_1 \oplus 0$ is normal. If possible, suppose $T_2 \neq 0$.

Since $T_2$ is a non-zero normaloid operator on $H = H$, there exists $u \in \nu(T_2)$ such that $|u| = \|T_2\|$. As above, $u \in \nu(T_2)$.

hence $\overline{u} \in \nu(T_2^*)$. i.e. $N_{T_2}(u) \cap N_{T_2^*}(\overline{u}) = \{0\}$,

a contradiction.

COROLLARY 4.4 : Let $T$ be an operator of class $(M)$ of the form (*). Then $T$ is normal.

PROOF : The corollary follows from the fact that operators of class $(M)$ are restriction - normaloids.

REMARK : Theorem 2.1 of [34] follows as a corollary from Theorem 4.5.
Let $\mathcal{K}(H)$ be the two-sided ideal of compact operators. The quotient algebra $B(H)/\mathcal{K}(H)$, which is a C*-algebra, is known as the Calkin algebra of $H$. Let $\hat{T}$ be the canonical image of $T$ in the Calkin algebra. We know that, if $T$ is hyponormal (or operator of class $(N)$), $\hat{T}$ is hyponormal (or operator of class $(N)$) [18, 34]. Also $\hat{T}$ is an operator of class $(M)$ (or quasi-hyponormal), whenever $T$ is an operator of class $(M)$ (or quasi-hyponormal). In [34], it is proved that a hyponormal Riesz operator is compact. We extend this theorem to include quasi-hyponormal operators and operators of class $(M)$.

**Theorem 4.6**: Let $T$ be an operator of the form (*). If $\hat{T}$ is normaloid then $T$ is compact.

**Proof**: Since $T = C + Q$, $\hat{T} = \hat{Q}$. Hence $\sigma(\hat{T}) = \sigma(\hat{Q}) = \{0\}$. Now $\hat{T}$ being normaloid,

$$||\hat{T}|| = \sup \{ |z| : z \in \sigma(\hat{T}) \} = 0,$$

i.e. $\hat{T} = 0$, hence $T$ is compact.

**Corollary 4.5**: If $T$ is a Riesz operator and hyponormal then $T$ is compact.

**Proof**: Since $T$ is a Riesz operator, it is of the form (*). $T$ being hyponormal, $\hat{T}$ is hyponormal and hence the result follows from the theorem.
COROLLARY 4.6: If $T$ is a Riesz operator and quasi-hyponormal (operator of class $(M)$), then $T$ is compact.

PROOF: Since the proof is exactly parallel to that of corollary 4.5, we omit it.

In [1], T. Andô has proved that every completely continuous (compact) hyponormal operator is necessarily normal. This result has been generalized for operators of class $(N)$ in [35]. The following theorem is a slight generalization of it.

THEOREM 4.7: Let $T$ be a restriction - normaloid operator such that $T^{*p_1 q_1} \ldots T^{*p_m q_m}$ is compact for some non-negative integers $p_1, q_1, \ldots, p_m, q_m$. Then $T$ is normal.

PROOF: To simplify the notations, we shall treat the case where $T^{*p q}$ is compact for some non-negative integers $p$ and $q$. Since $T$ is normaloid, there exists $z \in \sigma(T)$ such that $|z| = ||T||$. By [54, Lemma 4.4], $z \in \nu_o(T)$ and $N_T(z) = N_T^*(z)$. The family $F = \left\{ N_T(z) \cap N_T^*(z) : z \in \nu_o(T) \right\}$ is non-void. Let

$$H_0 = \Sigma + \left\{ N_T(z) \cap N_T^*(z) \right\}$$
where \( z \in \sigma(T) \). Then \( H_0 \) reduces \( T \) and \( T_1 = T/H_0 \) is normal.

We assert that \( T_2 = T/H_0 \) is normal. In the case contrary to this, \( T_2 \) is a non-zero normaloid operator on \( H_0 \) such that \( T_2^*T_2 \) is a compact operator.

There exists \( u \in \sigma(T_2) \) such that \( |u| = ||T_2|| \). As above, \( u \in \sigma(T_2) \) and \( N_{T_2}(u) = N_{T_2}^*(\overline{u}) \). Therefore, \( N_{T_2}(u) \cap N_{T_2}^*(\overline{u}) \neq \{0\} \), a contradiction.

**Remark:** This theorem is true when \( p = 0 \).

**Corollary 4.7:** Let \( T \) be a restriction-normaloid operator and \( M \) be an invariant subspace of \( T \) such that \( T^p/M \) is compact for some integer \( p \geq 1 \), then \( T/M \) is normal.

**Proof:** From the hypothesis, \( T/M \) is normaloid and \( (T/M)^p = T^p/M \) is compact. Hence \( T/M \) is normal by this theorem.

3. **Remark on a paper of T. Saitô**

In [55], T. Saitô has proved the following theorem.

**Theorem P:** If \( T \) is an isometry such that its adjoint \( T^* \) is paranormal, then \( T \) is unitary.
At first, we give a simple proof of this theorem and then derive it as a corollary from two different generalizations.

**THEOREM 4.8**: If $T$ is an isometry and $T^*$ is an operator of class $(N)$, then $T$ is unitary.

**PROOF**: Since $T^*$ is an operator of class $(N)$ and $T^*T = I$, for $x \in H$,

$$
\| x \|^2 = \| T^*Tx \|^2 \leq \| T^{2k}x \| \cdot \| Tx \| = \| T^*x \| \cdot \| x \| \leq \| x \|^2
$$

i.e. $\| Tx \| = \| x \| = \| T^*x \|$ for all $x$ in $H$. Hence $T$ is unitary.

In Theorem P, $T^*$ is a left inverse of $T$. Hence in order to prove that $T$ is unitary, we shall prove that $T$ is invertible. In this direction we prove the following theorem.

**THEOREM 4.9**: Let $T$ be a left invertible operator with left inverse $T^l$. If $T^l$ is an operator of class $(N, k)$, then $T$ is invertible.

**PROOF**: Since $T^l$ is an operator of class $(N, k)$ for $x \in H$,

$$
\| x \|^k = \| T^lTx \|^k \leq \| T^kT^lTx \| \cdot \| Tx \|^k = \| T^lTx \| \cdot \| Tx \|^k
$$
Thus $T^x x = \theta$ implies $x = \theta$, i.e. $T^x$ is one-one.

Now, $T^x$ being onto, it is invertible. Hence $T$ is invertible.

REMARK: If $T$ is an isometry then $T^* T = I$. Moreover, if $T^*$ is paranormal, then $T^*$ is invertible by this theorem. $T$ being isometry, $T$ is unitary which is Theorem P.

COROLLARY 4.8: Let $T$ be a left invertible operator with left inverse $T^x$. If $T^x$ is paranormal (hyponormal, normal, self-adjoint) then (i) $T$ is invertible and (ii) $T^{-1}$ is paranormal (hyponormal, normal, self-adjoint).

PROOF: As the proof is simple, we omit it.

COROLLARY 4.9: If $T$ is similar to an isometry, $T$ has a left inverse $T^x$ and $T^x$ is an operator of class $(N)$, then $T$ is unitary.

PROOF: Here $\sigma(T)$ lies on the unit circle or $\sigma(T)$ is the unit disc. By Corollary 4.8, $T$ is an invertible operator of class $(N)$. Hence $\sigma(T)$ lies on the unit circle. Now, $T$ is unitary by [35].

We also derive Theorem P as a corollary from Corollary 4.3. In Theorem P, if $S = T^*$, then $S$ is both, partial isometry and paranormal. Also, $N(S^*) = N(T) = \{\theta\}$. Now $S$ is unitary by Corollary 4.3. Hence $T$ is unitary, as desired.