CHAPTER II
CONDITIONS IMPLYING NORMALITY

In this Chapter we study conditions under which a quasi-hyponormal operators turns out to be normal. C.R. Putnam has proved that if the spectrum of a hyponormal operator has a plane measure zero, it is normal. We establish this result for quasi-hyponormal operators in Section 1. The proof is completely different from that of Putnam. In Section 2, we study conditions implying normality of quasi-hyponormal operators. We obtain extensions of the results proved by Berberian and Patel in Section 3.

1. PUTNAM'S RESULT FOR QUASI-HYPONORMAL OPERATORS

Gilfeather has proved that if $T$ is seminormal and $\sigma(T)$ is countable, it is normal (hence diagonal). More generally, C.R. Putnam has shown that a seminormal operator whose spectrum has a plane measure zero, is normal. Here we extend this result for quasi-hyponormal operators. We begin by establishing several lemmas.
LEMMA 2.1: If T is quasi-hyponormal, then 
\[ \sigma(T) \subseteq \sigma(PTP) \cup \{0\} \]
where P is the orthogonal projection on \( R(T) \).

PROOF: Since T is quasi-hyponormal, by Theorem C,
\[ \pi(T) \subseteq \pi(T^*) \cup \{0\} \]. Therefore, it suffices to exhibit that
\[ \pi(T^*) \subseteq \sigma(PTP) \cup \{0\} \]. Let \( z \neq 0 \) be a complex number in \( \pi(T^*) \). Then, there exists a sequence of unit vectors \( x_n \)
such that \( \| (T^*-z)x_n \| \to 0 \). Since \( H = R(T) \oplus N(T^*) \),
\[ x_n = u_n + v_n \], where \( u_n \in R(T) \) and \( v_n \in N(T^*) \). Then
\[
\| PT^*Pz u_n \| = \| PT^*u_n - z P u_n \|
= \| PT^*x_n - z P x_n \|
\leq \| T^*x_n - z x_n \| \to 0.
\]
Here, \( \| u_n \| \geq M \) for some \( M > 0 \) (otherwise \( u_n \to 0 \)
and hence \( |z| = \lim \| z x_n \| = \lim \| T^*x_n \| = \lim \| T^*u_n \| = 0 \),
an absurdity). In view of this fact,
\[
\| (PT^*Pz u_n) / \| u_n \| \| \leq \frac{1}{M} \| (PT^*Pz) u_n \| \to 0.
\]
This implies that \( z \in \sigma(PTP) \). Thus \( \pi(T^*) \subseteq \sigma(PTP) \cup \{0\} \),
as desired.
LEMMA 2.2: If \( T \) is quasi-hyponormal, then
\[
\nu_\infty(PTP) \subseteq \nu_\infty(T).
\]

PROOF: Let \( z \in \nu_\infty(PTP) \). Then \( z \) is isolated in \( \sigma(PTP) \) and \( 0 < \dim N(PTP-zI) < \infty \). Since \( Tx = zx \iff Tx = zx \) and \( x \in \overline{R(T)} \iff PTPx = zx \), we have \( N(T-zI) = N(PTP-zI) \). Thus \( 0 < \dim N(T-zI) < \infty \) and \( z \in \sigma(T) \). As \( z \) is isolated in \( \sigma(PTP) \), by Lemma 2.1, it is also isolated in \( \sigma(T) \). Consequently \( z \in \nu_\infty(T) \). Hence the lemma.

LEMMA 2.3: If \( T \) is quasi-hyponormal, then
\[
w(PTP) \subseteq w(T) \cup \{0\}.
\]

PROOF: Let \( z \neq 0 \) be not in \( w(T) \). Then (i) \( R(T-zI) \) is closed and (ii) \( 0 < \dim N(T-zI) = \dim N(T^*zI) < \infty \). We show that \( z \notin w(PTP) \). For that, we first prove that \( R(PTP-zI) \) is closed. Consider the sequence of vectors \( y_n \) in \( R(PTP-zI) \) such that \( y_n \to y \), for some \( y \) in \( H \). Now, \( y_n \in R(PTP-zI) \) implies \( y_n = (PTP-zI)x_n = (T-zI)x_n \), where \( x_n \in \overline{R(T)} \) and so \( y_n \in R(T-zI) \). Since \( R(T-zI) \) is closed, \( y \in R(T-zI) \). i.e. \( y = (T-zI)x \) for some \( x \) in \( H \). Moreover, \( Py = y \) and thus \( y \in \overline{R(T)} \) as \( Py_n = P(T-zI)x_n = (T-zI)x_n = y_n \). Therefore, \( x = z^{-1}(Tx-y) \in \overline{R(T)} \). This gives us \( y = (T-zI)x = (PTP-zI)x \) or \( y \in R(PTP-zI) \) and hence \( R(PTP-zI) \) is closed.
Next we assert that $0 < \dim N(PTP_{-zI}) = \dim N(PTP_{-zI}) < \infty$. As $T$ is quasi-hyponormal and $z \neq 0$, $N(T_{-zI}) \subseteq N(T^*_{-\bar{z}})$. This, together with (ii), yields

(a) $N(T_{-zI}) = N(T^*_{-\bar{z}})$ and

(b) $N(T_{-zI}) = N(PTP_{-zI})$, as seen in the proof of Lemma 2.2.

We now establish that $N(PTP_{-zI}) \subseteq N(T^*_{-\bar{z}})$. Let $x \in \overline{R(T)}$ such that $(PTP_{-zI})x = 0$. Then $P(T^*_{-\bar{z}})x = 0$, which is equivalent to $(T^*_{-\bar{z}})x \in N(T^*)$. Clearly, then $T^*x \in N(T^*_{-\bar{z}})$. Since $N(T^*_{-\bar{z}}) = N(T_{-zI})$, $N(T_{-zI}) = N(T^*x)$.

In consequence, $\|T^*x\|^2 = z(T^*x, x) \leq |z| \|T^*x\| \|x\|$. As $x \in \overline{R(T)}$ and $T^*x \neq 0$, the last inequality reduces to $\|T^*x\| \leq |z| \|x\|$. Now, using this relation we get

$$\|T^*x-zx\|^2 = \|T^*x\|^2 + |z|^2 \cdot \|x\|^2 - 2 \text{Re}(T^*x, \bar{zx})$$

$$= \|T^*x\|^2 + |z|^2 \cdot \|x\|^2 - 2 \text{Re}(PT^*Px, \bar{zx})$$

$$= \|T^*x\|^2 - |z|^2 \cdot \|x\|^2$$

$$\leq 0$$

i.e. $T^*x = \bar{zx}$. This proves $N(PT^*P_{-\bar{z}}) \subseteq N(T^*_{-\bar{z}})$. By the hyponormality of $PTP$, $N(PTP_{-zI}) \subseteq N(PT^*P_{-\bar{z}})$. This together with (a) and (b) yields $N(T^*_{-zI}) = N(PT^*P_{-\bar{z}})$. This together with (a) and (b) yields $N(T^*_{-zI}) = N(PT^*P_{-\bar{z}})$.
Thus we have established that \( R(PTP - zI) \) is closed and \( 0 < \dim N(PT^*P - zI) = \dim N(PTP - zI) < \infty \), i.e. \( z \not\in \sigma(P^{*}P) \). This completes the proof of lemma.

It is known that Weyl's theorem holds for any hyponormal operator [16], indeed, for any seminormal operator [6, Example 6]. Moreover, if Weyl's theorem holds for \( T \), then \( \sigma(T) = \sigma_r(T) \cup \sigma_{\infty}(T) \) [8]. We establish this result for quasi-hyponormal operators and derive several corollaries implying normality. For the proofs of these results, we shall use quite often the following result of C.R. Putnam [48, Theorem 1].

**Theorem 2.1:** If \( T \) is quasi-hyponormal, then
\[
\sigma(T) = \sigma_r(P^{*}P, T, T) \cup \sigma_{\infty}(T) .
\]

**Proof:** If \( 0 \not\in \sigma(T) \), then \( T \) turns to be hyponormal and in this case, the result is true. Assume then \( 0 \not\in \sigma(T) \). Since \( P^{*}P \) is hyponormal, \( \sigma(P^{*}P) = \sigma_r(P^{*}P, F) \cup \sigma_{\infty}(P^{*}P, F) \) [48, Theorem 2.1]. Applying Lemmas 2.1, 2.2 and 2.3, we get
\[
\sigma(T) \subset \sigma_r(P^{*}P, T, T) \cup \{ 0 \} = \sigma_r(T) \cup \sigma_{\infty}(T) \cup \{ 0 \} \subset \sigma(T) .
\]

i.e. \( \sigma(T) = \sigma_r(T) \cup \sigma_{\infty}(T) \cup \{ 0 \} \).
To complete the proof, we should assert that
0 ∈ \( w(T) \cup \pi_\infty(T) \). In case contrary to this, 0 must be a
non-isoloid eigenvalue of finite multiplicity. Since
\( \sigma(T) = w(T) \cup \pi_\infty(T) \cup \{ 0 \} \) and \( w(T) \) is closed, 0 must be
a limit point of \( \pi_\infty(T) \). But \( \pi_\infty(T) \subseteq \partial \sigma(T) \) and
\( \partial \sigma(T) \subseteq \hat{\sigma}(T) \cup \pi_\infty(T) \). The closeness of \( \partial \sigma(T) \) implies
0 ∈ \( \hat{\sigma}(T) \cup \pi_\infty(T) \). As 0 ∉ \( \pi_\infty(T) \), 0 ∉ \( \hat{\sigma}(T) \). But
\( \hat{\sigma}(T) \subseteq w(T) \). Thus 0 ∈ \( w(T) \), which is contradictory to
our assumption. Hence the theorem.

We now establish the theorem of Putnam for quasi-
hyponormal operators.

**Theorem 2.2** : If \( T \) is quasi-hyponormal and \( \sigma(T) \)
has planer measure zero, then \( T \) is normal.

**Proof** : As seen in the proof of Theorem 2.1,
\( \sigma(T) = \sigma(PTP) \cup \{ 0 \} \). Therefore, under the hypothesis,
\( \sigma(PTP) \) has planer measure zero. Since \( PTP \) is hyponormal, it
is normal \( \subseteq \). We now claim that \( \overline{R(T)} \) reduces \( T \), whence
\( T \) is normal. For this, it is enough to show that \( T^*x \in \overline{R(T)} \)
whenever \( x \in \overline{R(T)} \). By the quasi-hyponormality of \( T \),
\( || T^*x || \leq || Tx || \). Since \( PTP \) is normal,
\( || Tx || = || PTPx || = || PT^*Px || = || PT^*x || \leq || T^*x || \). Thus
\( || Tx || = || T^*x || \).
For \( y \in \overline{R(T)} \), \((T^*x, y) = (T^*Px, Py) = (PT^*Px, y)\) and hence \( T^*x - PT^*Px \in \overline{R(T)} \). This in turn implies

\[
||T^*x||^2 = ||T^*x - PT^*Px||^2 + ||PT^*Px||^2
\]

\[
= ||T^*x - PT^*Px||^2 + ||PTP||^2
\]

\[
= ||T^*x - PT^*Px||^2 + ||Tx||^2
\]

Since \( ||T^*x|| = ||Tx|| \), we get \( T^*x = PT^*Px \).

Clearly, then \( T^*x \in \overline{R(T)} \).

An immediate consequences of this theorem are the following:

**COROLLARY 2.1:** If \( T \) is quasi-hyponormal and \( \sigma(T) \) is countable, then \( T \) is a diagonal normal operator.

**PROOF:** Since \( \sigma(T) \) is countable, its planer measure is zero. Hence \( T \) is normal by Theorem 2.2. From [8, Theorem 1], it follows that \( T \) is diagonal.

**REMARKS:**

(i) From Theorem 2.1, if \( T \) is quasi-hyponormal then \( \sigma(T) \) is countable if and only if \( w(T) \) is countable. Hence Corollary 2.1 holds, if \( w(T) \) is countable. Particularly, when \( w(T) \) is finite, \( T \) is a diagonal normal and polynomially compact operator [10, Theorem 6.4].
(ii) Since $w(T) \subseteq \sigma(\hat{T})$, $w(T)$ is countable whenever $\sigma(\hat{T})$ is countable. Therefore, Corollary 2.1 holds also when $\sigma^{-1}(\hat{T})$ is countable.

COROLLARY 2.2: If $T$ is quasi-hyponormal and all but finite number of points of $\sigma(\hat{T})$ are real, then $T$ is normal.

PROOF: The planer measure of $\sigma(T)$ is zero.

If all points of $\sigma(T)$ are real then we have the following corollary:

COROLLARY 2.3: If $T$ is quasi-hyponormal such that $\sigma(T)$ is real, then $T$ is self-adjoint.

PROOF: $T$ is normal by Theorem 2.2. Now, since $\sigma(T)$ is real, $T$ is self-adjoint.

REMARK: Corollary 2.3 extends Theorem 4.1 of [54].

2. CONDITIONS IMPLYING NORMALITY.

An operator $T$ is said to be polynomially compact if there exists a non-zero polynomial $p$ such that $p(T)$ is compact (by infinite dimensionality, $p$ is even non constant). The structure of such operators has been described recently in [27]. This structure is useful in showing symptoms of normality. We prove following theorems.
THEOREM 2.3 [46]: If \( T \) is quasi-hyponormal and \( \alpha T + \beta T^* \) (\( \alpha \) or \( \beta \) is non-zero) is polynomially compact, then \( T \) is normal.

PROOF: Let \( z \neq 0 \) be in \( \sigma_h^*(T) \). Then there exists a sequence of unit vectors \( x_n \) such that \( x_n \to 0 \) weakly in \( H \) and \( ||(T-zI)x_n|| \to 0 \) \([19]\). By Theorem C, \( ||(T^*-zI)x_n|| \to 0 \). Clearly, then \( \alpha z + \beta \bar{z} \in \sigma_h^*(\alpha T + \beta T^*) \). Now \( \alpha T + \beta T^* \) being polynomially compact, \( \sigma_h^*(\alpha T + \beta T^*) \) and hence \( \sigma_h^*(\alpha T + \beta T^*) \) is finite. Thus, the set \( Z = \{ \alpha z + \beta \bar{z} : z \neq 0 \text{ and } z \in \sigma_h^*(T) \} \) is finite. If \( |\alpha| \neq |\beta| \), then \( \alpha z_1 + \beta \bar{z}_1 + \alpha z_2 + \beta \bar{z}_2 \), whenever \( z_1 \neq z_2 \). Therefore, in this case the finiteness of \( Z \) will imply that \( \sigma_h^*(T) \) is finite. Since \( \sigma(T) \subseteq \sigma_h^*(T) \cup \pi_\infty(T) \) and \( \pi_\infty(T) \) is countable, \( \sigma(T) \) and hence \( \sigma_h^*(T) \) is countable. Thus, \( T \) is normal by Corollary 2.1.

Next, consider the case \( |\alpha| = |\beta| \). Then, for some \( \theta \),

\[
\beta = e^{i\theta} \alpha \quad \text{and} \quad \alpha T + \beta T^* = \alpha e^{\frac{i\theta}{2}} (e^{-\frac{i\theta}{2}} T + e^{\frac{i\theta}{2}} T^*)
\]

\[
= \alpha e^{\frac{i\theta}{2}} (S + S^*) \quad \text{where} \quad S = e^{\frac{i\theta}{2}} T .
\]

Since \( \sigma_h^*(\alpha T + \beta T^*) \) is finite, so is \( \sigma_h^*(S + S^*) \). This shows that \( \sigma_h^*(S) \) lies on a finite number of vertical lines. As
\[ \partial \sigma(S) \subseteq \sigma_k(S) \cup \pi_\infty(S) , \quad \sigma(S) = \partial \sigma(S) = \sigma_k(S) \cup \pi_\infty(S) , \]
which has planer measure zero. Since \( S \) is quasi-hyponormal, Theorem 2.2 assures us that \( S \) and hence \( T \) is normal.

**Theorem 2.4**  
If \( T = H + iJ \) is quasi-hyponormal and \( HJ \) is polynomially compact, then \( T \) is normal.

**Proof:** Let \( z \neq 0 \) be in \( \sigma_k(T) \). Then \( \text{Re}(z), \text{Im}(z) \in \sigma_k(HJ) \). Since \( HJ \) is polynomially compact, \( \sigma_k(HJ) \) is finite. Consequently, \( \sigma_k(T) \) lies on a finite number of rectangular hyperbolas. This implies that \( \sigma(T) = \sigma_k(T) \cup \pi_\infty(T) \) and hence \( \sigma(T) \) has planer measure zero. Now, normality of \( T \), follows from Theorem 2.2.

For any compact operator \( T \), we know that \( w(T) = \{ 0 \} \) \cite{10, Example 2.12}. The converse is false (consider any non-compact quasinilpotent operator). However, for quasi-hyponormal operator, converse holds. We prove the following theorem:

**Theorem 2.5**  
If \( T \) is quasi-hyponormal and \( w(T) = \{ 0 \} \), then \( T \) is normal and compact.

**Proof:** By Theorem 2.1, \( \sigma(T) = w(T) \cup \pi_\infty(T) = \{ 0 \} \cup \pi_\infty(T) \).

Now, normality of \( T \) follows from Theorem 2.2 and compactness from Theorem 6.4 of \cite{10}.
COROLLARY 2.4: If $T$ is quasi-hyponormal and $T = K + N$ where $K$ is compact and $\sigma^-(N) = \{0\}$, then $T$ is normal and compact.

PROOF: Since $w(T) = w(K+N) = w(N) \subseteq \sigma^-(N) = \{0\}$, $T$ is normal and compact by Theorem 2.5.

COROLLARY 2.5: If $T$ is quasi-hyponormal and $\sigma^-(T) = \{0\}$, then $T$ is normal and compact.

PROOF: $\emptyset \subseteq \sigma^-(T) = \{0\}$ implies $w(T) = \{0\}$ and hence the conclusion follows by Theorem 2.5.

REMARK: As the eigenspaces are not reducing for quasi-hyponormal operators, Theorem 2.5 and Corollaries 2.4, 2.5 do not follow directly from the results proved by Berberian in [9, 10]. Also, Theorem 2.5 does not follow from [18, Theorem 2.2], as quasi-hyponormal operators are not convexoids.

3. EXTENSIONS OF SOME RESULTS.

Berberian [9] has proved the following result for seminormal operators:

**THEOREM H:** If $T$ is a seminormal operator such that $T^p = S T^p S^{-1} + C$, where $p$ is a positive integer, $C$ is compact and $0 \not\in \overline{w(S)}$, then $T$ is normal.
We extend this result. The proof is parallel.

THEOREM 2.6: If $T$ is a quasi-hyponormal operator such that $T^p = ST^*S^{-1} + C$, where $p$ is a positive integer, $C$ is compact and $0 \not\in \overline{W(S)}$, then $T$ is normal.

PROOF: From the proof of Theorem 1, $w(T)$ has zero area, i.e. $\text{meas } w(T) = 0$. From Theorem 2.1, $\text{meas } \sigma(T) = 0$ and hence by Theorem 2.2, $T$ is normal.

I. Istrătescu [33, Lemma 1] has shown that if a hyponormal operator is the sum of a self-adjoint operator and a compact operator, then it is normal. To generalize this result we prove the following theorem:

THEOREM 2.7: Let $T$ be a quasi-hyponormal operator such that $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$. If $T = A + K$, where $A$ is an operator with real spectrum and $K$ is a compact operator, then $T$ is normal.

PROOF: Let us decompose $T = T_1 \oplus T_2$ as in [10, Proposition 4.1]. We assert that $T_2$ is self-adjoint, whence $T$ is normal. Let $z \in \partial \sigma(T_2)$. Now $\partial \sigma(T_2) \subseteq \sigma^\wedge(T_2) \cup \pi_0(T_2)$ [48]. Therefore, $z \in \sigma^\wedge(T_2)$ or $z \in \pi_0(T_2)$. We claim that $z \in \sigma^\wedge(T_2)$. In the case contrary to this, $z \in \pi_0(T_2)$ and hence $\mathcal{N}(T_2 - zI) \subseteq \mathcal{N}(T_2^* - zI)$. Moreover $\mathcal{N}(T_2^* - zI) \subseteq \mathcal{N}(T^* - zI)$. 
Consequently, by \([10, \text{Proposition } 4.1]\), \(z \in \delta\), a contradiction. Thus \(z \in \sigma_{\hat{f}}(T_2) \subseteq \sigma_{\hat{f}}(T) = \sigma_{\hat{f}}(A)\). Hence \(z\) is real and as a result, \(\sigma(T_2)\) is real. Since \(T_2\) is quasi-hyponormal, \(T_2\) is self-adjoint by Corollary 2.3. This completes the proof of the theorem.

V. Istrătesch and I. Istrătescu \([36, \text{Theorem } 3.1]\) have proved the following result:

**Theorem I**: If \(T\) is a seminormal operator such that \(ST^p = S^*S + K\), where \(0 \not\in \mathcal{W}(S)\) and \(K\) is an essentially compact self-adjoint operator, then \(T\) is normal.

S.M. Patel \([43]\) has generalized this result as follows:

**Theorem J**: If \(T\) is a seminormal operator such that \(ST^p = T^qS + K\), \(0 \not\in \mathcal{W}(S)\), then \(T\) is normal.

We establish this result for quasi-hyponormal operator.

**Theorem 2.3**: If \(T\) is a quasi-hyponormal operator such that \(ST^p = T^qS + K\), \(0 \not\in \mathcal{W}(S)\), then \(T\) is normal.

**Proof**: In view of Theorem I, it suffices to prove the result for \(p \neq q\). Let us decompose \(T = T_1 \oplus T_2\) as in \([10]\). Then \(T_1\) is normal. We assert that \(T_2\) is normal, whence \(T = T_1 \oplus T_2\).
is normal. Take $\delta = \pi_o(T) - \{0\}$ [10, Proposition 4.1].

Case (i): Assume $0 \notin \pi_o(T)$. Hence $\pi_o(T^2) = \emptyset$. Let us first suppose that $T_2$ is non-singular. Since $\pi_{oo}(T^2) = \emptyset$, $\partial \sigma(T^2) \subseteq \sigma(T^2)$. If $z \in \partial \sigma(T^2)$ then there exists a sequence of unit vectors $x_n$ in $M^1$ such that $x_n \to 0$ weakly in $M^1$ and $\| (T^2 - zI) x_n \| \to 0$. Clearly, $x_n \to 0$ weakly in $H$ and $\| (T - zI) x_n \| \to 0$. From [41], $\partial \sigma(T^2) \subseteq C$, where $C$ denotes the unit circle. Similarly, as $\pi_o(T^2) = \phi$, $\partial \sigma(T^{-1}) \subseteq C$. Thus $\sigma(T^2) \subseteq C$. Since $T_2$ is quasi-hyponormal, $T_2$ is of class $(N)$. Now, $T_2$ is unitary by [35].

Next, suppose $T_2$ is singular. Then $p$ and $q$ are both non-negative. Since $|p| = p \neq q = |q|$, it follows that $\sigma(T)$ is finite and hence $\sigma(T_2)$ is finite. As $\pi_{oo}(T_2) = \phi$, $\partial \sigma(T_2)$ and $\sigma(T_2)$ is finite. $T_2$ being quasi-hyponormal, $T_2$ is normal by Corollary 2.1.

Case (II): Assume $0 \notin \pi_o(T)$. Then $\pi_o(T^2) = \{0\}$ and hence $T_2$ is singular. Now, as in Case (i), $T_2$ is normal.

This completes the proof of the theorem.