CHAPTER - I

QUASI-HYPERNORMAL OPERATORS

This Chapter and the next Chapter are devoted to the study of quasi-hyponormal operators. Present Chapter consists of three sections. In section 1, we study relations of quasi-hyponormal operators with other classes of operators. Moreover, we study conditions that imply hyponormality. In section 2, we obtain relations related to ascent of an operator. In section 3, we study spectral properties of quasi-hyponormal operators.

1. QUASI-HYPERNORMAL OPERATORS

DEFINITION: An operator $T$ defined on Hilbert space $H$ is said to be quasi-hyponormal if $T^*T^2 \geq (T^*)^2$ or equivalently if $||T^*Tx|| \leq ||TTx||$ for all $x$ in $H$.

Hyponormal operators are quasi-hyponormal and quasi-hyponormal operators are operators of class (N). We have the
following inclusion relations for these classes of operators.

\[ \text{Hyponormals} \subseteq \text{Quasi-hyponormals} \subseteq \text{Class (N)}. \]

Sheth [68] has shown that the inclusion relation on the right is proper. To show the other inclusion relation proper, we give an example of a non-hyponormal quasi-hyponormal operator.

An example: Let \( K \) be the direct sum of a denumerable number of copies of \( H \). For the given positive operators \( A \) and \( B \) defined on \( H \), define the operator \( T_{A,B,n} \) on \( K \) as follows:

\[
T_{A,B,n} (x_1, x_2, \ldots) = (0, A x_1, \ldots, A x_n, B x_{n+1}, \ldots)
\]

The operator \( T_{A,B,n} \) is hyponormal if and only if \( B^2 \geq A^2 \) and quasi-hyponormal if and only if \( AB^2A - A^4 \geq 0 \).

Now, if we take \( H \) to be a two-dimensional Hilbert space with \( A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \), then \( T_{A,B,n} \) is quasi-hyponormal but not hyponormal.

It is well-known that hyponormal operators are normaloids [1, 35, 67, 70] and convexoids [35, 46, 53, 71]. Hence spectraloids [22, 29]. From the recent work of T. Andô [3], Izumi Nshitani and Yasuo Watatani [39] it follows that quasi-hyponormal operators are not convexoids. Moreover, the operator
where $M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $N$ is a normal operator with spectrum as the closed disc $D$ with centre 0 and radius $1/2$. [29, solution of problem 174] is convexoid but not quasi-hyponormal. Thus the class of quasi-hyponormal operators is independent of the class of convexoid operators. This makes the study of quasi-hyponormal operators interesting.

If $T$ is quasi-hyponormal then for $x \in R(T)$,

\[ ||T^*x|| \leq ||Tx||. \]

Using the continuity of $T$ and $T^*$, we get \[ ||T^*x|| \leq ||Tx|| \] for all $x \in \overline{R(T)}$. Thus if $R = T/\overline{R(T)}$, then for $x \in \overline{R(T)}$,

\[ ||R^*x|| \leq ||T^*x|| \leq ||Tx|| = ||Rx|| \]

i.e. $R$ is hyponormal.

Thus the restriction of a quasi-hyponormal operator to the closure of its range is hyponormal. Hence the class of quasi-hyponormal operators seems to be quite close to that of hyponormal operators, and so, it is possible that it may definitely retain some important properties of the class of hyponormal operators. We, now proceed to establish some important properties of quasi-hyponormal operators, enjoyed by hyponormal operators.
Two obvious consequences of the fact mentioned above are:

(i) if $T$ is quasi-hyponormal and invertible then $T$ is hyponormal.

(ii) if $T$ is non-hyponormal quasi-hyponormal then $\overline{R(T)} \neq H$, i.e. $0 \in \sigma_v (T^*)$.

Translate of a quasi-hyponormal operator is not quasi-hyponormal \cite{68}. Regarding the product, only commutativity is not sufficient to ensure quasi-hyponormality of the product of two quasi-hyponormal operators. In \cite{68}, an example of a quasi-hyponormal operator is given whose square is not quasi-hyponormal. However, for the product we prove the following theorems.

**Theorem 1.1 \cite{58}**: If $T$ and $S$ are doubly commutative quasi-hyponormal operators then $TS$ is quasi-hyponormal.

**Proof**: For $x \in H$,

$$|| (TS)^* (TS) x || = || S^* T^* Tx || = || S^* S T^* T x || \leq || S S^* T T^* x || = || T^* T S S x || \leq || T T S S x || = || (TS)^2 x || .$$

i.e. $TS$ is quasi-hyponormal.

**Theorem 1.2 \cite{58}**: If $T$ is quasi-hyponormal and
commutes with an isometric operator \( S \), then \( TS \) is quasi-hyponormal.

**PROOF:** For \( x \in H \),

\[
\|(TS)^*(TS)x\| = \|S^*T^*TSx\| \leq \|T^*TSx\| \leq \|TTSx\| = \|(TS)^2x\|.
\]

i.e. \( TS \) is quasi-hyponormal.

**REMARK:** In Theorem 1.2, if we take \( T \) to be an operator of class \((N)\) (normaloid) then the product comes out to be an operator of class \((N)\) (normaloid) \([58, Theorems 6, 7]\).

Now, we study conditions under which a quasi-hyponormal operator turns out to be hyponormal.

**THEOREM 1.3** \([62]\): If \( T \) is quasi-hyponormal and \( N(T^*) \subseteq N(T) \) then \( T \) is hyponormal.

**PROOF:** Since \( H = N(T^*) \oplus \overline{R(T)} \), for any \( x = u + v \), \( u \in N(T^*) \) and \( v \in \overline{R(T)} \), we have

\[
\|T^*x\| = \|T^*v\| \leq \|Tv\| = \|Tx\|
\]

i.e. \( T \) is hyponormal.

**REMARK:** If \( T \) is quasi-hyponormal and \( N(T) \subseteq N(T^*) \) then \( T \) need not be hyponormal \([15]\).
Following corollaries are immediate.

**COROLLARY 1.1**: If $T$ is hyponormal and $T^*$ is quasi-hyponormal then $T$ is normal.

**PROOF**: Hyponormality of $T$ implies $N(T) \subseteq N(T^*)$.

From theorem 1.3, $T^*$ is hyponormal and hence $T$ is normal.

**COROLLARY 1.2**: If $T$ and $T^*$ are quasi-hyponormal and their null-spaces are comparable then $T$ is normal. (Two sets are comparable if one is a subset of the other).

**PROOF**: Suppose $N(T^*) \subseteq N(T)$. The result follows from Theorem 1.3 and Corollary 1.1.

**COROLLARY 1.3**: If $T$ is quasi-hyponormal and $0$ is an extreme point of $W(T)$ then $T$ is hyponormal.

**PROOF**: If $T$ is hyponormal then there is nothing to prove. If $T$ is non-hyponormal quasi-hyponormal then $0 \in \pi(T^*)$.

Suppose $T^*x = 0$ for $x \neq 0 \in N(T^*)$. For $T = H+iJ$, this implies that $(Hx, x) = 0$. Since $\alpha T$, $\alpha$ scalar, is also quasi-hyponormal, without loss of generality, we can assume that $\Re W(T) \geq 0$. i.e. $H \geq 0$. Together with $(Hx, x) = 0$ we get $Hx = 0$. Since $T^*x = 0$, $Tx = 0$. i.e. $N(T^*) \subseteq N(T)$.

From Theorem 1.3, $T$ is hyponormal.

**REMARK**: In this Corollary, $0$ is an eigenvalue. Thus, if $T$ is quasi-hyponormal and $0$ is an extreme point of $W(T)$ then $0$ is an eigenvalue.
**COROLLARY 1.4:** An idempotent binomial operator is hyponormal.

**PROOF:** \( T \) being binormal \( TT^*T = T^*TT^* \implies TT^*T = T^*TT^* \implies N(T) = N(T^*) \). Now for \( x \in H \),

\[
|TT^*x|^2 = (TT^*x, TT^*x) = (T^*TT^*x, T^*x) = (TT^*Tx, T^*x)
\]

\[
= (TT^*Tx, x) = (T^*TT^*x, x)
\]

i.e. \( |TT^*x|^2 \leq |TT^*x||T^*x|| \), \( ||T^*Tx|| \leq ||TT^*x|| \) \( \cdots (i) \)

Also

\[
||T^*Tx||^2 = (T^*Tx, T^*Tx) = (TT^*Tx, Tx) = (T^*TT^*x, Tx)
\]

\[
= (T^*TT^*x, x)
\]

i.e. \( ||T^*Tx|| \leq ||TT^*x|| \). \( ||T^*Tx|| \leq ||TT^*x|| \) \( \cdots (i) \). (by (i)).

Thus \( T \) is quasi-hyponormal and hence hyponormal by the above theorem.

**REMARKS:** (i) From \([24, 26]\), in this corollary \( T \) turns out to be a projection. (ii) Theorem-5 of \([24]\) does not hold for a binormal operator \( T \). For \( T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \) is binormal and \( T^k = T \) for \( k = 3, 5, 7, \ldots \), but \( T \) is not normal.
2. RESULTS RELATED TO ASCENT OF AN OPERATOR.

For any operator $T$, we have $N(T) \subseteq N(T^2) \subseteq \ldots$. Also $N(T) = N(T^*)$ and $N(T^*) = N(TT^*) \ [74]$. The ascent of an operator $T$ is defined to be the smallest positive integer $n$ for which $N(T^n) = N(T^{n+k})$, for all positive integers $k$. Thus, $T$ is of ascent 0 or 1 if and only if $N(T) = N(T^n)$, for all positive integers $n \geq 1 \ [51, 74]$. It is wellknown that every operator of class $(N, k)$ and hence, quasi-hyponormal operator is of ascent 0 or 1. Here, we obtain some results related to ascent of an operator. At first, we prove the following Lemma.

**Lemma 1.1** [62]: If the ascent of $T$ is 0 or 1, then $N(T) = N(T^*)$ if and only if $N(T^n) = N(T^*_n)$.

**Proof**: (i) $N(T) = N(T^*) \Rightarrow N(T^n) = N(T^*_n)$: Assume that $N(T^k) = N(T^*_k)$ for some positive integer $k$. Then $N(T^{k+1}) = N(T) = N(T^*) \subseteq N(T^*_k)$. To prove the reverse inclusion, let $x \in N(T^*_k+1)$. Then $T^*_x \in N(T^*_k) = N(T^k) = N(T)$. i.e. $x \in N(T^*_k) = N(T^*_k) = N(T) = N(T^{k+1})$. Thus $N(T^*_k+1) \subseteq N(T^*+1)$. This clears (i).

(ii) $N(T^n) = N(T^*_n) \Rightarrow N(T) = N(T^*)$: We have $N(T^n) = N(T^*_n)$ for $n \geq 2$. For $n = 1$, the conclusion
follows immediately. Now for \( n \geq 2 \), \( N(T^n) \subseteq N(T^n) \) \( = N(T^n) = N(T) \) i.e. \( N(T^n) \subseteq N(T) \). Also if \( x \in N(T) \), then \( x \in N(T^n) = N(T^n) = N(T^n) \) i.e. \( T^n x \in N(T^n) = N(T^n) \) \( = N(T^n) = N(T) \) i.e. \( x \in N(T^n) = N(T^n) \). Thus \( N(T) \subseteq N(T^n) \).

Hence \( N(T) = N(T^n) \).

REMARK : If the ascent of \( T \) is 0 or 1 and \( N(T^n) = N(T^n) \), it follows from the proof of (ii) that \( N(T^n) = N(T^n) \) for all positive integers \( n \).

We derive the following corollary:

COROLLARY 1.5 : If the ascent of \( T \) is 0 or 1 and \( T^n \) is normal for some positive integer \( k \), then \( N(T^n) = N(T^n) \) for all positive integers \( n \).

J. G. Stampflil [70] has proved the following theorem:

THEOREM A : If \( T \) is hyponormal and \( T^n \) is normal for some positive integer \( n \), then \( T \) is normal.

In [2], T. Andô has shown that the above theorem is also true for operators of class \( (N) \). He has also proved the following theorem:

THEOREM B : If \( T \) and \( T^* \) are operators of class \( (N) \) and \( N(T) = N(T^*) \), then \( T \) is normal.
Using Lemma 1.1, we prove Theorem A under a weaker condition on $T^n$. In Theorem B, we consider $T^{*n}$ to be an operator of class (N) instead of $T^*$, and prove the theorem under the condition that their null-spaces are comparable.

**Theorem 1.4**: If $T$ is hyponormal and $T^{*n}$ is quasi-hyponormal for some $n \geq 1$, then $T$ is normal.

**Proof**: Since $T$ is hyponormal, $N(T^n) = N(T) \subseteq N(T^*) \subseteq N(T^{*n})$ i.e. $N(T^n) \subseteq N(T^{*n})$. Now $T^{*n}$ being quasi-hyponormal, by Theorem 1.3 $T^{*n}$ is hyponormal. Hence $N(T^{*n}) \subseteq N(T^n)$. Consequently $N(T^n) = N(T^{*n})$.

Now $T^n$ and $T^{*n}$ are operators of class (N) and $N(T^n) = N(T^{*n})$. Therefore $T^n$ is normal by Theorem B. Now the conclusion follows by Theorem A.

**Theorem 1.5**: If $T$ and $T^{*n}$ are operators of class (N) and their null-spaces are comparable, then $T$ is normal.

**Proof**: Suppose that $N(T) \subseteq N(T^{*n})$. Let $H = N(T^n) \oplus R(T^{*n})$ and $x = u + v$, $u \in N(T^n)$, $v \in R(T^{*n})$ and $\|x\| = 1$. Then
Hence $N(T^*) \subseteq N(T)$. Since $N(T) = N(T^n)$, we have $N(T^n) = N(T^*)$. Now $T$ is normal as in Theorem 1.4.

**Remark** : Corollaries 1.1 and 1.2 are immediate consequences of Theorems 1.4 and 1.5.

### 3. Spectral Properties

In this section we take up the study of spectral properties of quasi-hyponormal operators. For hyponormal operators, eigenspaces are reducing and mutually orthogonal. Further, every isolated point of its spectrum is an eigenvalue. These properties are widely used in discussing conditions implying normality of hyponormal operators. Naturally we are led to examine whether these properties are enjoyed by quasi-hyponormal operators.

Firstly we collect the following basic spectral properties of quasi-hyponormal operators [68]

**Theorem C** : Let $T$ be a quasi-hyponormal operator.
(i) If $H_0$ is invariant under $T$, then $T/H_0$ is quasi-hyponormal.

(ii) If $z \not\in 0 \in \pi_0(T)$ then $\bar{z} \in \pi_0(T^*)$ and $N_T(z) \subseteq N_{T^*}(\bar{z})$.

(iii) If $z_1 \not\in \pi_0(T)$ then $N_T(z_1) \subseteq N_T(z_2)$.

(iv) If $z \not\in 0 \in \pi(T)$ and $x_n \rightarrow 0$ for a sequence of unit vectors $x_n$, then $\bar{z} \in \pi(T^*)$ and $T^*x_n - \bar{z}x_n \rightarrow 0$.

(v) If $\dim H < \infty$, then $T$ is normal.

(vi) If $H$ is spanned by its eigenvectors then $T$ is normal.

(vii) $r(T^*) = \emptyset$


Remarks: (a) In (ii) we take $z \not\in 0$. For $z = 0$, it is not true. i.e. For a quasi-hyponormal operator $N(T)$ need not be reducing. In the example, considered at beginning of this Chapter, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$, then $T_{A,B,n}$ is the non-hyponormal quasi-hyponormal operator. Let $\bar{x} = \{x_n\}_{-\infty}^{\infty}$ where $x_n = 0$ if $n \not\in 1$ and $x_1 = (-4, 2)$. Since $y_n = Ax_{n-1}$ for $n \leq 1$ and $y_n = Bx_{n-1}$ for $n \geq 2$, we have $Tx = \emptyset$ but
(b) $T$ may be quasi-hyponormal with reducing subspace without being hyponormal $[15]$. If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

then $T_{A,B,n}$ is non-hyponormal, quasi-hyponormal. Since $N(A) \neq \{0\}$ and $N(B) = \{0\}$, we have $N(T) \neq \{0\}$ and $N(T) \subseteq N(T^*)$.

For an operator satisfying condition $G_1$, J.G. Stampfli $[71]$ has shown that every isolated point of its spectrum is an eigenvalue. S.K. Berberian has generalized this result for restriction convexoid operators $[12]$. As the class of quasi-hyponormal operators is independent that of convexoid operators, it is natural to inquire whether the above result holds for quasi-hyponormal operators. Here also the answer is in affirmative. i.e. Quasi-hyponormal operators retain this property as the following theorem shows.

**Theorem 1.6** $[62]$ : If $T$ is quasi-hyponormal and $z$ is an isolated point of $\sigma(T)$, then $z$ is an eigenvalue. Further if $z \neq 0$, then $z$ is a normal eigenvalue.

**Proof** : Choose a positive number $\epsilon$ sufficiently small such that $z$ is the only point of $\sigma(T)$ contained in or on the circle $|z - z'| = \epsilon$. If
\[ E = \frac{1}{2\pi i} \oint_{|z-z|=r} (T - zI)^{-1} \, dz \]

then \( E \) is a non-zero projection. Since it commutes with \( T \), \( E(H) \) is invariant under \( T \). Hence \( T/E(H) \) is quasi-hyponormal and \( \sigma(T/E(H)) = \sigma(T) \cap \left\{ |z - z| \leq r \right\} = \left\{ z \right\} \).

Now the following two cases arise:

(i) \( z = 0 \). Since \( T/E(H) \) is normaloid, \( T/E(H) = 0 \)
i.e. for all \( x \in E(H) \), \( Tx = 0 \). Hence \( 0 \in \pi_0(T) \).

(ii) For \( z \neq 0 \), \( T/E(H) \) is invertible and hence hyponormal. Now \( T/E(H) - zI/E(H) \) is also hyponormal and \( T/E(H) - zI/E(H) = 0 \). This gives us \( T/E(H) = zI/E(H) \), or \( Tx = zx \) for all \( x \in E(H) \).
i.e. \( z \in \pi_0(T) \). From Theorem C, \( z \neq 0 \) is a normal eigenvalue.

Following corollaries are trivial.

**COROLLARY 1.6**: If \( T \) is quasi-hyponormal and \( 0 \) is the only limit point of \( \sigma(T) \), then \( T \) is normal.

**COROLLARY 1.7**: A compact quasi-hyponormal operator is normal.

For normal operators, we have the well-known spectral mapping theorem.
\[ \sigma(f(T)) = f(\sigma(T)) = \left\{ f(z, \overline{z}) : z \in \sigma(T) \right\}, \]

for every polynomial \( f(z, \overline{z}) \). Particularly, for \( f(z, \overline{z}) = \frac{1}{2}(z + \overline{z}) = \text{Re} \, z \), the relation

\[(*) \quad \sigma(\text{Re}(T)) = \text{Re} \, \sigma(T)\]

holds for every normal operator. Putnam [46], has proved that the relation (*) holds for seminormal operators and particularly for hyponormal operators. Berberian [11] has proved this spectral mapping relation for certain more general convexoid operators. He has specifically proved the following:

**Theorem D:** If \( T \) satisfies the growth condition \( G_1 \) and \( \sigma(T) \) is connected, then (*) holds.

**Theorem E:** If \( T \) is a convexoid operator such that both \( \sigma(T) \) and \( \sigma(\text{Re} \, T) \) are connected, then (*) holds.

Our next theorem shows that Theorem D is true for operators of class \( R \), a class of operators introduced by Luecke [37]. According to Luecke, an operator \( T \) is of class \( R \) if and only if

\[ ||(T - zI)^{-1}|| = \frac{1}{d(z, \overline{W(T)})} \]

for all \( z \notin \overline{W(T)} \). Luecke [37, Theorem-1] has characterized operators of class \( R \) as follows:
THEOREM 2: An operator $T$ is of class $R$ if and only if $\partial W(T) \subseteq \sigma(T)$.

THEOREM 1.7: If $T \in R$ then (*) holds.

PROOF: Since $T \in R$, $T$ satisfies condition $G_1$. Hence from [11, Lemma 8], $\text{Re } \sigma^-(T) \subseteq \sigma^-(\text{Re } T)$. Suppose $u \in \sigma^-(\text{Re } T)$. If $T = H + iJ$, then $u \in \sigma(H)$. By the faithful $*$-representation $T \to T^0$ of $B(H)$ into $B(H^0)$, we can suppose $\sigma(H) = \pi_o(H^0)$ [5]. Then $u \in \nu_o(H^0)$. Hence there exists $x \in \theta \subseteq H^0$ such that $H^0x = ux$. i.e.

$$(H^0x, x) = u \text{ if } ||x|| = 1.$$ Let $v = (J^0x, x)$. Then

$$(T^0x, x) = (H^0x, x) + i(J^0x, x) = u + iv = z, \text{ say.}$$

Then $z \in W(T^0) = \overline{W(T)}$. Hence $z \in \partial W(T)$ or $z$ is an interior point of $W(T)$. If $z \in \partial W(T)$, then $z \in \sigma^-(T)$ and hence $\text{Re } z = u \in \text{Re } \sigma^-(T)$. If $z$ is an interior point of $W(T)$, then there is $z' \in \partial W(T)$ such that $\text{Re } z = \text{Re } z' = u$. Hence $u \in \text{Re } \sigma^-(T)$. This completes the proof.

REMARC: After the completion of our work, we learned that Patel [42] has also obtained this result. The present proof is independent of and different from that of Patel.

In the following theorems we establish the relation (*) for certain quasi-hyponormal operators.
**Theorem 1.8**: If $T$ is a quasi-hyponormal convexoid operator and $\sigma(T)$ is connected, then (*) holds.

**Proof**: Let $z = \alpha + i\beta \in \pi(T)$ and $\{x_n\}$ be a sequence of unit vectors such that $Tx_n - zx_n \to 0$. From Theorem C, 

$$T^*x_n - \overline{zx_n} \to 0.$$ 

If $T = H + iJ$ then $Hx_n - \alpha x_n$

$$= \frac{1}{2}(T + T^*)x_n - \frac{1}{2}(z + \overline{z})x_n \to 0.$$ i.e. $\alpha \in \sigma(H)$.

This proves that $\text{Re}\, \pi(T) \subseteq \sigma(H)$. Now, to prove $\text{Re}\, \sigma(T) \subseteq \sigma(\text{Re}\, T)$, let $z_0 \in \sigma(T)$. The vertical line $\text{Re} \, z = \text{Re} \, z_0$ exits the spectrum at a boundary point $u$ of $\sigma(T)$. Since $\partial \sigma(T) \subseteq \pi(T)$, we have $\text{Re} \, z_0 = \text{Re} \, u \in \sigma(\text{Re} \, T) = \sigma(H)$.

Thus, $\text{Re}\, \sigma(T) \subseteq \sigma(\text{Re} \, T)$. Since $T$ is convexoid and $\sigma(T)$ is connected, the inclusion relation $\sigma(\text{Re} \, T) \subseteq \text{Re} \, \sigma(T)$ follows from [11, Lemma 5]. This completes the proof.

**Theorem 1.9**: If $T$ is a quasi-hyponormal convexoid operator and $\sigma(T)$ is convex then (*) holds.

**Proof**: As in the proof of Theorem 1.8, $\text{Re} \, \sigma(T) \subseteq \sigma(\text{Re} \, T)$. To prove the reverse inclusion relation, let $T = H + iJ$ and $u \in \sigma(\text{Re} \, T) = \sigma(H)$. By faithful $*$-representation [5], we can suppose $\sigma(H) = \pi_o(H^0)$. Then $u \in \pi_o(H^0)$. Hence there exists $x \neq 0 \in H^0$ such that $H^0x = ux$. i.e. $(H^0x, x) = u$ if $||x|| = 1$. Let $v = (J^0x, x)$. Then $(T^0x, x) = u + iv = z$. 

\[ \text{THEOREM 1.8} \text{ : If } T \text{ is a quasi-hyponormal convexoid operator and } \sigma(T) \text{ is connected, then (*) holds.} \]

\[ \text{PROOF : Let } z = \alpha + i\beta \in \pi(T) \text{ and } \{x_n\} \text{ be a sequence of unit vectors such that } Tx_n - zx_n \to 0. \quad \text{From Theorem C, } \]

\[ T^*x_n - \overline{zx_n} \to 0. \quad \text{If } T = H + iJ \text{ then } Hx_n - \alpha x_n \]

\[ = \frac{1}{2}(T + T^*)x_n - \frac{1}{2}(z + \overline{z})x_n \to 0. \quad \text{i.e. } \alpha \in \sigma(H). \]

This proves that $\text{Re}\, \pi(T) \subseteq \sigma(H)$. Now, to prove $\text{Re}\, \sigma(T) \subseteq \sigma(\text{Re} \, T)$, let $z_0 \in \sigma(T)$. The vertical line $\text{Re} \, z = \text{Re} \, z_0$ exits the spectrum at a boundary point $u$ of $\sigma(T)$. Since $\partial \sigma(T) \subseteq \pi(T)$, we have $\text{Re} \, z_0 = \text{Re} \, u \in \sigma(\text{Re} \, T) = \sigma(H)$. 

Thus, $\text{Re}\, \sigma(T) \subseteq \sigma(\text{Re} \, T)$. Since $T$ is convexoid and $\sigma(T)$ is connected, the inclusion relation $\sigma(\text{Re} \, T) \subseteq \text{Re} \, \sigma(T)$ follows from [11, Lemma 5]. This completes the proof.

**Theorem 1.9** : If $T$ is a quasi-hyponormal convexoid operator and $\sigma(T)$ is convex then (*) holds.

**Proof** : As in the proof of Theorem 1.8, $\text{Re} \, \sigma(T) \subseteq \sigma(\text{Re} \, T)$. To prove the reverse inclusion relation, let $T = H + iJ$ and $u \in \sigma(\text{Re} \, T) = \sigma(H)$. By faithful $*$-representation [5], we can suppose $\sigma(H) = \pi_o(H^0)$. Then $u \in \pi_o(H^0)$. Hence there exists $x \neq 0 \in H^0$ such that $H^0x = ux$. i.e. $(H^0x, x) = u$ if $||x|| = 1$. Let $v = (J^0x, x)$. Then $(T^0x, x) = u + iv = z$. 

\[ \text{THEOREM 1.8} \text{ : If } T \text{ is a quasi-hyponormal convexoid operator and } \sigma(T) \text{ is connected, then (*) holds.} \]

\[ \text{PROOF : Let } z = \alpha + i\beta \in \pi(T) \text{ and } \{x_n\} \text{ be a sequence of unit vectors such that } Tx_n - zx_n \to 0. \quad \text{From Theorem C, } \]

\[ T^*x_n - \overline{zx_n} \to 0. \quad \text{If } T = H + iJ \text{ then } Hx_n - \alpha x_n \]

\[ = \frac{1}{2}(T + T^*)x_n - \frac{1}{2}(z + \overline{z})x_n \to 0. \quad \text{i.e. } \alpha \in \sigma(H). \]

This proves that $\text{Re}\, \pi(T) \subseteq \sigma(H)$. Now, to prove $\text{Re}\, \sigma(T) \subseteq \sigma(\text{Re} \, T)$, let $z_0 \in \sigma(T)$. The vertical line $\text{Re} \, z = \text{Re} \, z_0$ exits the spectrum at a boundary point $u$ of $\sigma(T)$. Since $\partial \sigma(T) \subseteq \pi(T)$, we have $\text{Re} \, z_0 = \text{Re} \, u \in \sigma(\text{Re} \, T) = \sigma(H)$. 

Thus, $\text{Re}\, \sigma(T) \subseteq \sigma(\text{Re} \, T)$. Since $T$ is convexoid and $\sigma(T)$ is connected, the inclusion relation $\sigma(\text{Re} \, T) \subseteq \text{Re} \, \sigma(T)$ follows from [11, Lemma 5]. This completes the proof.
say. Hence \( z \in \mathcal{W}(T^0) = \overline{\mathcal{W}(T)} = \Sigma^-(T) = \sigma^-(T) \). i.e. \( u \in \Re \sigma(T) \). This completes the proof.

Next, we proceed to consider spectra of polar factors of operators. An operator \( T \) will be said to have polar factorization if \( T = UP \), where \( U \) is unitary and \( P \) is a non-negative self-adjoint operator \([50]\). If \( T = UP \), then \( T^* = PU^* \) and \( T^*T = P^2 \), hence \( P = (T^*T)^{1/2} \).

Here we prove Theorems 1 and 3 of \([50]\) for a quasi-hyponormal operators. The proofs are parallel to that of Theorems 1 and 3.

THEOREM 1.10 : Let \( T \) be quasi-hyponormal and \( z \in \mathcal{O} \sigma^-(T) \). Then \( |z| \in \sigma(TT^\ast)^{1/2} \cap \sigma(T^\ast T)^{1/2} \).

PROOF : Since \( z \in \mathcal{O} \sigma^-(T) \), \( z \in \mathcal{W}(T) \). Hence, there exists a sequence of unit vectors \( x_n \) such that \( T^nx_n - zx_n \to 0 \).

By Theorem C, \( T^nx_n - zx_n \to 0 \). Hence, \( T^nx_n - |z|^2 x_n \to 0 \) and \( TT^nx_n - |z|^2 x_n \to 0 \). Consequently, \( (T^*T)^{1/2} x_n - |z| x_n \to 0 \) and \( (TT^\ast)^{1/2} x_n - |z| x_n \to 0 \). Hence the conclusion follows.

THEOREM 1.11 : Let \( T \) be a quasi-hyponormal operator with a polar factorization \( T = UP \). Suppose \( z \neq 0 \in \mathcal{O} \sigma(T) \) and
z = |z| e^{i\theta}. Then e^{i\theta} \in \sigma(U)

PROOF: Let \( z = r e^{i\theta} \), where \( r = \max \{ |z| : z \in \sigma(T) \} \). Then \( z \in \sigma(T) \) and hence \( z \in \sigma(\pi(T)). \) Therefore, there exists a sequence of unit vectors \( x_n \) such that \( T x_n - z_1 x_n \to 0 \). Hence \( T^* x_n - \overline{z_1} x_n \to 0 \) and consequently, \( (T^* T)^{1/2} x_n - \overline{r} x_n \to 0 \). Then \( T x_n - z_1 x_n = U(T^* T)^{1/2} x_n - \overline{r} x_n \to 0 \). As \( r > 0 \), we get \( U x_n - e^{i\theta} x_n \to 0 \). i.e. \( e^{i\theta} \in \sigma(U) \).