CHAPTER 4

METRIZATION OF QUASI NAGATA SPACES
4.1 : Introduction and Definitions

Developable spaces, Nagata spaces and $\gamma$-spaces are important generalized metric spaces. An ingenious approach to study the generalized metric spaces was introduced by R.N. Heath [36] and pursued by R.E. Hodel [29], [30]. This approach describes a generalized metric property of a space $X$ by a sequence of open cover of $X$.

Quasi-Nagata and quasi-$\gamma$-spaces were introduced by R.W. Martin in [39] and [40]. They are generalizations of $\aleph$-spaces and $\gamma$-spaces respectively of R.E. Hodel [30]. We study these further and derive their metrization theorems.
In [30], H. N. Martin proved that a $T_2$ quasi-Nagata, $\gamma$-space is metrizable. In this chapter it is shown that ' $\gamma$-space' in the above result can be replaced by ' $\mathbb{E}$-space' of [33].

In this chapter, whenever we consider a sequence
\[ \{ g(n, x) : x \in X \} \]
for \( n \leq \infty \) of open covers of a space \( X \), it is assumed to satisfy the following:

1. \( x \in g(n, x) \) for each \( n \in \mathbb{N} \) and \( x \in X \).
2. \( g(n+1, x) \subseteq g(n, x) \) for each \( n \in \mathbb{N} \), \( x \in X \).

The following definitions of developable spaces, $u\Delta$-spaces, Nagata spaces, semi-stratifiable spaces and first countable spaces are not original but they are their characterizations proved in the reference indicated with them.

In view of remark 2.1 of [30] if a phrase ' \( p \) is a cluster point of \( \{ x_n \} \)' appears in the following definitions then we mean that \( \{ x_n \} \) converges to \( p \).
Definition 4.1.1: A space $X$ is called a developable space $\square 30 \square$ (uA -space $\square 30 \square$ ) if there is a sequence 
\[ \{g(n, x) : x \in X \} \] of open covers of $X$ such that 
\[ \forall n \in \mathbb{N} \]
if $\{p, x_n\} \subseteq g(n, y_n)$ for $n = 1, 2, 3, \ldots$, then $p$ is a cluster point of $\{x_n\}$ ( $\{x_n\}$ has a cluster point ).

Definition 4.1.2: A space $X$ is called a $\gamma$-space ($\gamma$-space) if there is a sequence 
\[ \{g(n, x) : x \in X \} \] of open covers of $X$ such that 
\[ \forall n \in \mathbb{N} \]
if $y_n \subseteq g(n, p)$ and $x_n \subseteq g(n, y_n)$ for $n = 1, 2, \ldots$, then $p$ is a cluster point of $\{x_n\}$ ( $\{x_n\}$ has a cluster point ).

Definition 4.1.3: A space $X$ is called a Nagata space $\square 30 \square$ ( all-space $\square 30 \square$ ) if there is a sequence 
\[ \{g(n, x) : x \in X \} \] of open covers of $X$ such that if 
\[ g(n, p) \cap g(n, x_n) \neq \emptyset \] for $n = 1, 2, \ldots$ then $p$ is a cluster point of $\{x_n\}$ ( $\{x_n\}$ has a cluster point ).
Definition: 4.1.4: \( \square \) : A space \( X \) is called a quasi-\( Y \)-space if there is a sequence \( \{ g(n, x) : x \in E \}_{n=1}^{\infty} \) of open covers of \( X \) such that if a sequence \( \{ Y_n \} \) converges and \( x_n \in g(n, Y_n) \) for \( n = 1, 2, \ldots \) then \( \{ x_n \} \) has a cluster point.

Definition: 4.1.5: \( \square \) : A space \( X \) is called quasi-\( Y \)-space if there is a sequence \( \{ g(n, x) : x \in X \}_{n=1}^{\infty} \) of open covers of \( X \) such that if \( \{ Y_n \} \) converges and \( Y_n \in g(n, x_n) \) for \( n = 1, 2, \ldots \) then \( \{ x_n \} \) has a cluster point.

Definition: 4.1.6: A space \( X \) is called a semi-stratifiable space \( \square \) \( ( \mathcal{Y} \)-space \( \square \) ) if there is a sequence \( \{ g(n, x) : x \in X \}_{n=1}^{\infty} \) of open covers of \( X \) such that if \( p \in g(n, x_n) \) for \( n=1, 2, \ldots \) then \( p \) is a cluster point of \( \{ x_n \} \) ( \( \{ x_n \} \) has a cluster point ).
Definition 4.1.7: A space $X$ is called a first countable space $\square 39\square$ ( $\omega$-space $\square 43\square$ ), if there is a sequence $\{g(n, x) : x \in X\}^\infty_{n=1}$ of open covers of $X$ such that if $x_n \in g(n, p)$ for $n = 1, 2, \ldots$, then $p$ is a cluster point of $\{x_n\}$ ($\{x_n\}$ has a cluster point).

Definition 4.1.8: A space $X$ is called a $\Omega$-space $\square 22\square$ if there is a sequence $\{g(n, x) : x \in X\}^\infty_{n=1}$ of open covers of $X$ such that if $\{x_n : p\} \subset g(n, x_n)$ for $n = 1, 2, \ldots$ and $\{y_n\}$ has a cluster point then $p$ is a cluster point of $\{x_n\}$ ($\{x_n\}$ has a cluster point).

Definition 4.1.9: A space $X$ is called a $\omega$-stratifiable space if there is a sequence $\{g(n, x) : x \in X\}^\infty_{n=1}$ of open covers of $X$ such that if $K$ is a compact subset of $X$ and $y \notin K$ then there is an integer $n$ such that $y \notin \bigcup\{g(n, x) : x \in K\}$.
A space $X$ is called \textit{c-Nagata} if it is \textit{c}-stratifiable and first countable.

\textbf{Definition 4.1.10} \cite{32}. A space $X$ is called \textit{c-semistratifiable} if there is a sequence

$$\{g(n, x) : x \in X\}_{n=1}^{\infty}$$

of open covers of $X$ such that

if $X$ is a compact subset of $X$ and $y \notin X$ then there is an integer $n$ such that $y \notin \bigcup\{g(n, x) : x \in X\}$.

\textbf{Definition 4.1.11} \cite{30}. A space $X$ is called a \textit{c-space} if there is a sequence

$$\{g(n, x) : x \in X\}_{n=1}^{\infty}$$

of open covers of $X$ such that

(a) $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}$ for each $x \in X$ and

(b) if $y \in g(n, x)$ then $g(n, y) \subseteq g(n, x)$.

\section*{4.3: Quasi-Nagata spaces, Quasi-$\gamma$-spaces and their metrization theorems}

\textbf{Theorem 4.3.1:} A first countable space $X$ is well

iff it is Quasi-Nagata.
Proof : Necessity is proved in \( \square \).

To prove sufficiency, let \( \{g(n, x) : x \in \mathbb{K}\}_{n=1}^{\infty} \) and
\( \{h(n, x) : x \in X\}_{n=1}^{\infty} \) be quasi-Nagata and first countable sequences for \( X \).

Let \( k(n, x) = g(n, x) \cap h(n, x) \) for each \( n \in \mathbb{N} \)
and \( x \in X \). Then \( \{k(n, x) : x \in X\}_{n=1}^{\infty} \) is both a first countable and a quasi-Nagata sequence for \( X \).

Suppose \( x_n \in k(n, p) \cap k(n, x_n) \) for each \( n \in \mathbb{N} \).
Then \( \{x_n\} \) converges to \( p \) and hence \( \{x_n\} \) has a cluster point.

Theorem : 4.2.3 : If a space \( X \) is first-countable and quasi-\( \gamma \), then it is a \( \nu \gamma \)-space.

Proof : Similar to that of theorem 4.2.1.

Theorem : 4.2.3 : \( \square \). Every quasi-Nagata, \( \gamma \)-space is metrizable.

Proof : Nodel \( \square \) proved that a \( \gamma \)-, \( \omega \)-space is metrizable. Since \( \gamma \)-spaces are first countable, the result follows using theorem 4.2.1.
In [30], K.B. Lee proved that a space $X$ is a Nagata space ($\gamma$-space) iff it is a c-Nagata and cell-space ($w\gamma$-space). In view of this result and theorems 4.2.1 and 4.2.2 the following theorems are obvious.

Theorem 4.2.4: A space $X$ is Nagata iff it is c-Nagata and quasi-Nagata.

Theorem 4.3.3: A space $X$ is a $\gamma$-space iff it is a c-Nagata and a quasi-$\gamma$-space.

K.B. Lee [30] also proved that a regular space $X$ is a $\gamma$-space iff it is a c-stratifiable, $w\gamma$-space.

This result can be generalised to the following:

Theorem 4.3.6: A regular space $X$ is a $\gamma$-space iff it is a c-stratifiable, quasi-$\gamma$-space.

Proof: We note that

(i) every quasi-$\gamma$-space is a $q$-space
(ii) each point is $G_\delta$ in a c-stratifiable space
(iii) Lutzer showed that a regular, c-space in which each point is $G_3$, is first countable $\square 37\square$.

By (i), (ii) and (iii) $X$ is a c-Nagata space. Now apply theorem 4.3.6.

Theorem 4.3.7: If $X$ is a regular, semi-stratifiable, quasi-$\gamma$-space then it is a Moore space.

**Proof:** Using the technique of the proof of theorem 4.2.1 it can be shown that a semi-stratifiable, quasi-$\gamma$-space is a $\omega\alpha$-space. Hodel $\square 30\square$ proved that a regular, semi-stratifiable, $\omega\alpha$-space is developable and hence a Moore space. Hence $X$ is a Moore space.

Theorem 4.3.8: A regular, c-semi-stratifiable, quasi-Nagata, quasi-$\gamma$-space $X$ is metrizable.

**Proof:** $X$ being quasi-Nagata is a $\delta$-space. Hence by theorem 3 of Martin $\square 41\square$ it is semi-stratifiable. By theorem 4.3.7, $X$ is developable. Moreover $X$ is $\omega\beta$. Hence it is metrizable.

**Corollary:** 4.3.9: A regular, c-semi-stratifiable, $\omega\beta$-space $X$ is a Nagata space.
Proof: By the same argument as in the proof of the above theorem, $X$ is semi-stratifiable. Hence it is an $\alpha$-space. Nadler [34] proved that a regular, ultra-$\alpha$-space is always a Nagata space. Hence $X$ is Nagata.

Corollary: 4.2.10: If a space $X$ is quasi-$\gamma$ and Nagata then it is metrizable.

Proof: Since $X$ is first countable it is $\omega r$. R.H. Lee [36] proved that a Nagata, $\omega r$-space is metrizable. Hence $X$ is metrizable.

In [23], Fletcher and Lindgren proved that if a space $X$ is a $\Omega$-space and an $\alpha$-space then it is developable.

Theorem: 4.3.11: If a space $X$ is a $\Omega$-space and a quasi-Nagata space then it is metrizable.

Proof: By the above result of Fletcher and Lindgren, $X$ is developable. Also $X$ is ultra-$\Omega$. Hodel [35] proved that a developable, ultra-$\Omega$-space is metrizable. Hence $X$ is metrizable.

The above theorem generalises a well known theorem of R.H. Hodel [30] which states that a ultra-$\gamma$-space is metrizable.