Chapter 6

Markovian Feedback Queueing Models with Balking, Reneging, and Retention of Reneged Customers

In this chapter, we study finite capacity Markovian feedback queueing models with balking, reneging, and retention of reneged customers. It consists of two sections. Section 6.1 deals with an M/M/1/N feedback queueing model with balking, reneging and retention of reneged customers. The steady-state solution of the model is obtained. Performance measures are derived and special cases of the model are also discussed. In section 6.2, an M/M/c/N feedback queueing model with balking, reneging and retention of reneged customers is described and solved. Some important performance measures and special cases of the model are derived. Numerical illustration is provided to study the effect of probability of retaining a reneging customer on different performance measures. Moreover, the economic analysis of the models is also performed. The work based on this chapter is published in Sharma and Kumar (2012b). The models discussed in this chapter may be useful in modeling various production and service processes involving feedback and impatient customers.
6.1 Single-Server Markovian Feedback Queueing Model with Balking, Reneging, and Retention of Reneged Customers.

6.1.1 Model Construction

The arrival process is Poisson with average arrival rate \( \lambda \). The inter-arrival times are independently, identically and exponentially distributed with parameter \( \lambda \). There is one server and the service time is independently, identically and exponentially distributed with parameter \( \mu \). After completion of each service, the customer may either join at the end of the queue with probability \( p_1 \) or he may leave the system with probability \( q_1 \), where \( p_1 + q_1 = 1 \). The customers both newly arrived and those that are fed back are served in order in which they join the tail of original queue. The queue discipline is first come first served (FCFS). We do not distinguish between the regular arrival and feedback arrival. The capacity of the system is taken as finite, say \( N \). Each customer upon joining the queue will wait a certain length of time, say \( T \) for his service to begin, if it does not begin by then, he may get reneged and may leave the queue without getting service with probability \( p_2 \) and may remain in the queue for his service with probability \( q_2 \) \((= 1 - p_2)\) if certain customer retention strategy is applied. The times \( T \) follow exponential distribution with parameter \( \xi \). An arriving customer balks with probability \( \frac{n}{N} \), where \( n \) is the number of customers in system and joins the system with probability \( 1 - \frac{n}{N} \).

6.1.2 Steady-state Equations of the Model

Let \( P_n \) be the probability that there are \( n \) customers in the system. Using the Markov chain theory, the steady-state equations of the model are

\[
\lambda P_0 = \mu q_1 P_1 
\]

\[
\left[ \left( 1 - \frac{n}{N} \right) \lambda + \mu q_1 + (n - 1) \xi p_2 \right] P_n = [\mu q_1 + n \xi p_2] P_{n+1} 
\]

\[
+ \left[ \left( 1 - \frac{n-1}{N} \right) \lambda \right] P_{n-1}, \quad 1 \leq n \leq N - 1 
\]

\[
\left[ \left( 1 - \frac{N-1}{N} \right) \lambda \right] P_{N-1} = [\mu q_1 + (N - 1) \xi p_2] P_N 
\]

6.1.3 Steady-state Solution of the Model

From equation (6.1), we have

\[
q_1 \mu P_1 = \lambda P_0 
\]
Substitute \( n = 1 \) in (6.2), we get

\[
[\mu q_1 + \xi p_2]P_2 = \left[ (1 - \frac{1}{N}) \lambda + \mu q_1 \right]P_1 - \lambda P_0
\]

\[
[\mu q_1 + \xi p_2]P_2 = \left[ \left(1 - \frac{1}{N}\right) \lambda \right] P_1 \quad \{ \therefore \text{of (6.1)} \}
\]

\[
P_2 = \frac{N - 1}{N} \frac{\lambda}{\mu q_1 + \xi p_2} P_0 \quad \{ \therefore \text{of (6.4)} \}
\]

\[
P_2 = \prod_{k=1}^{2} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} P_0 \quad (6.5)
\]

For \( n = 2 \) in equation (6.2), we have

\[
[\mu q_1 + 2\xi p_2]P_3 = \left[ \left(1 - \frac{2}{N}\right) \lambda + \mu q_1 + \xi p_2 \right] P_2 - \left(1 - \frac{1}{N}\right) \lambda P_1
\]

\[
[\mu q_1 + 2\xi p_2]P_3 = \left[ \left(1 - \frac{2}{N}\right) \lambda \right] P_2 + \{ [\mu q_1 + \xi p_2]P_2 \} - \left(1 - \frac{1}{N}\right) \lambda P_1 \quad \{ \text{Using (*)} \}
\]

\[
P_3 = \frac{N - 2}{N} \frac{\lambda}{\mu q_1 + 2\xi p_2} \prod_{k=1}^{2} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} P_0 \quad \{ \text{Using (6.5)} \}
\]

\[
P_3 = \prod_{k=1}^{3} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} P_0 \quad (6.6)
\]

Proceeding in the same way for \( 3 \leq n \leq N - 1 \), we get

\[
P_n = \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} P_0 \quad (6.7)
\]

Similarly, for \( n = N \), equation (6.3) follows

\[
[\mu q_1 + (N - 1)\xi p_2]P_N = \left(1 - \frac{N - 1}{N}\right) \lambda P_{N-1}
\]

\[
P_N = \frac{N - (N - 1)}{N} \frac{\lambda}{(\mu q_1 + (N - 1)\xi p_2)} P_{N-1}
\]
\[ P_N = \frac{N - (N - 1)}{N} \frac{\lambda}{\mu q_1 + (N - 1) \xi p_2} \prod_{k=1}^{N-1} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1) \xi p_2} P_0 \] {Using (6.7)}

Hence for \( 1 \leq n \leq N \), we get

\[ P_n = \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1) \xi p_2} P_0 \] (6.8)

Using \( \sum_{n=0}^{N} P_n = 1 \), we get

\[ P_0 + \sum_{n=1}^{N} P_n = 1 \]

\[ P_0 + \sum_{n=1}^{N} \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1) \xi p_2} P_0 = 1 \]

\[ \left[ 1 + \sum_{n=1}^{N} \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1) \xi p_2} \right] P_0 = 1 \]

\[ P_0 = \frac{1}{1 + \sum_{n=1}^{N} \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1) \xi p_2}} \] (6.10)

Hence, the steady-state probabilities of the system size are derived explicitly.

### 6.1.4 Performance Measures

(i) The Expected System Size \( (L_s) \)

\[ L_s = \sum_{n=1}^{N} n P_n \]

\[ L_s = \sum_{n=1}^{N} n \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1) \xi p_2} \right\} P_0 \]

(ii) The Expected Number of Customers Served \( (E(C.S.)) \)

\[ E(C.S.) = \mu q_1 \sum_{n=1}^{N} P_n \]

81
\[ E(C.S.) = \mu q_1 \sum_{n=1}^{N} \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} P_0 \]

(iii) Average Reneging Rate \((R_r)\)

\[ R_r = \sum_{n=1}^{N} (n - 1)\xi p_2 P_n \]

\[ R_r = \sum_{n=1}^{N} (n - 1)\xi p_2 \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} \right\} P_0 \]

(iv) Average Balking Rate \((R_b)\)

\[ R_b = \sum_{n=1}^{N} n\lambda P_n \]

\[ R_b = \sum_{n=1}^{N} n\lambda \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} \right\} P_0 \]

(v) Average Retention Rate \((R_R)\)

\[ R_R = \sum_{n=1}^{N} (n - 1)\xi q_2 P_n \]

\[ R_R = \sum_{n=1}^{N} (n - 1)\xi q_2 \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{\mu q_1 + (k - 1)\xi p_2} \right\} P_0 \]

where \(P_0\) is computed in (6.10).
Numerical Results

Here we provide numerical results for different measures of performance. We study the variation in $L_s, E(C.S.), R_r, R_R$ and $R_b$ with respect to $q$ (table-6.1). When $N = 10, \lambda = 4, \mu = 4.5, q_1 = 0.95, \xi = 0.4$, we have the following observations:

Table 6.1: Measures of Effectiveness Vs. $q_2$

<table>
<thead>
<tr>
<th>S.No.</th>
<th>$q_2$</th>
<th>$L_s$</th>
<th>$E(C.S.)$</th>
<th>$R_r$</th>
<th>$R_R$</th>
<th>$R_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.664</td>
<td>3.168</td>
<td>0.3692</td>
<td>0</td>
<td>0.666</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>1.695</td>
<td>3.18</td>
<td>0.342</td>
<td>0.038</td>
<td>0.678</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>1.728</td>
<td>3.206</td>
<td>0.313</td>
<td>0.0783</td>
<td>0.692</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>1.763</td>
<td>3.226</td>
<td>0.283</td>
<td>0.121</td>
<td>0.705</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>1.801</td>
<td>3.247</td>
<td>0.249</td>
<td>0.166</td>
<td>0.721</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>1.842</td>
<td>3.269</td>
<td>0.215</td>
<td>0.215</td>
<td>0.736</td>
</tr>
<tr>
<td>7</td>
<td>0.6</td>
<td>1.885</td>
<td>3.293</td>
<td>0.178</td>
<td>0.267</td>
<td>0.754</td>
</tr>
<tr>
<td>8</td>
<td>0.7</td>
<td>1.933</td>
<td>3.317</td>
<td>0.138</td>
<td>0.324</td>
<td>0.773</td>
</tr>
<tr>
<td>9</td>
<td>0.8</td>
<td>1.984</td>
<td>3.343</td>
<td>0.096</td>
<td>0.385</td>
<td>0.794</td>
</tr>
<tr>
<td>10</td>
<td>0.9</td>
<td>2.039</td>
<td>3.369</td>
<td>0.0501</td>
<td>0.451</td>
<td>0.816</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>2.102</td>
<td>3.398</td>
<td>0</td>
<td>0.523</td>
<td>0.841</td>
</tr>
</tbody>
</table>

Table-6.1 indicates that as we increase the probability of retaining a reneging customer there is a steady increase in the average system size, average number of customers served and average retention rate. The average balking rate increases with the increase in average system size. Here, the average reneging rate decrease subsequently on increasing $q_2$. Thus, one can study the effect of different probabilities of retaining reneging customers on the different performance measures. When $q_2 = 0$, there is no customer retention and therefore $R_R = 0$. Also, when $q_2 = 1$, $R_r = 0$, that is, all reneging customers are retained.

6.1.5 Economic Analysis of the Model

We now perform economic analysis of the model. We use the following terminology to derive total expected cost (TEC), total expected revenue(TER) and total expected profit(TEP) respectively.

- $1/\lambda$ = mean inter arrival time.
- $1/\mu$ = mean service time.
- $\lambda_{lost} = \lambda P_N$ = mean rate at which a unit is lost.
- $P_N$ = probability that the system is full.
- $L_s$ = expected number of units in the system.
- $R_r$ = average rate of reneging.
- $R_R$ = average rate of Retention.

83
$R_b =$ average rate of balking.

$C_s =$ cost per service per unit time.

$C_h =$ holding cost per unit per unit time.

$C_l =$ cost associated to each lost unit per unit time.

$C_r =$ cost associated to each reneged unit per unit time.

$C_b =$ cost associated to each balked unit per unit time.

$C_R =$ per unit cost associated with a retained unit per unit time.

$R =$ earned revenue by providing service to each unit per unit time.

$TEC =$ total expected cost of the system.

$TEP =$ total expected profit of the system.

$TER =$ total expected revenue of the system.

The total expected cost (TEC) of the system is

$$TEC = C_s \mu q_1 + C_h L_s + C_l \lambda P_N + C_r R_r + C_b R_b + C_R R_R$$

The total expected revenue (TER) of the system is given by

$$TER = R L_s - R \lambda P_N - RR_r - RR_b$$

Now, total expected profit (TEP) of the system is defined as:

$$TEP = TER - TEC$$

Thus,

$$TEP = (R - C_h) L_s - (R + C_l) \lambda P_N - (R + C_r) R_r - (R + C_b) R_b - C_s \mu q_1 - C_R R_R$$
Numerical Results

The cost-profit aspects of the model are studied numerically by using MS-Excel. The following tables present the numerical results associated with the economic analysis of the model. When \( N = 10, \lambda = 4, \xi = 0.4, \mu = 4.5, q_1 = 0.95, C_l = 100, C_s = 10, C_h = 8, C_r = 50, C_b = 30 \) and \( R = 500 \), we have the following observations:

Table 6.2: Variations in TEC, TER and TEP with the variation in probability of retaining reneging customer\((q_2)\)

<table>
<thead>
<tr>
<th>S.No.</th>
<th>( q_2 )</th>
<th>( C_R )</th>
<th>TEC</th>
<th>TER</th>
<th>TEP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>96.742</td>
<td>314.602</td>
<td>217.861</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>5</td>
<td>96.185</td>
<td>337.564</td>
<td>241.378</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>10</td>
<td>95.994</td>
<td>361.901</td>
<td>265.907</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>15</td>
<td>96.201</td>
<td>387.757</td>
<td>291.557</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>20</td>
<td>96.847</td>
<td>415.305</td>
<td>318.458</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>25</td>
<td>97.978</td>
<td>444.743</td>
<td>346.765</td>
</tr>
<tr>
<td>7</td>
<td>0.6</td>
<td>30</td>
<td>99.647</td>
<td>476.306</td>
<td>376.659</td>
</tr>
<tr>
<td>8</td>
<td>0.7</td>
<td>35</td>
<td>101.917</td>
<td>510.277</td>
<td>408.361</td>
</tr>
<tr>
<td>9</td>
<td>0.8</td>
<td>40</td>
<td>104.866</td>
<td>547.001</td>
<td>442.135</td>
</tr>
<tr>
<td>10</td>
<td>0.9</td>
<td>45</td>
<td>108.581</td>
<td>586.894</td>
<td>478.313</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>50</td>
<td>113.178</td>
<td>630.484</td>
<td>517.305</td>
</tr>
</tbody>
</table>

We can see in table-6.2 that the total expected revenue and total expected profit of the system increases as the probability of retaining the reneging customers increases. Balking also has a very negative effect on the cost and revenue of the system. The symbol \( C_R \) is the cost of customer retention strategy associated with the corresponding probability of retaining reneging customers\((q_2)\). It is obvious from the values of TEC and TER that the total expected profit increases with the increase in probability of retaining a reneging customer. Thus, one can have an estimate of the costs and profits associated with the customer retention strategies of the system in the long run.

6.1.6 Special Cases

1 In the absence of feedback (i.e. for \( q_1 = 1 \), the queueing model reduces to an \( M/M/1/N \) queueing model with balking, reneging and retention of reneged customers as studied by Kumar and Sharma (2012b).

2 In the absence of retention of reneged customer (i.e. for \( q_2 = 0 \), our model reduces to an \( M/M/1/N \) queueing model with feedback, balking, and reneging.
3 In absence of balking, model resembles with an M/M/1/N feedback queueing model with reneging and retention of reneged customers as studied by Sharma and Kumar (2012a).

4 In absence of balking and feedback, our model reduces to one studied in Kumar and Sharma (2013c).
6.2 Multi-Server Feedback Queueing Model with Balking, Reneging, and Retention of Reneged Customers

6.2.1 Model Construction

The arrival process is Poisson with average arrival rate $\lambda$. The inter-arrival times are independently, identically and exponentially distributed with parameter $\lambda$. There is $c$ server and the service time is independently, identically and exponentially distributed with parameter $\mu$. The mean service rate is given by $\mu_n = \begin{cases} n\mu, & 0 \leq n \leq c-1 \\ c\mu, & c \leq n \leq N \end{cases}$ where $n$ is the number of customers in the system and $N$ is the capacity of the system. After completion of each service, the customer may either join at the end of the queue with probability $p_1$ or he may leave the system with probability $q_1$, where $p_1 + q_1 = 1$. The customers both newly arrived and those that are fed back are served in order in which they join the tail of original queue. The queue discipline is first come first served (FCFS). We do not distinguish between the regular arrival and feedback arrival. The capacity of the system is taken as finite, say $N$. Each customer upon joining the queue will wait a certain length of time, say $T$ for his service to begin, if it does not begin by then, he may get reneged and may leave the queue without getting service with probability $p_2$ and may remain in the queue for his service with probability $q_2 (= 1 - p_2)$ if some customer retention strategy is applied. The times $T$ follow exponential distribution with parameter $\xi$. An arriving exponential distribution with parameter $\xi$. An arriving customer balks with probability $\frac{n}{N}$, where $n$ is the number of customers in system and joins the system with probability $1 - \frac{n}{N}$.

6.2.2 Steady-state Equations of the Model

Let $P_n$ be the probability that there are $n$ customers in the system. Using the Markov chain theory, the steady-state equations of the model are

$$\lambda P_0 = \mu q_1 P_1$$

$$\left[ (1 - \frac{n}{N}) \lambda + n \mu q_1 \right] P_n = [(n+1)\mu q_1] P_{n+1} + \left[ \left( 1 - \frac{n-1}{N} \right) \lambda \right] P_{n-1}, \ 1 \leq n < c$$

$$\left[ (1 - \frac{n}{N}) \lambda + \mu q_1 + (n-c)\xi p_2 \right] P_n = [c\mu q_1 + \{(n+1) - c\}]\xi p_2] P_{n+1}$$

$$+ \left[ \left( 1 - \frac{n-1}{N} \right) \lambda \right] P_{n-1}, \ c \leq n \leq N - 1$$

$$\left[ \left( 1 - \frac{N-1}{N} \right) \lambda \right] P_{N-1} = [c\mu q_1 + (N-c)\xi p_2] P_N,$$
6.2.3 Steady-state Solution of the Model

From equation (6.11), we have
\[ \mu P_1 = \lambda P_0 \]

\[ P_1 = \left( \frac{\lambda}{\mu q} \right) P_0 \]  
(6.15)

Substitute \( n = 1 \) in equation (6.12), we get
\[ 2\mu q_1 P_2 = \left[ \left( 1 - \frac{1}{N} \right) \lambda + \mu q_1 \right] P_1 - \lambda P_0 \]
\[ 2\mu q_1 P_2 = \left[ \left( 1 - \frac{1}{N} \right) \lambda \right] P_1 + [\mu q_1 P_1 - \lambda P_0] \quad \{ \text{Using (6.11)} \} \]
\[ 2\mu q_1 P_2 = \frac{N - 1}{N} \lambda P_1 \]  
(\( \ast \ast \))

\[ P_2 = \prod_{k=1}^{2} N - (k - 1) \frac{\lambda^2}{2!(\mu q_1)^2} P_0 \quad \{ \text{Using (6.22)} \} \]  
(6.16)

For \( n = 2 \) in equation (6.12), we have
\[ 3\mu q_1 P_3 = \left[ \left( 1 - \frac{2}{N} \right) \lambda + 2\mu q_1 \right] P_2 - \left( 1 - \frac{1}{N} \right) \lambda P_1 \]
\[ 3\mu q_1 P_3 = \left[ \left( 1 - \frac{2}{N} \right) \lambda \right] P_2 + \left[ 2\mu q_1 P_2 - \left( 1 - \frac{1}{N} \right) \lambda P_1 \right] \]
\[ 3\mu P_3 = \left[ \left( 1 - \frac{2}{N} \right) \lambda \right] P_2 \quad \{ \because \text{of} \ (\ast \ast) \} \]
\[ P_3 = \prod_{k=1}^{3} N - (k - 1) \frac{\lambda}{\mu q_1 + (k - 1)\xi_P} P_0 \]  
(6.17)

Proceeding in the same way for \( 3 \leq n \leq c - 1 \), we get
\[ P_n = \prod_{k=1}^{n} N - (k - 1) \frac{\lambda^n}{N n!(\mu q_1)^n} P_0, \quad 1 \leq n \leq c \]  
(6.18)

Now for \( n = c \), equation (6.13) become
\[ (c\mu q_1 + \xi_P)P_{c+1} = \left[ \left( 1 - \frac{c}{N} \right) \lambda + c\mu q_1 \right] P_c - \left( 1 - \frac{c - 1}{N} \right) \lambda P_{c-1} \quad \{ \because c\mu q_1 P_c = \frac{N - (c - 1)}{N} \lambda P_{c-1} \} \]
\[ P_{c+1} = \prod_{k=c+1}^{c+1} N - (k - 1) \frac{\lambda}{c\mu q_1 + (k - c)\xi_P} P_c \]  
(6.19)
Similarly, for \( c + 1 \leq n \leq N - 1 \), we get
\[
P_n = \prod_{k=c+1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c\mu q_1 + (k - c)\xi p_2} P_c \quad (6.20)
\]

Also from equation (6.14), the value of \( P_N \) is derived as
\[
[c\mu q_1 + (N - c)\xi p_2]P_N = \left(1 - \frac{N - 1}{N}\right)\lambda P_{N-1}
\]
\[
P_N = \frac{N - (N - 1)}{N} \frac{\lambda}{c\mu q_1 + (N - c)\xi p_2} P_{N-1} \quad \text{Using (6.20)}
\]
\[
P_N = \prod_{k=c}^{N} \frac{N - (k - 1)}{N} \frac{\lambda}{c\mu q_1 + (k - c)\xi p_2} P_c \quad (6.21)
\]

Thus, the steady-state solution of the model is
\[
P_n = \begin{cases} 
\prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda^n}{n!(\mu q_1)^n} P_0, & 1 \leq n < c \\
\prod_{k=c}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c\mu q_1 + (k - c)\xi p} \left\{ \prod_{r=1}^{c-1} \frac{N - (r - 1)}{N} \frac{\lambda^{c-1}}{(c-1)!(\mu q_1)^{c-1}} \right\} P_0, & c \leq n \leq N 
\end{cases}
\]
\[
(6.22)
\]

We know that \( \sum_{n=0}^{N} P_n = 1 \), we get
\[
P_0 = \frac{1}{1 + Q_1 + Q_2} \quad (6.23)
\]

\[
Q_1 = \sum_{n=1}^{c-1} \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda^n}{n!(\mu q_1)^n} P_0
\]

and
\[
Q_2 = \sum_{n=c}^{N} \prod_{k=c}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c\mu q_1 + (k - c)\xi p_2} \left\{ \prod_{r=1}^{c-1} \frac{N - (r - 1)}{N} \frac{\lambda^{c-1}}{(c-1)!(\mu q_1)^{c-1}} \right\}
\]

Hence, the steady-state probabilities of the system size are derived explicitly.
6.2.4 Performance Measures

(i) The Expected System Size ($L_s$)

\[
L_s = \sum_{n=1}^{c-1} nP_n + \sum_{n=c}^{N} nP_n
\]

\[
L_s = \sum_{n=1}^{c-1} n \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda^n}{n!(\mu q_1)^n} \right\} P_0
\]

\[
+ \sum_{n=c}^{N} n \left\{ \prod_{k=c}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c \mu q_1 + (k - c) \xi p_2} \right\} P_{c-1}
\]

(ii) The Expected Number of Customers Served ($E(C.S.)$)

\[
E(C.S.) = c \sum_{n=1}^{N} n \mu q_1 P_n + \sum_{n=c+1}^{N} c \mu q_1 P_n
\]

\[
E(C.S.) = \sum_{n=1}^{c} n \mu q_1 \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda^n}{n!(\mu q_1)^n} \right\} P_0
\]

\[
+ \sum_{n=c+1}^{N} c \mu q_1 \left\{ \prod_{k=c+1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c \mu q_1 + (k - c) \xi p_2} \right\} P_c
\]

(iii) Average Reneging Rate ($R_r$)

\[
R_r = \sum_{n=1}^{N} (n - c) \xi p_2 P_n
\]

\[
R_r = \sum_{n=c}^{N} (n - c) \xi p_2 \left\{ \prod_{k=c}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c \mu q_1 + (k - c) \xi p_2} \right\} P_{c-1}
\]

(iv) Average Balking Rate ($R_b$)

\[
R_b = \sum_{n=1}^{N} \frac{n}{N} \lambda P_n
\]

\[
R_b = \sum_{n=1}^{c-1} \frac{n \lambda}{N} \left\{ \prod_{k=1}^{n} \frac{N - (k - 1)}{N} \frac{\lambda^n}{n!(\mu q_1)^n} \right\} P_0
\]

\[
+ \sum_{n=c}^{N} \frac{n \lambda}{N} \left\{ \prod_{k=c}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c \mu q_1 + (k - c) \xi p_2} \right\} P_{c-1}
\]
(v) Average Retention Rate ($R_R$)

$$R_R = \sum_{n=1}^{N} (n - c)\xi q_2 P_n$$

$$R_R = \sum_{n=c}^{N} (n - c)\xi q_2 \left\{ \prod_{k=c}^{n} \frac{N - (k - 1)}{N} \frac{\lambda}{c\mu q_1 + (k - c)\xi q_2} \right\} P_{c-1}$$

where $P_0$ is computed in (6.23).

### 6.2.5 Special Cases

1. In the absence of feedback (i.e. for $q_1 = 1$, the queueing model reduces to an M/M/c/N queueing model with balking, reneging and retention of reneged customers as studied by Kumar and Sharma (2013b).

2. In absence of balking and feedback, our model reduces to an M/M/c/N queueing model with reneging, and retention of reneged customers as studied by Kumar and Sharma (2013a).

3. In absence of balking, model resembles with an M/M/c/N feedback queueing model with reneging and retention of reneged customers as studied in Kumar and Sharma (2014e).

4. In the absence of retention of reneged customer (i.e. for $q_2 = 0$), our model reduces to an M/M/c/N queueing model with feedback, balking and reneging.