Chapter 2

2-PRIMAL RINGS

When one undertakes the study of non-commutative rings, it rapidly becomes apparent that the Mathematical landscape has changed significantly from that which one encounters in the commutative setting. Things that are obvious in the commutative setting can often turn into difficult problems without commutativity. One of the best examples of this is the construction of the field of fractions of a ring $R$. When $R$ is a commutative integral domain the process is relatively straightforward. The extension of this concept to non-commutative rings, however, is highly non-trivial and is indeed still a topic of current research.

2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. The study of 2-primal rings was initiated...
by Shin [60], and he proved that a ring $R$ is 2-primal if and only if every minimal prime ideal of $R$ is completely prime in Proposition (1.11) of [60]. The study of 2-primal condition was continued by Birkenmeier-Heatherly-Lee [15], Hirano [35] and Sun [62], etc.

Shin also showed that every proper ideal of a ring $R$ is 2-primal if and only if every prime ideal of $R$ is completely prime in Proposition (1.13) of [60]. Birkenmeier-Heatherly-Lee provided various examples relating to this equivalent condition in [15]. Some of the fundamental properties of 2-primal rings are developed in [35], [60] and [62]. (N. B. The terminology is not uniform: 2-primal rings are called “N-rings” in [35], and, under an equivalent definition, called “weakly symmetric” in [62]).

This chapter aims to recall some known definitions and results that lead to further discussion.
2.1 Minimal prime ideals and Completely prime ideals

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [43]. 2-primal near rings have been discussed by Argac and Groenewald in [1].

Recall that a proper ideal $P$ in a commutative ring $R$ is prime if, whenever $a, b \in R$ such that $ab \in P$ implies $a \in P$ or $b \in P$; equivalently, $P$ is a prime ideal if and only if the factor ring $R/P$ is a domain.

Prime ideals are also quite useful in the study of non-commutative rings, but are not defined as in commutative case. Such ideals are known in the literature as completely prime ideals or, alternatively, strongly prime ideals. In non-commutative setting an integral domain is defined as in commutative case, but it turns out to be restrictive for the ideal $P$ such that $R/P$ is a domain.

**Definition 2.1.1.** A prime ideal in a ring $R$ is any ideal $P$ of $R$ ($P \neq R$) such that, whenever $A$ and $B$ are ideals of $R$ with $AB \subseteq P$, either $A \subseteq P$ or $B \subseteq P$.

**Definition 2.1.2.** The set of prime ideals in a ring $R$ is called the *prime spectrum* of $R$, denoted as $Spec(R)$. 
Definition 2.1.3. An ideal $P$ of $R$ is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. The set of all completely prime ideals of $R$ is denoted by $C.Spec(R)$.

In commutative case completely prime ideal and prime have the same meaning. In general (non-commutative) situation every completely prime ideal of a ring $R$ is a prime ideal, but converse need not be true.

For instance, let $R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right) = M_2(\mathbb{Z})$. If $p$ is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of $R$, but is not completely prime, since for $a = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ and $b = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

Definition 2.1.4. Let $R$ be a domain with quotient field $K$. A prime ideal $P$ of $R$ is called a strongly prime ideal if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$. The set of all strongly prime ideals of $R$ is denoted by $S.Spec(R)$.

Clearly, any strongly prime ideal is prime. But the converse need not be true.

Note 2.1.1. The zero ideal in the ring of all $n \times n$ matrices over a division ring is a prime ideal but not strongly prime.

Definition 2.1.5. A prime ideal $P$ is said to be a minimal prime ideal over an ideal $I$ if there are no prime ideals strictly contained in $P$ that contain $I$. A prime
ideal is said to be a minimal prime ideal if it is a minimal prime ideal over the zero ideal. In a commutative Artinian ring, every maximal ideal is a minimal prime ideal.

**Definition 2.1.6.** The set of minimal prime ideals in a ring $R$ is called *minimal prime spectrum* of $R$, denoted as $\text{MinSpec}(R)$.

**Definition 2.1.7.** A non-zero right module $M$ over a ring $R$ is prime if the annihilator of $M$ is the same as the annihilator of $N$ for every non-zero submodule $N \subseteq M$.

It can be shown that if $M$ is prime, then $\text{Ann}(M)$ is a prime ideal. The converse is not true; for example, the $\mathbb{Z}$ module $M = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ has $\text{Ann}(M) = 0$, but is not a prime module if $n \geq 2$.

**Definition 2.1.8. Associated prime ideals:** Let $M$ be a right module over a ring $R$. An ideal $P$ of $R$ is called an associated prime ideal of $M$ if there exists a prime submodule $N \subseteq M$ such that $P = \text{Ann}(N)$. The set of associated primes of $M$ is denoted by $\text{Ass}(M_R)$.

**Example 2.1.1.** The $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ has two associated primes, 0 and $2\mathbb{Z}$, and 0 is certainly not maximal among the annihilators of nonzero submodules of this module.
2.2 Automorphisms and Derivations

A ring $R$ always means an associative ring with identity. Let $R$ be a ring and \( \sigma \) an automorphism of $R$. Let $I$ be an ideal of $R$ such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$. We denote $\cap_{i=1}^{m} \sigma^i(I)$ by $I^0$. For any two ideals $I, J$ of $R$, $I \subset J$ means that $I$ is strictly contained in $J$.

**Definition 2.2.1. Nilpotent element:** An element $a \in R$ is said to be nilpotent if there is a positive integer $n$ (depending on $a$) such that $a^n = 0$. $N(R)$ denotes the set of nilpotent elements of $R$. (In any ring $R$, 0 is a nilpotent element, called the *trivial nilpotent element*.)

**Definition 2.2.2. Prime radical:** The prime radical of a ring $R$ is the intersection of all the prime ideals of $R$. It is denoted by $P(R)$. Also a ring is semiprime if and only if its prime radical is zero.

**Remark 2.2.1.** Any nilpotent element in $R$ is in all prime ideals of $R$ i.e., if $N = \{a \in R \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}$ and $N' = \cap_{P} P$ where the intersection is taken over all prime ideals of $R$, then $N \subseteq N'$.

We now have the following important Theorem which establishes a relationship between the set of nilpotent elements and the prime ideals.

**Theorem 2.2.1.** The set of all nilpotent elements in a commutative ring $R$ with 1 is the intersection of all prime ideals, i.e., $N = N'$. 
Definition 2.2.3. Derivation: Let $R$ be a ring, a map $\delta : R \to R$ is called a $\delta$-derivation if for every $a, b \in R$

1. $\delta(a + b) = \delta(a) + \delta(b)$

2. $\delta(a \cdot b) = \delta(a) \cdot b + a \cdot \delta(b)$

Definition 2.2.4. Derivation (endomorphism type): Let $R$ be a ring and $\sigma$ an endomorphism of $R$, a mapping $\delta : R \to R$ is called a $\sigma$-derivation if

$$\delta(a \cdot b) = \delta(a) \cdot \sigma(b) + a \cdot \delta(b).$$

Proposition 2.2.2. Let $R$ be a ring $\sigma$ be an automorphism of $R$. Then $\sigma | Z(R)$ is an automorphism.

Lemma 2.2.3. (Gabriel, Lemma 3.4 of [25]). If $R$ is a Noetherian $\mathbb{Q}$-algebra and $\delta$ is a derivation of $R$, then $\delta(P) \subseteq P$ for all $P \in \text{MinSpec}(R)$.

Let $\sigma$ be an automorphism of $R$. Recall that an ideal $I$ of a ring $R$ is called completely semiprime if $a^2 \in I$ implies $a \in R$. With this we have the following definitions.

2.3 $\sigma(*)$-rings and 2-primal rings

Another related area of interest since recent past has been the study of 2-primal rings. Krempa in [45] introduced $\sigma$-rigid rings; Kwak in [48] introduced $\sigma(*)$-rings
and Ouyang in [57] introduced weak $\sigma$-rigid rings, where $\sigma$ is an endomorphism of ring $R$. These rings are related to 2-primal rings.

**Definition 2.3.1. $\delta$-Ring:** Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then $R$ is a $\delta$-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Note that a $\delta$-ring is a ring without identity, as $1\delta(1) = 0$, but $1 \notin P(R)$, as $1 \neq 0$.

**Example 2.3.1.** Let $S = 2\mathbb{Z}$ and $R = S \times S$ with $P(R) = \{0\}$. Then $\sigma : R \to R$ is an endomorphism defined by $\sigma((a, b)) = (b, a)$. For any $s \in R$, define $\delta_s : R \to R$ by $\delta_s(a, b) = (a, b)s - s\sigma(a, b)$. Then $R$ is a $\delta$-ring.

**Definition 2.3.2. $\sigma$-Rigid:** A ring $R$ is $\sigma$-rigid if there exists an endomorphism of $R$ with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$.

**Example 2.3.2.** Let $R = \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a + ib) = a - ib; a, b \in \mathbb{R}$. Then $\sigma$ is a rigid endomorphism.

**Definition 2.3.3. $\sigma(*)$-Ring:** (Kwak [48]). Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $R$ is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$. 
Example 2.3.3. Let \( R = M_2(\mathbb{Z}_3) \) be the 2 \( \times \) 2 matrix ring over the field \( \mathbb{Z}_3 \).

Let \( \sigma : R \rightarrow R \) be an automorphism defined by

\[
\sigma\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.
\]

Then \( R \) is a \( \sigma(*) \)-ring.

Definition 2.3.4. 2-Primal Rings: A ring \( R \) is 2-primal if and only if the set of nilpotent elements and the prime radical of \( R \) are the same if and only if the prime radical is a completely semiprime ideal.

We note that a reduced ring (i.e., a ring with no non-zero nilpotent elements) is 2-primal and a commutative ring is also 2-primal.

Example 2.3.4. Let \( \mathbb{Z}_2 \) be the ring of integers modulo 2 and \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Then \( R \) is a commutative reduced ring with \( P(R) = \{(0, 0)\} \).

Proposition 2.3.1. (Bhat, Proposition 2 of [10]). Let \( R \) be a ring and \( \sigma \) an automorphism of \( R \). Then \( R \) is a \( \sigma(*) \)-ring implies \( R \) is 2-primal.

Proof. Let \( a \in R \) be such that \( a^2 \in P(R) \).

Then \( a\sigma(a)\sigma(a) = a\sigma(a)\sigma(a^2) \in \sigma(P(R)) = P(R) \).

Therefore \( a\sigma(a) \in P(R) \) and hence \( a \in P(R) \).

Remark 2.3.1. Every \( \sigma(*) \)-ring is a 2-primal ring but the converse need not be true.
For instance, let $R = F[x]$ be the polynomial ring over the field $F$. Then $R$ is 2-primal with $P(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then $R$ is not a $\sigma(\ast)$-ring.

### 2.4 Weak $\sigma$-rigid rings and $SI$-rings

**Definition 2.4.1.** (Ouyang [57]). Let $R$ be a ring and $\sigma$ an endomorphism of $R$ such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then $R$ is called a weak $\sigma$-rigid ring.

**Example 2.4.1.** Let $R$ be the ring of Eisenstein’s integers, where Eisenstein’s integers are the complex numbers of the form $z(\omega) = a + b\omega$, where $a$ and $b$ are integers and $\omega = \frac{1}{2}(-1 + \sqrt[3]{3}) = e^{\frac{2\pi i}{3}}$ is a primitive (non-real) cube root of unity. Then $R$ is a commutative ring of algebraic integer in the algebraic field $Q(\omega)$- the third cyclotomic field. If we define an endomorphism $\sigma : R \to R$ by

$$\sigma(a + be^{\frac{2\pi i}{3}}) = (a - be^{\frac{2\pi i}{3}}).$$

Then $R$ is a weak $\sigma$-rigid ring.

Ouyang has proved in [57] that if $\sigma$ is an endomorphism of a ring $R$, then $R$ is $\sigma$-rigid if and only if $R$ is weak $\sigma$-rigid and reduced.

**Definition 2.4.2.** $SI$-rings: (Shin [60]). Let $R$ be a ring. Then $R$ is called an $SI$-ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. 

Example 2.4.2. (Gosani and Bhat, Example 1.2 of [34]).

Let $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}$, $a, b \in \mathbb{Z}$.

The only matrices $A$ and $B$ satisfying $AB = 0$ are of the type

$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$; $a, b \in \mathbb{Z}$.

i.e., $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$.

Now for all $K = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R$,

$AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

implies $AKB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This implies $R$ is an $SI$-ring.

Shin showed that $SI$-rings are 2-primal in Theorem (1.5) of [60], and so reduced rings are 2-primal.

Note that a ring $R$ is 2-primal if and only if the zero ideal of $R$ is 2-primal.
Corollary 2.4.1. (Shin, Corollary 1.9 of [60]). If $P(R)$ coincides with the set of all nilpotent elements of $R$, then for each $P \in \text{Spec}R$,

$$N(P) = \bigcap\{Q \in \text{Spec}R : N(P) \subseteq Q\} = \bigcap\{Q \in \text{Spec}R : Q \subseteq P\}.$$  

Corollary 2.4.2. (Shin, Corollary 1.10 of [60]). If $P(R)$ coincides with the set of all nilpotent elements of $R$, then for each $P \in \text{Spec}R$ the following are equivalent:

(a) $P$ is a minimal prime ideal.

(b) $N(P) = P$.

(c) For any $a \in P$, $ab$ is nilpotent for some $b \in R \setminus P$.

Proposition 2.4.3. (Shin, Proposition 1.11 of [60]). For any ring $R$, the following are equivalent:

(a) $P(R)$ coincides with the set of all nilpotent elements of $R$.

(b) Every minimal prime ideal is completely prime.

We used the above results to show that the prime radical of a weak $\sigma$-rigid ring is completely semiprime. This result also holds for an SI-ring.

Proposition 2.4.4. (Gosani and Bhat, Proposition 2.3 of [33]). Let $R$ be a ring. Then $R$ is an SI-ring implies that $P(R)$ is completely semiprime.
Proof. Since $R$ is an SI-ring. So, by Proposition (1.5) of Shin [60] $R$ is 2-primal implies that $P(R)$ is completely semiprime.

**Theorem 2.4.5.** *(Gosani and Bhat, Theorem 2.5 of [33]).* Let $R$ be a commutative Noetherian ring. Then $R$ is an SI-ring implies that $N(R)$ is completely semiprime.

Proof. The proof is obvious by Proposition (2.4.4) and Theorem (1.5) of Shin [60].

**Proposition 2.4.6.** *(Neetu-Gosani-Bhat, Proposition 3 of [47]).* Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$. Then $R$ is a weak $\sigma$-rigid ring implies that $N(R)$ is completely semiprime.

Proof. First of all we show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of $R$. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So $I = \sigma^{-1}(N(R)) = \{a \in R \text{ such that } \sigma(a) = n \in N(R)\}$ is an ideal of $R$. Now $I$ is nilpotent, therefore $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$ Now let $R$ be a weak $\sigma$-rigid ring. We will show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R).$$

Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.
The converse of the above Proposition need not be true which can be seen from the following example.

**Example 2.4.3.** (Kwak [48]). Let $K$ be a field, $R = K \times K$ and the automorphism $\sigma$ of $R$ defined by $\sigma((a, b)) = (b, a)$, $a, b \in K$. Then $R$ is a reduced ring and so $N(R) = 0$ is completely semiprime. But the ring $R$ is not a weak $\sigma$-rigid ring since $(1, 0)\sigma((1, 0)) = (0, 0)$ but $(1, 0) \notin N(R)$. 