Chapter 1

INTRODUCTION

‘Mathematics is the art of all the arts

and the science of all the sciences.’

In Mathematics there is a perfect reality, a realm of God, of which our familiar world is but an imperfect reflection. One of the many reasons the subject of Mathematics is so beautiful is the continuing process of building one theorem, algorithm, or conjecture upon another. This can be compared to the construction of a cathedral, where each stone gets laid upon those that came before it with great care. As each mason lays his stone he can only be sure to put it in its proper place, and see that it rests plumb, level, and square, with its neighbors. Prior to the nineteenth century, algebra meant the study of the solution of polynomial equations. By the twentieth century algebra came to encompass the study of abstract, axiomatic systems such as groups, rings, and fields. Abstract Mathematics is different from other sciences. In laboratory sciences such as chemistry
and physics, scientists perform experiments to discover new principles and verify theories. Although Mathematics is often motivated by physical experimentation or by computer simulations, it is made rigorous through the use of logical arguments. Abstract algebra is really a form of “Meta-Mathematics”, where it studies the structure of Mathematics itself.

*Pure mathematics is, in its way, the poetry of logical ideas*

*....Albert Einstein*

1.1 Background and Motivation

**Polynomials** not only help us to construct new and useful examples of rings and fields but are of interest in themselves. The “ring” of polynomials in one variable has two binary operations - addition and multiplication. In Mathematics, especially in the field of abstract algebra, a polynomial ring is a ring formed from the set of polynomials in one or more variables with coefficients in another ring. Polynomial rings have influenced much of Mathematics, from the ‘Hilbert basis Theorem’, to the construction of splitting fields, and to the understanding of a linear operator.
Many important conjectures involving polynomial rings, such as Serre’s problem, have influenced the study of other rings, and have influenced even the definition of other rings, such as group rings and rings of formal power series. The polynomial ring $K[X]$ is remarkably similar to the ring $\mathbb{Z}$ of integers in many respects. This analogy and the arithmetic of the ring of polynomials were thoroughly investigated by Gauss and his theory served as a model for development of abstract algebra in the second half of the nineteenth century in the works of Kummer, Kronecker, and Dedekind.

The first property of the polynomial ring is elementary and says that a product of two non-zero polynomials need not be a non-zero polynomial. The next property of the polynomial ring is much deeper. Already Euclid noted that every positive integer can be uniquely factored into a product of primes; this statement is now called the ‘Fundamental Theorem of Arithmetic’. Polynomial rings have been generalized in a great many ways, including polynomial rings with generalized exponents, power series rings, non-commutative polynomial rings, and skew polynomial rings.
An important class of modules and rings, (‘Artinian and Noetherian’), which have some very special properties have also been studied. In fact, the conditions for Artinian rings and Noetherian rings, called respectively the Descending Chain Conditions (DCC) and Ascending Chain Conditions (ACC) (often termed the minimum and maximum conditions) have some measure of finiteness associated with them. These properties make Artinian and Noetherian rings of interest to an algebraist. Furthermore, these two types of rings are related. In 1921, Emmy Noether introduced the ACC for the first time in Mathematics literature. She was considering ideals in commutative rings. After Noether’s introduction of the ACC, work in this area of ring theory exploded. Her results were expanded to non-commutative settings. In addition, other similar conditions for ideals in a ring were introduced. In particular, Emil Artin formulated the DCC in 1927, which provided a minimum condition to complement the maximum condition given by the ACC. It was later discovered, first by Noether herself and then more formally by Hopkins and Levitzki, that the DCC is actually the stronger condition.

The study of prime ideals has been an area of active research. In recent past a considerable work has been done in this direction. Associated prime ideals,
completely prime ideals and minimal prime ideals of certain types of skew polynomial rings have been characterized.

The subject, Noetherian rings is an exciting one for its own sake as well as for its applications in related areas, especially the representation theories of groups and Lie Algebras. Non-commutative Noetherian rings are presently the subject of very active research in modern development of Mathematics, particularly in Algebra. The classical study of any commutative Noetherian ring is done by studying its primary decomposition. Further there are other structural properties of rings, for example the existence of quotient rings or more particularly the existence of Artinian quotient rings etc. which can be nicely tied to primary decomposition of Noetherian rings.

However, many non-commutative notions that were developed earlier in the century are now being looked at again using modern tools. One of the fundamental differences between the theories of commutative and non-commutative rings, is that the former arise naturally as rings of functions whereas the later arise naturally as rings of operators. For example, early in the twentieth century, some
of the first non-commutative rings that received serious study were certain rings of differential operators. In many modern applications, in turn, it is essential to regard non-commutative rings as rings of transformations or operators of various kinds. This has led us to emphasize the role of modules when studying a ring, for modules are simple ways of representing a ring in terms of endomorphisms of abelian groups.

Although the Noetherian condition is very natural in commutative ring theory, since it holds for the rings of integers in algebraic number fields and for the coordinate rings crucial to algebraic geometry, it was originally less clear that this condition would be useful in the non-commutative setting. For instance, Jacobson’s definitive book of 1956 makes only minimal mention of Noetherian rings. Similarly, prime ideals, essential in the commutative theory, seemed to have relatively less importance for non-commutative rings; in fact, because of the fundamental role of representation-theoretic ideas in the development of the non-commutative theory, the initial emphasis in the subject was almost exclusively on irreducible representations (i.e., simple modules) and primitive ideals (i.e., annihilators of irreducible representations).
In the meantime, however, it has turned out that various important types of non-commutative rings - in particular, certain infinite group rings and the enveloping algebras of finite-dimensional Lie algebras - are in fact Noetherian. This has been used to good effect in recent work on the representation theory of the corresponding groups and Lie algebras, just as the theory of finite-dimensional algebras and Artinian rings have played a key role in research on the representations of finite groups. Also as soon as Noetherian rings and their modules received serious attention, prime ideals forced themselves into the picture, even in contexts where the original interest had been entirely in primitive ideals. The first important result in the theory of non-commutative Noetherian rings was proved relatively in [19], [20]. This was Goldie’s Theorem, which gives an analog of a field of fractions for factor ring \( R/P \), where \( R \) is a Noetherian ring and \( P \) a prime ideal of \( R \).

Let \( R \) be a ring, \( \sigma \) an endomorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Then the skew polynomial ring (also known as Ore extension) \( R[x; \sigma, \delta] \) is the set of right polynomials
\[ \{ \sum_{i=0}^{n} x^i a_i, a_i \in R, n \in \mathbb{N} \} \]

with usual addition of polynomials and multiplication subject to the relation
\[ ax = x\sigma(a) + \delta(a) \] for all \( a \in R \). We denote \( R[x; \sigma, \delta] \) by \( O(R) \). In case \( \sigma \) is
the identity map, we denote the ring of differential operators \( R[x; \delta] \) by \( D(R) \). If
\( \delta \) is the zero map, we denote the skew polynomial ring \( R[x; \sigma] \) by \( S(R) \). (Detailed
construction of Ore extensions is given in Chapter 3).

Ore extensions have invited attention of Mathematicians and in this direc-
tion a considerable work has been done. Many notions from commutative set
up have been generalised to non-commutative set up via Ore extensions. The
characterization of ideals and prime ideals (in particular associated prime ideals,
completely prime ideals and minimal prime ideals), and 2-primal property of Ore
extensions have been discussed in [47], [32], [33]. Ore extensions constitute an
important class of rings, appearing in several natural contexts, including skew
and differential polynomial rings [31], group algebras of polycyclic groups [27],
universal enveloping algebras of solvable Lie algebras [41], [22], and coordinate
rings of quantum groups [42].
Another related area of interest since recent past has been the study of 2-primal rings. This involves the notions of prime radical and the set of nilpotent elements of a ring. Furthermore, the concept of completely prime ideals and the completely semiprime ideals are also studied in this area. Recall that a ring $R$ is 2-primal if $N(R) = P(R)$, i.e., if the prime radical is a completely semiprime ideal. An ideal $I$ of a ring $R$ is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We also note that a reduced ring is 2-primal and a commutative ring is also 2-primal. Some of the fundamental properties of 2-primal rings are developed in [35], [60] and [62]. The study of 2-primal rings was initiated by Shin [60].

Motivated by these developments, the aim of this work is to investigate the Ore extensions over 2-primal Noetherian rings $R$. In the first instance, we shall investigate the nature of minimal prime ideals of Ore extensions $O(R)$ over a Noetherian ring $R$ and their relation with those of the coefficient ring $R$. Using this, we investigate the Ore extensions of 2-primal Noetherian rings.
1.2 Importance

In trying to understand the ideal theory of a commutative ring, one quickly sees that it is important to first understand the prime ideals. The importance of prime ideals is perhaps clearest in the setting of algebraic geometry, for if \( R \) is the coordinate ring of an affine algebraic variety, the prime ideals of \( R \) correspond to irreducible subvarieties.

The most relaxed definition for the concept of a prime ideal in the non-commutative case, was first purposed in 1928 by Krull. The most important class of prime ideals that arises in this way is the class of primitive ideals. A basic principle of algebraic geometry is to study algebraic varieties via rings of functions on them. A key part of the theory is a correspondence between certain ideals and subvarieties that arises from “annihilators”. (Such terminology is commonly used in reference to any process that results in a zero.)

Invariance of minimal primes under derivations was first proved for associated primes as well as minimal primes in commutative Noetherian \( \mathbb{Q} \)-algebras
by Seidenberg [[59], Theorem I]. For minimal completely prime ideals in non-commutative $\mathbb{Q}$-algebras it follows from a result of Dixmier [[21], Lemma 6.1] and his argument was extended to arbitrary minimal primes in $\mathbb{Q}$-algebras by Gabriel [[25], Lemma 3.4].

Since their formal introduction in the 1930’s by Oystein Ore, Ore extensions and their iterated constructions have been the subject of many studies. It quickly appeared that their “manageable” non-commutativity offers a very good tool for constructing counter-examples. For instance:

1. They were used by Bergman to produce a left but not right primitive ring.

2. They enable Cohn and Schofield to construct division rings having different left and right dimension over some subdivision ring.

After their introduction by Ore, the structure theory of Ore extensions was further developed by N. Jacobson, S. A. Amitsur, P. M. Cohn, G. Cauchon, T. Y. Lam, A. Leroy, J. Matczuk and many many others.

Ore extensions are also an essential tool in the theory of quantum groups.
Many quantum groups can be presented using iterated Ore extensions. In this case one powerful tool is what is called the “erasing of derivations” process due to G. Cauchon. (Cf. [18]). Ore extensions have both a “ring theoretical aspect”: characterization of simplicity, description of prime ideals, passage of properties from the base ring to the Ore extensions (often with the aim of giving examples of a non left/right symmetric behaviour) and a more “arithmetical aspect” mainly related to the factorization of polynomials, computation of the roots and also in relation with special matrices, in particular Vandermonde and Wronskian matrices. The link between these two aspects is given, in particular, by special types of polynomial (invariant, semi-invariant, complex valued, irreducible, Wedderburn, fully reducible). Another important feature of the Ore extensions is their relation with differential equation and operator theory. This was the origin of their study even before their formal definition given by Ore.

Recently Ore extensions have been successfully applied in many areas, for example:

1. Solving ordinary differential equations [17], [20], [38], [59].

2. Control theory [24], [26], and
1.3 Literature survey

The past decade has seen a large increase in interest regarding non-commutative algebraic structures. The fact that every prime contains a minimal prime was first proved in the commutative case by Krull [[46], Satz 5]. Towards the generalization of prime ideals in non-commutative setup a considerable work has been done. Ever since the appearance of Ore’s fundamental paper “Theory of non-commutative polynomials”, Ore extensions (the skew polynomial rings) have played an important role in non-commutative ring theory and many non-commutative ring theorists have investigated Ore extensions from different points of view.

We initially study the non-commutative polynomial extensions invented by Ore in the early 1930’s. This work is loosely tied (indirectly related) to the problem of determining the primes of a ring \( R \), which extend to the commutative polynomial extension \( R[x] \). It is interesting to note the commutative result showing that, for each prime \( P \subseteq R \), the ideal \( P[x] \subseteq R[x] \) is prime was not proved until the early 1970’s by Brewer and Heinzer [17]. Although more recent proofs by Faith [23] in

3. Coding theory [50].
2000 offered techniques which could be used for non-commutative rings. Moreover a relation between prime ideals of a ring \( R \) and those of Ore extensions \( O(R) \) has been investigated.

In section (4) of [28] Goodearl and Letzter have proved that if \( R \) is a Noetherian ring, then for each prime ideal \( P \) of Ore extension \( O(R) \), the prime ideals of \( R \) minimal over \( P \cap R \) are contained within a single \( \sigma \)-orbit of \( \text{Spec}(R) \). In [2], it is shown that if \( R \) is embeddable in a right Artinian ring and if the characteristic of \( R \) is zero, then the differential operator ring \( D(R) \) embeds into a right Artinian ring, where \( \delta \) is a derivation of \( R \). It is also shown in [3] that if \( R \) is a commutative Noetherian ring and \( \sigma \) an automorphism of \( R \), then \( S(R) \) embeds into an Artinian ring.

Diximier in [22], proved that if \( U(g) \) is the enveloping algebra of a finite dimensional solvable Lie algebra \( g \) over a field of characteristic zero, then every prime factor ring of \( U(g) \) is a domain. Lorenz in [52], shows that under certain conditions on the ring \( R \), the iterated differential operator ring \( T = R[\theta_1; \delta_1]...R[\theta_n; \delta_n] \) has
the same property. Sigurdsson in [61] proved that if $R$ is a commutative Noetherian $\mathbb{Q}$-algebra, then every prime factor ring of $T$ is a domain. He also proved that, if $P$ is a prime ideal of $T$, then the Goldie dimension of $T/P$ is uniformly bounded, provided $R$ satisfies certain conditions.

In [48], Kwak has extended the $\sigma$-rigid property of a ring $R$ to the prime radical $P(R)$ of $R$. Some characterizations of a ring $R$ whose unique maximal nil ideal $N_r(R)$ coincides with the set of all its nilpotent elements $N(R)$ by using its minimal strongly prime ideals are given by Hong and Kwak in [37]. In [45] some elementary but important examples of reduced rings are exhibited by Krempa. Special attention is paid to Ore extensions, semigroup rings and their generalizations.

Shin, and later S. H. Sun (in [62]), proved that the 2-primal condition entails elegant properties for the prime spectrum of a ring. Shin’s work on sheaf representations was inspired by J. Lambek’s investigation of symmetric rings in [49], where “Sheaf Representation Theorems” by A. Grothendieck and J. Dieudonné,
for commutative rings, and by K. Koh, for reduced rings, are generalized to symmetric rings. Shin’s study of prime ideals and stalks in sheaf representations was extended recently by G. F. Birkenmeier, J. Y. Kim, and J. K. Park in [16]. This paper contains a detailed analysis of the properties of, and interrelations among, 2-primal, (PS 1), (S 1), symmetric, duo, and numerous other conditions.

In [9], Bhat has proved that if $R$ is a Noetherian ring satisfying the $\sigma(\ast)$-property, then $S(R)$ is a 2-primal ring. Also in [11] he proved that if $\sigma$ is an endomorphism and $\delta$ is a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$, then for a Noetherian $\sigma(\ast)$-ring, which is also an algebra over $\mathbb{Q}$, the Ore extension $O(R)$ is 2-primal Noetherian.

The above discussion leads to the investigation of prime ideals, prime radicals etc. in non-commutative set up. We also investigate these structures for polynomial rings over Noetherian rings, i.e., we investigate the 2-primal property of a polynomial ring in particular Ore extensions and this leads to the question: how 2-primal rings behave in various types of ring extensions; in particular, the Ore extensions. This thesis concerns the same question. We try to connect some
earlier results and generalize some.

1.4 Structure of work

This thesis is divided into six chapters.

Chapter 1 is devoted to the introduction and importance followed by literature survey.

Chapter 2 covers notations, definitions and basic results that will be used throughout the thesis. It is almost entirely expository. In this chapter we recall some definitions and basic results which include Noetherian rings, automorphism and derivations and many other rings such as \(\sigma(*)\)-rings, 2-primal rings, weak \(\sigma\)-rigid rings, \(\delta\)-rings, \(SI\)-rings. We also show that for a weak \(\sigma\)-rigid ring \(R\), where \(\sigma\) is an automorphism of \(R\) and \(R\) is a Noetherian ring, \(N(R)\) is completely semiprime.

In Chapter 3 we discuss Ore extensions. We recall the construction of the Ore extensions and mention some known results and generalize some. We also find a
relation between completely prime ideals and minimal prime ideals of a ring $R$ and that of the Ore extensions $O(R)$. In the last section of this chapter, we study the associated prime ideals of Ore extension $O(R)$ and we prove the following in this direction:

Let $R$ be a right Noetherian $SI$-ring, which is also an algebra over $\mathbb{Q}$. Then $P$ is an associated prime ideal of $O(R)$ (viewed as a right module over itself) if and only if there exists an associated prime ideal $U$ of $R$ such that $O(P \cap R) = P$ and $P \cap R = U$.

In Chapter 4 we define the $S1$ condition and trivial extension for the Ore extension $O(R)$. We have also generalized some results of Bhat for the ring $O(R)$ and give a necessary and sufficient condition for the Ore extension $O(R)$ to be a 2-primal ring. We find a relation between prime radicals of a ring $R$ and that of the Ore extensions $O(R)$. We also find a relation between an SI-ring $O(R)$ and the trivial extension ring $T(O(R), M)$, where $M$ is an $(R, R)$ bimodule. In this chapter we show that if $R$ is a Noetherian ring such that $\sigma(P) = P$ for all minimal prime ideal $P$ of $R$, then $O(R)$ is 2-primal.
In Chapter 5 we begin by concentrating on the case where the twisting is done by an automorphism. We give a relation between a $\sigma(*)$-ring and a weak $\sigma$-rigid ring and also discuss the skew polynomial rings $S(R)$ over weak $\sigma$-rigid rings. We also give a characterization for $S(R)$ to be a $\overline{\sigma}(*$)-ring and an $SI$-ring, where $\overline{\sigma}$ is an extension of $\sigma$. With this we show that if $R$ is a Noetherian ring and $\sigma$ an automorphism of $R$, then $R$ is a weak $\sigma$-rigid ring if and only if $S(R)$ is a weak $\overline{\sigma}$-rigid ring. This chapter is concluded by giving the necessary and sufficient condition for a skew polynomial ring $S(R)$ to be 2-primal.

Finally in Chapter 6 we give the conclusion of our work and some problems for future work.