CHAPTER 8

FRACTIONAL INTEGRAL TRANSFORM

In this chapter†, we study the fractional integral transform for functions which are fractional differentiable but may not be differentiable, called fractional Elzaki’s transform. We also establish the duality relations among other fractional transforms.

8.1 Introduction

Alike the fractional Laplace transform [55] and the fractional Sumudu transform [40], the fractional Elzaki transform are based on modified Riemann-Liouville derivative. The fractional Elzaki transform can also be applied to functions which are fractional differentiable but not differentiable. In this section, we introduce some basic definitions of fractional derivative and fractional integral transforms.

†Contents of this chapter is communicated for publication as a paper entitled “Fractional Elzaki’s Transform”, (Under review).
8.1.1 Modified Riemann-Liouville derivative

Jumarie [53–58] introduced and studied a modification on the Riemann-Liouville fractional derivative by means of fractional difference, which are fractional differentiable but may not be differentiable.

Fractional derivative via fractional difference is defined in following way [53]:

**Definition 8.1.1.** Let \( f : \mathbb{R} \to \mathbb{R}, \ x \to f(x) \), denote a continuous (but not necessarily differentiable) function, and let \( h > 0 \) denote a constant discretization span. Then the forward operator \( FW(h) \) is defined by the equality

\[
FW(h)f(x) := f(x + h); \quad (8.1.1)
\]

and for the fractional difference of order \( \alpha, \ 0 < \alpha < 1 \), of \( f(x) \) is defined by the expression

\[
\Delta^\alpha f(x) := (FW - 1)^\alpha f(x)
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \left( \frac{\alpha}{k} \right) f[x + (\alpha - k)h], \quad (8.1.2)
\]

and its fractional derivative is the limit

\[
f^{(\alpha)}(x) := \lim_{h \to 0} \frac{\Delta^\alpha[f(x) - f(0)]}{h^\alpha}. \quad (8.1.3)
\]

The \((\alpha \atop n)\) is defined as:

**Definition 8.1.2.** The extended binomial coefficient \((\alpha \atop n)\), \((a \in \mathbb{R}, \ n \in \mathbb{N})\) is defined by

\[
\left( \begin{array}{c} \alpha \\ n \end{array} \right) = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} & \text{if } n > 0, \\ 1 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases} \quad (8.1.4)
\]

**Definition 8.1.3** (Modified Riemann Liouville derivative [53]). Let \( f : \mathbb{R} \to \mathbb{R}, \ x \to f(x) \), denote a continuous (but not necessarily differentiable) function.
i. Assume that $f(x)$ is a constant $K$. Then its fractional derivative of order $\alpha$ is

$$D_x^\alpha K = K \Gamma(1 - \alpha)x^{-\alpha}, \quad \alpha \leq 0,$$

$$= 0, \quad \alpha > 0. \quad (8.1.5)$$

ii. When $f(x)$ is not a constant, then we will set

$$f(x) = f(0) + (f(x) - f(0)),$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(x) = D_x^\alpha f(0) + D_x^\alpha(f(x) - f(0)),$$

in which, for negative $\alpha$, one has

$$D_x^\alpha(f(x) - f(0)) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha - 1}f(\xi)d\xi, \quad \alpha < 0, \quad (8.1.7)$$

while for positive $\alpha$, one has

$$D_x^\alpha(f(x) - f(0)) = D_x^\alpha f(x) = D_x(f^{(\alpha-1)}(x)). \quad (8.1.8)$$

When $n \leq \alpha < n + 1$, we will set

$$f^{(\alpha)}(x) := (f^{(\alpha-n)}(x))^{(n)}, \quad n \geq 1. \quad (8.1.9)$$

Then the following differential relation is useful for finding the fractional derivative of compound functions

$$d^\alpha f \cong \Gamma(1 + \alpha) df, \quad 0 < \alpha < 1. \quad (8.1.10)$$

Some basic properties of the modified Riemann-Liouville derivative are as follows:

**Corollary 8.1.4 ([56]).** The following equalities hold, which are:

if $0 < \alpha < 1$, then

$$D_x^\alpha x^\gamma = \Gamma(\gamma + 1)\Gamma^{-1}(\gamma + 1 - \alpha)x^{\gamma-\alpha}, \quad \gamma > 0, \quad (8.1.11)$$

or, if $\alpha = n + \theta, \ n \in \mathbb{N}$, then

$$D_x^{n+\theta} x^\gamma = \Gamma(\gamma + 1)\Gamma^{-1}(\gamma + 1 - n - \theta)x^{\gamma-n-\theta}, \quad 0 < \theta < 1. \quad (8.1.12)$$
Fractional product laws

\[(uv)^{(\alpha)} = u^{(\alpha)}v + uv^{(\alpha)},\]  
\[(f[u(x)])^{(\alpha)} = f'_u(u)u^{(\alpha)}(x),\]
\[= f'^{\alpha}_u(u)\left(\frac{d}{dx}\right)^{\alpha}u,\]

where \(u(x)\) is non-differentiable in (8.1.13) and (8.1.14) and differentiable in (8.1.15), \(v(x)\) is non-differentiable, and \(f(u)\) is differentiable in (8.1.14) and non-differentiable in (8.1.15).

### 8.1.2 Integration with respect to \((dt)^{\alpha}\)

The integral with respect to \((dx)^{\alpha}\) is defined as the solution of the fractional differential equation [55]

\[
\frac{dy}{dx} = f(x)(dx)^{\alpha}, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha < 1.
\]  

The integration by part formula can be obtained by combining (8.1.10) and (8.1.13) as

\[
\int_a^b u^{(\alpha)}(x)v(x)(dx)^{\alpha} = \Gamma(\alpha + 1) [u(x)v(x)]_a^b - \int_a^b u(x)v^{(\alpha)}(x)(dx)^{\alpha}.
\]  

**Lemma 8.1.5.** Let \(f(x)\) denote a continuous function; then the solution \(y(x), y(0) = 0\) of (8.1.16) is defined by the equality

\[
y = \int_0^x f(\xi)(d\xi)^\alpha
\]
\[= \alpha \int_0^x (x - \xi)^{\alpha - 1}f(\xi)d\xi, \quad 0 < \alpha < 1.
\]  

When \(f(x) = x^\beta\), then from (8.1.18) one obtains

\[
\int_0^x \xi^\beta(d\xi)^\alpha = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)}x^{\alpha + \beta}, \quad 0 < \alpha < 1.
\]  

\[(8.1.19)\]
8.1. INTRODUCTION

The following formula does not hold for Riemann-Liouville definition, but can be applied to the modified Riemann-Liouville definition [55]

\[ D^\alpha_x E_\alpha (u^\alpha(x)) = E_\alpha (u^\alpha) (u'(x))^\alpha, \]
\[ D^\alpha_x E_\alpha (\lambda x) = \lambda \alpha^{-\alpha} x^{1-\alpha} E_\alpha (\lambda x), \]
\[ D^\alpha_x E_\alpha (\lambda x^\alpha) = \lambda E_\alpha (\lambda x^\alpha), \]
\[ \int f^{(\alpha)}(x) \, (dx)^\alpha = \int d^\alpha f = \alpha! f(x), \]
\[ \int \frac{(dx)^\alpha}{x} = \frac{x^{\alpha-1} - 1}{(\alpha - 1)\Gamma(2 - \alpha)}, \]
\[ \Gamma_\alpha(x) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty E_\alpha(-t^\alpha) t^{(x-1)\alpha}(dt)^\alpha, \]
\[ \Gamma_\alpha(x + 1) = \Gamma_\alpha(x + 1) x \Gamma_\alpha(x), \]
\[ \Gamma_\alpha(n + 1) = \Gamma_\alpha(n + 1) n \Gamma_\alpha(n + 1), \quad n \in \mathbb{N}, \]
\[ B_\alpha(x, y) = \int_0^1 (1 - t)^{(x-1)\alpha} t^{(y-1)\alpha}(dt)^\alpha, \]
\[ B_\alpha(x, y) = \frac{\Gamma_\alpha(x) \Gamma_\alpha(y)}{\Gamma_\alpha(x + y)}. \]

8.1.3 Fractional integral transform

Recently, Jumarie [55] introduced and studied the Laplace’s transform of fractional order via Mittag-Leffler function and modified Riemann-Liouville derivative in following manner:

**Definition 8.1.6.** Let \( f(t) \) denote a function which vanishes for negative values of \( t \). Its Laplace’s transform \( \mathcal{L}_\alpha \{ f(t) \} \) of order \( \alpha \) is defined by the following expression, when it is finite:

\[ \mathcal{L}_\alpha \{ f(t) \} := F_\alpha(v) := \int_0^\infty E_\alpha(-v^\alpha t^\alpha) f(t)(dt)^\alpha, \]
\[ := \lim_{M \to \infty} \int_0^M E_\alpha(-v^\alpha t^\alpha) f(t)(dt)^\alpha, \]

where \( v \in \mathbb{C} \), and \( E_\alpha(t) \) is the Mittag-Leffler function.
8.2. ELZAKI’S TRANSFORM

Gupta et al. [40] defined and studied the fractional Sumudu transform for fractional differentiable function as:

**Definition 8.1.7.** Let \( f(t) \) denote a function which vanishes for negative values of \( t \). Its Sumudu’s transform \( \mathcal{G}_\alpha \{ f(t) \} \) of order \( \alpha \) is defined by the following expression, when it is finite:

\[
\mathcal{G}_\alpha \{ f(t) \} := G_\alpha(v) := \int_0^\infty E_\alpha(-t^\alpha) f(vt)(dt)^\alpha,
\]

\[
:= \lim_{M \to \infty} \int_0^M E_\alpha(-t^\alpha) f(vt)(dt)^\alpha, \quad (8.1.31)
\]

where \( v \in \mathbb{C} \), and \( E_\alpha(t) \) is the Mittag-Leffler function.

8.2 Elzaki’s transform

In a series of papers Elzaki et al. [23–29] introduced and studied a new integral transform, which is a modified general Laplace and Sumudu transform, referred as Elzaki’s transform. The Elzaki’s transform was successfully applied to integral equation, ordinary and partial differential equations. The origin of Elzaki’s transform was traced back to the classical Fourier integral.

Consider a class of functions defined as follows:

**Definition 8.2.1.** For a given function in the set \( \mathcal{A} \), defined as

\[
\mathcal{A} = \{ f(t) \mid \exists M, k_1 \text{ and/or } k_2 > 0, \text{ such that } |f(t)| < Me^{|t|/k_j}, \text{ if } t \in (-1)^j \times [0, \infty) \},
\]

where the constant \( M \) must be finite, while \( k_1 \) and \( k_2 \) need not simultaneously exist, and each may be infinite.

**Definition 8.2.2.** For a function \( f(t) \) in class \( \mathcal{A} \), the Elzaki’s transform is defined as:

\[
\mathcal{E}\{ f(t) \} := \mathcal{T}(v) := v \int_0^\infty e^{-\frac{t}{v}} f(t)dt, \quad v \in (-k_1, k_2).
\]
8.2. ELZAKI’S TRANSFORM

\[ := v^2 \int_0^\infty e^{-t} f(vt)dt. \] (8.2.2)

The sufficient conditions for the existence of the Elzaki’s transform are that of exponential order [27].

For the Mittag-Leffler function satisfies the following inequality [55]:

\[ E_\alpha (\lambda (x+y)\alpha) = E_\alpha (\lambda x^\alpha) E_\alpha (\lambda y^\alpha), \] (8.2.3)

where \( \lambda \) is a constant.

8.2.1 Fractional Elzaki’s transform

Definition 8.2.3. Let \( f(t) \) denote a function, which vanishes for negative values of \( t \).

Then the fractional Elzaki’s transform \( \mathcal{E}_\alpha \{ f(t) \} \) is defined by the following expression, when it is finite:

\[ \mathcal{E}_\alpha \{ f(t) \} := \mathcal{T}_\alpha (v) := v^\alpha \int_0^\infty E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) f(t)(dt)^\alpha, \quad 0 < \alpha < 1, \]

\[ := v^{2\alpha} \int_0^\infty E_\alpha (-t^\alpha) f(vt)(dt)^\alpha, \]

\[ := \lim_{M \to \infty} v^{2\alpha} \int_0^M E_\alpha (-t^\alpha) f(vt)(dt)^\alpha, \] (8.2.4)

where \( v \in \mathbb{C} \) and \( E_\alpha(t) \) is the Mittag-Leffler function.

Analogously, the double fractional Elzaki’s transform can be defined in the following way:

Definition 8.2.4. Let \( f(x,t) \) denote a function which vanishes for the negative value of \( x \) and \( t \). Then the double fractional Elzaki’s transform is defined as

\[ \mathcal{E}_{\alpha,\beta} \{ f(t,x) \} := \mathcal{T}_{\alpha,\beta} (u,v) = u^\alpha v^\beta \int_0^\infty \int_0^\infty E_\alpha (-t^\alpha) E_\beta (-x^\beta) f(ut,vx)(dt)^\alpha(dx)^\beta, \] (8.2.5)

where \( 0 < \alpha, \beta < 1 \), \( u,v \in \mathbb{C} \), and \( E_\alpha(t) \) is the Mittag-Leffler function.
8.2. ELZAKI’S TRANSFORM

In particular when $\alpha = \beta$

$$\mathcal{E}_{\alpha,\alpha}\{f(t,x)\}_{u,v} := \mathcal{T}_{\alpha,\alpha}(u,v) = (uv)^\alpha \int_0^\infty \int_0^\infty E_{\alpha}\left(-(t+x)^\alpha\right)f(ut,vx)(dt)^\alpha(dx)^\alpha.$$

(8.2.6)

8.2.2 Duality of fractional order

In this section, the duality relation among the fractional Laplace’s transform (8.1.30), fractional Sumudu transform (8.1.31) and fractional Elzaki’s transform (8.2.1) are discussed.

**Theorem 8.2.5.** If $F_\alpha(v)$ denotes the fractional Laplace’s transform of a function $f(t)$ and $\mathcal{T}_\alpha(v)$ denotes the fractional Elzaki’s transform of the same function, provided the integral exist, then

$$\mathcal{T}_\alpha(v) = v^\alpha F_\alpha\left(\frac{1}{v}\right),$$

(8.2.7)

where $0 < \alpha < 1$.

**Proof.** Let $f(t) \in \mathcal{A}$. Then by the definition of fractional Elzaki’s transform is given by

$$\mathcal{T}_\alpha(v) = \mathcal{E}_\alpha\{f(t)\}_v$$

$$= v^\alpha \int_0^\infty E_{\alpha}\left(-\frac{t^\alpha}{v^\alpha}\right)f(t)(dt)^\alpha.$$

Since $0 < \alpha < 1$, taking $1/v = s$, it follows

$$\mathcal{T}_\alpha(v) = \frac{1}{s^\alpha} \int_0^\infty E_{\alpha}\left(-s^\alpha t^\alpha\right)f(t)(dt)^\alpha$$

$$= v^\alpha F_\alpha\left(\frac{1}{v}\right).$$

This completes the proof. \(\square\)

In fact, the connection of fractional Elzaki’s transform with the fractional Laplace’s transform goes much deeper. Therefore the role of $\mathcal{T}_\alpha(v)$ and $F_\alpha(v)$ can be interchanged.
8.2. ELZAKI’S TRANSFORM

Corollary 8.2.6. Let \( f(t) \in \mathcal{A} \), having \( \mathcal{T}_\alpha(v) \) and \( F_\alpha(v) \) for fractional Elzaki’s transform and fractional Laplace’s transform respectively. Then

\[
F_\alpha(v) = v^\alpha \mathcal{T}_\alpha \left( \frac{1}{v} \right). \tag{8.2.8}
\]

Theorem 8.2.7. If \( G_\alpha(v) \) denotes the fractional Sumudu’s transform of a function \( f(t) \) and \( \mathcal{T}_\alpha(v) \) denotes the fractional Elzaki’s transform of the same function, provided the integral exist, then

\[
\mathcal{T}_\alpha(v) = v^{2\alpha} G_\alpha(v), \tag{8.2.9}
\]

where \( 0 < \alpha < 1 \).

Proof. Let \( f(t) \in \mathcal{A} \). Then by Definition 8.2.3 and Definition 8.1.7, we find that

\[
\mathcal{T}_\alpha(v) = v^{2\alpha} \int_0^{\infty} E_\alpha(-t^\alpha) f(vt) \, dt^\alpha
= v^{2\alpha} \mathcal{S}\{f(t)\}_v
= v^{2\alpha} G_\alpha(v).
\]

This completes the proof. \( \square \)

8.2.3 Basic operational properties

Let \( f(t) \in \mathcal{A} \). Then the following operational formula can be easily obtained:

\[
\mathcal{E}_\alpha \{ t^n \}_v = v^{(n+2)\alpha} \Gamma(n+1) \Gamma(n+\alpha), \quad n \in \mathbb{N}, \tag{8.2.10}
\]

\[
\mathcal{E}_\alpha \{ f(at) \}_v = \frac{\mathcal{T}_\alpha(av)}{a^{2\alpha}}, \quad a \in \mathbb{R}_+, \tag{8.2.11}
\]

\[
\mathcal{E}_\alpha \{ f(t-b) \}_v = E_\alpha(-b^\alpha) \mathcal{T}_\alpha(v), \tag{8.2.12}
\]

\[
\mathcal{E}_\alpha \{ E_\alpha(-a^\alpha t^\alpha) \}_v = \frac{v^{2\alpha} \Gamma(n+\alpha)}{(1+av)^{\alpha}}, \tag{8.2.13}
\]
\[ E_\alpha \{ E_\alpha (-a^{\alpha} t^{\alpha}) f(t) \}_v = (1 + av)^\alpha T_\alpha \left( \frac{v}{1 + av} \right), \]  
\( (8.2.16) \)

\[ E_\alpha \{ t^\alpha f(t) \}_v = v^{2\alpha} D_v^\alpha T_\alpha (v) - v^\alpha \Gamma(\alpha + 1) T_\alpha (v), \]  
\( (8.2.17) \)

\[ E_\alpha \{ t^{2\alpha} f(t) \}_v = v^{4\alpha} D_v^{2\alpha} T_\alpha (v) + \left\{ \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} - 2\Gamma(\alpha + 1) \right\} v^{3\alpha} D_v^\alpha T_\alpha (v), \]  
\( (8.2.18) \)

\[ E_\alpha \{ f^{(\alpha)}(t) \}_v = \frac{T_\alpha (v)}{v^\alpha} - v^\alpha \Gamma(\alpha + 1) f(0), \]  
\( (8.2.19) \)

\[ E_\alpha \{ t^{(n\alpha)} f(t) \}_v = \frac{T_\alpha (v)}{v^{n\alpha}} - \Gamma(1 + \alpha) \sum_{k=0}^{n-1} v^{(2-n+k)\alpha} f^{(k\alpha)} (0), \]  
\( (8.2.20) \)

\[ E_\alpha \left\{ \int_0^t f(u)(du)^\alpha \right\}_v = v^\alpha \Gamma(\alpha + 1) T_\alpha (v). \]  
\( (8.2.21) \)

**Proof of (8.2.10).** By using Definition 8.2.3 to obtain
\[ E_\alpha \{1\}_v = v^{2\alpha} \int_0^\infty E_\alpha (-t^{\alpha}) (dt)^\alpha. \]

Now, using relation (8.1.25) and (8.1.27), we find that
\[ E_\alpha \{1\}_v = v^{2\alpha} \alpha! \Gamma_\alpha (1) = v^{2\alpha} \Gamma(\alpha + 1). \]

**Proof of (8.2.11).** By using the Definition 8.2.3, one has
\[ E_\alpha \{ t^{n\alpha}\}_v = v^{(n+2)\alpha} \int_0^\infty t^{n\alpha} E_\alpha (-t^{\alpha}) (dt)^\alpha \]
\[ = v^{(n+2)\alpha} \int_0^\infty t^{(n+1)\alpha - \alpha} E_\alpha (-t^{\alpha}) (dt)^\alpha. \]

Now applying (8.1.25) and (8.1.27), we obtain
\[ E_\alpha \{ t^{n\alpha}\}_v = v^{(n+2)\alpha} \Gamma(\alpha + 1) \Gamma_\alpha (n + 1) = v^{(n+2)\alpha} \Gamma^{n+1}(\alpha + 1) \Gamma(n + 1), \]
which is the required result.

**Proof of (8.2.12).** The proof is similar to (8.2.11).
8.2. ELZAKI’S TRANSFORM

Proof of (8.2.13). We start from the equality

\[ \mathcal{E}_\alpha \{ f(at) \}_v = v^{2\alpha} \int_0^\infty E_\alpha (-t^\alpha) f(avt)(dt)^\alpha \]

\[ = \alpha v^{2\alpha} \lim_{M \to \infty} \int_0^M (M - t)^{\alpha-1} E_\alpha (-t^\alpha) f(avt)dt. \]

Put \( av = \tau \), it follows

\[ \mathcal{E}_\alpha \{ f(at) \}_v = \alpha \tau^{2\alpha} \lim_{M \to \infty} \int_0^M (M - t)^{\alpha-1} E_\alpha (-t^\alpha) f(\tau t)dt \]

\[ = \mathcal{T}_\alpha(\tau) \]

\[ = \frac{\mathcal{T}_\alpha(\tau)}{a^{2\alpha}}, \]

which yields the result. \( \square \)

Proof of (8.2.14). By using Definition 8.2.3 and (8.1.18), one can easily obtain

\[ \mathcal{E}_\alpha \{ f(t-b) \}_v = \lim_{M \to \infty} v^{2\alpha} \int_0^M E_\alpha (-t^\alpha) f(v(t-b))(dt)^\alpha \]

\[ = \alpha v^{2\alpha} \lim_{M \to \infty} \int_0^M (M - t)^{\alpha-1} E_\alpha (-t^\alpha) f(v(t-b))dt. \]

Changing variable \( t-b \to \tau \) and using the equality (8.2.3), it follows that

\[ \mathcal{E}_\alpha \{ f(t-b) \}_v = \alpha v^{2\alpha} \lim_{M \to \infty} \int_0^{M-b} (M - b - \tau)^{\alpha-1} E_\alpha (-\tau^\alpha) f(v(\tau))(d\tau)^\alpha \]

\[ = E_\alpha (-b^\alpha) v^{2\alpha} \lim_{M \to \infty} \int_0^\infty E_\alpha (-\tau^\alpha) f(v(\tau))(d\tau)^\alpha, \]

which yields the result. \( \square \)

Proof of (8.2.15). We started from the equality

\[ \mathcal{E}_\alpha \{ E_\alpha (-a^\alpha t^\alpha) \}_v = v^{2\alpha} \lim_{M \to \infty} \int_0^M E_\alpha (-t^\alpha) E_\alpha (-a^\alpha v^\alpha t^\alpha)(dt)^\alpha. \]

Changing of variable \( (1 + av)t \to \tau \) and according to (8.2.3) , one has the equality

\[ \mathcal{E}_\alpha \{ E_\alpha (-a^\alpha t^\alpha) \}_v = \alpha v^{2\alpha} \lim_{M \to \infty} \int_0^M (M - t)^{\alpha-1} E_\alpha (-1 + av)^\alpha t^\alpha dt \]
8.2. ELZAKI’S TRANSFORM

\[ \alpha v^{2\alpha} \lim_{M \uparrow \infty} \int_0^{M(1+av)} (M(1 + av) - \tau)^{\alpha-1} E_{\alpha}(-\tau^\alpha) d\tau \]

\[ = \frac{v^{2\alpha}}{(1 + av)^\alpha} \lim_{M \uparrow \infty} \int_0^{M(1+av)} E_{\alpha}(-\tau^\alpha) (d\tau)^\alpha \]

\[ = \frac{v^{2\alpha}}{(1 + av)^\alpha} \int_0^{\infty} E_{\alpha}(-\tau^\alpha) (d\tau)^\alpha. \]

Applying (8.1.25) and (8.1.27), one obtains the required result.

**Proof of (8.2.16).** By using Definition 8.2.3, we find that

\[ \mathcal{E}_\alpha \{ E_{\alpha}(-a^\alpha t^\alpha) f(t) \} \big|_v = v^{2\alpha} \lim_{M \uparrow \infty} \int_0^M E_{\alpha}(-t^\alpha) E_{\alpha}(-a^\alpha v^\alpha t^\alpha) f(vt) (dt)^\alpha \]

Applying the equality (8.2.3) and (8.1.18), one gets

\[ \mathcal{E}_\alpha \{ E_{\alpha}(-a^\alpha t^\alpha) f(t) \} \big|_v = \alpha v^{2\alpha} \lim_{M \uparrow \infty} \int_0^M (M - t)^{\alpha-1} E_{\alpha}(-(1 + av)^\alpha t^\alpha) f(vt) dt. \]

Changing variable \((1 + av)t \rightarrow \tau\) and , it reduces to

\[ \mathcal{E}_\alpha \{ E_{\alpha}(-a^\alpha t^\alpha) f(t) \} \big|_v = \alpha v^{2\alpha} \frac{\alpha v^{2\alpha}}{(1 + av)^\alpha} \lim_{M \uparrow \infty} \int_0^{M(1+av)} (M(1 + av) - \tau)^{\alpha-1} E_{\alpha}(-\tau^\alpha) \]

\[ \times f \left( \frac{v \tau}{1 + av} \right) d\tau. \]

\[ = \frac{v^{2\alpha}}{(1 + av)^\alpha} \lim_{M \uparrow \infty} \int_0^{M(1+av)} E_{\alpha}(-\tau^\alpha) f \left( \frac{v \tau}{1 + av} \right) (d\tau)^\alpha \]

\[ = (1 + av)^\alpha \left( \frac{v}{1 + av} \right)^{\alpha-1} \int_0^{\infty} E_{\alpha}(-\tau^\alpha) f \left( \frac{v \tau}{1 + av} \right) (d\tau)^\alpha \]

\[ = (1 + av)^\alpha T_{\alpha} \left( \frac{v}{1 + av} \right). \]

This completes the proof.

**Proof of (8.2.17).** On using the (8.1.20) and (8.1.11) to find

\[ \mathbf{D}_v^\alpha \left\{ v^\alpha E_{\alpha} \left( -\frac{t^\alpha}{v^\alpha} \right) \right\} = \Gamma(\alpha + 1) E_{\alpha} \left( -\frac{t^\alpha}{v^\alpha} \right) + \frac{t^\alpha}{v^\alpha} E_{\alpha} \left( -\frac{t^\alpha}{v^\alpha} \right). \]

(8.2.22)

Thus, by Definition 8.2.3, one gets

\[ \mathcal{E}_\alpha \{ t^\alpha f(t) \} \big|_v = v^\alpha \int_0^{\infty} E_{\alpha} \left( -\frac{t^\alpha}{v^\alpha} \right) t^\alpha f(t) (dt)^\alpha \]
8.2. ELZAKI’S TRANSFORM

\[ = v^{2\alpha} \int_0^\infty \frac{t^\alpha}{v^\alpha} E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) f(t)(dt)^\alpha. \]

Applying the equality (8.2.22), it reduces to

\[ \mathcal{E}_\alpha \{ t^\alpha f(t) \}_v = v^{2\alpha} \int_0^\infty \left[ D^\alpha_v \left\{ v^\alpha E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) \right\} - \Gamma(\alpha + 1) E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) \right] f(t)(dt)^\alpha \]

\[ = v^{2\alpha} D^\alpha_v \left\{ v^\alpha \int_0^\infty E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) f(t)(dt)^\alpha \right\} \]

\[ - \Gamma(\alpha + 1) v^{2\alpha} \int_0^\infty E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) f(t)(dt)^\alpha, \]

which yields the results.

\[ \square \]

**Proof of (8.2.18).** By using (8.2.17), one can easily establish the result.

**Proof of (8.2.19).** By Definition 8.2.3, we have

\[ \mathcal{E}_\alpha \{ f^{(\alpha)}(t) \}_v = v^\alpha \int_0^\infty E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) f^{(\alpha)}(t)(dt)^\alpha. \]

Applying the integration by part formula (8.1.17) and (8.1.22) lead us to the equation

\[ \mathcal{E}_\alpha \{ f^{(\alpha)}(t) \}_v = v^\alpha \left\{ \Gamma(1 + \alpha) \left[ f(t)E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) \right]_0^\infty + \frac{1}{v^\alpha} \int_0^\infty E_\alpha \left( -\frac{t^\alpha}{v^\alpha} \right) f(t)(dt)^\alpha \right\} \]

\[ - \frac{T_\alpha(v)}{v^\alpha} - v^\alpha \Gamma(1 + \alpha) f(0). \]

This completes the proof.

\[ \square \]

**Proof of (8.2.20).** First find the fractional Elzak’s transform of \( f^{(\alpha)}(t) \).

Let \( g(t) = f^{(\alpha)}(t) \). Then we have

\[ \mathcal{E}_\alpha \{ g^{(\alpha)}(t) \}_v = \mathcal{E}_\alpha \{ f^{(\alpha)}(t) \}_v = \frac{v^\alpha}{v^{2\alpha}} - v^\alpha \Gamma(\alpha + 1) f(0). \]

Hence,

\[ \mathcal{E}_\alpha \{ f^{(2\alpha)}(t) \}_v = \mathcal{E}_\alpha \{ f^{(\alpha)}(t) \}_v = v^\alpha \Gamma(\alpha + 1) f^{(\alpha)}(0). \]

Applying the relation (8.2.19), it follows that

\[ \mathcal{E}_\alpha \{ f^{(2\alpha)}(t) \}_v = \mathcal{E}_\alpha \{ f(t) \}_v - \Gamma(\alpha + 1) \left\{ f(0) - v^\alpha f^{(\alpha)}(0) \right\}. \]
8.2. ELZAKI’S TRANSFORM

Then the result is valid for \( n = 2 \). Suppose that the result is true for \( r \), i.e.

\[
\mathcal{E}_\alpha\{f^{(r\alpha)}(t)\}_v = \frac{T_\alpha(v)}{v^{r\alpha}} - \Gamma(1 + \alpha) \sum_{k=0}^{r-1} v^{(2-r+k)\alpha} f^{(k\alpha)}(0).
\]  
(8.2.23)

Next, we check whether the result is also true for \( r + 1 \).

Put \( g(t) = f^{(r\alpha)} \), then by relation (8.2.19), we obtain

\[
\mathcal{E}_\alpha\{g^{(\alpha)}(t)\}_v = \mathcal{E}\{g(t)\}_v - \Gamma(1 + \alpha) v^{\alpha} f^{(\alpha)}(0).
\]

Now using the above equality (8.2.23), we have

\[
\mathcal{E}_\alpha\{f^{((r+1)\alpha)}(t)\}_v = \frac{T_\alpha(v)}{v^{(r+1)\alpha}} - \Gamma(1 + \alpha) \sum_{k=0}^{r} v^{(2-(r+1)+k)\alpha} f^{(k\alpha)}(0).
\]

Hence by principle of mathematical induction the result (8.2.20) is true for all \( n \in \mathbb{N} \).

This completes the proof. \( \square \)

**Proof of** (8.2.21). By using fractional operational formula for the fractional Laplace’s transform [55], we have

\[
\mathcal{L}_\alpha\left\{ \int_0^t f(u) (du)^\alpha \right\}_v = v^{-\alpha} \Gamma(\alpha + 1) F_\alpha(v).
\]

Now, using the Laplace-Elzaki duality (8.2.7), yields the result. \( \square \)

Let \( f(t, x) \in \mathcal{A} \times \mathcal{A} \). Then some operational formulas for the double Elzaki’s transform are given as under:

\[
\mathcal{E}_{\alpha,\beta}\{f(at)g(bx)\}_{u,v} = \frac{T_{\alpha,\beta}(au)f T_{\alpha,\beta}(bv)g}{a^{2\alpha}b^{2\beta}},
\]  
(8.2.24)

\[
\mathcal{E}_{\alpha,\beta}\{f(at, bx)\}_{u,v} = \frac{T_{\alpha,\beta}(au, bv)}{a^{2\alpha}b^{2\beta}},
\]  
(8.2.25)

\[
\mathcal{E}_{\alpha,\beta}\{D_t^\alpha f(t, x)\}_{u,v} = \frac{T_{\alpha,\beta}(u, v)}{u^\alpha} - u^\alpha \Gamma(\alpha + 1)f(0, x),
\]  
(8.2.26)

\[
\mathcal{E}_{\alpha,\beta}\{D_x^{2\beta} f(t, x)\}_{u,v} = \frac{T_{\alpha,\beta}(u, v)}{v^{2\beta}} - \Gamma(\beta + 1)f(t, 0) - v^\beta \Gamma(\beta + 1)f^\beta(t, 0).
\]  
(8.2.27)

The above operational formulas can be easily derived from the above, so we will give only the statements.
8.2.4 Convolution theorem

**Proposition 8.2.8.** If the convolution of order $\alpha$ of the two functions $f(t)$ and $g(t)$ is denoted by the expression

$$ (f(t) * g(t))_\alpha := \int_0^t f(t - u)g(u)(du)^\alpha, \quad (8.2.28) $$

then the equality

$$ E_{\alpha} \left\{(f(t) * g(t))_\alpha\right\}_v = \frac{1}{v^\alpha} E_{\alpha} \{f(t)\}_v E_{\alpha} \{g(t)\}_v, \quad (8.2.29) $$

hold.

**Proof.** On starting from the Definition 8.2.3, one has

$$ E_{\alpha} \left\{(f * g)_\alpha\right\}_v = v^\alpha \int_0^\infty E_{\alpha} \left(-\frac{t^\alpha}{v^\alpha}\right) \int_0^t f(t - u)g(u)(du)^\alpha (dt)^\alpha $$

$$ = v^\alpha \int_0^\infty E_{\alpha} \left(-\frac{(t - u)^\alpha}{v^\alpha}\right) E_{\alpha} \left(-\frac{u^\alpha}{v^\alpha}\right) \int_0^t f(t - u)g(u)(du)^\alpha (dt)^\alpha. $$

Changing variable $t - u \rightarrow x$ and $u \rightarrow y$, to obtain

$$ E_{\alpha} \left\{(f * g)_\alpha\right\}_v = v^\alpha \int_0^\infty E_{\alpha} \left(-\frac{x^\alpha}{v^\alpha}\right) f(x)(dx)^\alpha \int_0^\infty E_{\alpha} \left(-\frac{y^\alpha}{v^\alpha}\right) g(y)(dy)^\alpha $$

$$ = \frac{1}{v^\alpha} E_{\alpha} \{f(t)\}_v E_{\alpha} \{g(t)\}_v. $$

8.2.5 Inversion formula

**Lemma 8.2.9** (cf. [55]). Define the function

$$ \delta_{\alpha}(x, \varepsilon) = \begin{cases} 
0, & x \notin [0, \varepsilon] \\
\varepsilon^{-\alpha}, & 0 < x \leq \varepsilon,
\end{cases} $$

then one has the limit

$$ \lim_{x \downarrow 0} \delta_{\alpha}(x, \varepsilon) = \delta_{\alpha}(x). $$
8.3. Applications

The relation between the generalized function $\delta_\alpha(x)$ of fractional order $\alpha$, $0 < \alpha < 1$ and Mittag-Leffler function $E_\alpha(x^\alpha)$ is clarified by the following result [55]:

**Lemma 8.2.10.** The following equality hold:

$$\frac{\alpha}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\omega x)^\alpha)(d\omega)^\alpha = \delta_\alpha(x), \quad (8.2.30)$$

where $M_\alpha$ is the period of the complex-valued Mittag-Leffler function defined by the equality $E_\alpha(i(M_\alpha)^\alpha) = 1$.

**Proposition 8.2.11** (cf. [55]). For $0 < \alpha < 1$ the fractional Elzaki’s transform

$$T_\alpha(v) = v^{2\alpha} \int_0^\infty E_\alpha(-t^\alpha)f(v t)(dt)^\alpha \quad (8.2.31)$$

one has the inversion formula

$$f(t) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{+i\infty} v^\alpha E_\alpha(v^\alpha t^\alpha) T_\alpha \left( \frac{1}{v} \right) (dv)^\alpha, \quad (8.2.32)$$

where $M_\alpha$ is the period of the complex-valued Mittag-Leffler function defined by the equality $E_\alpha(i(M_\alpha)^\alpha) = 1$.

**Proof.** By using complex inversion formula of fractional Laplace’s transform [55], we have

$$F_\alpha(v) = \int_0^\infty E_\alpha(-v^\alpha x^\alpha)f(x)(dx)^\alpha, \quad 0 < \alpha < 1,$$

then the inversion formula is given as

$$f(t) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{+i\infty} E_\alpha(v^\alpha x^\alpha)F_\alpha(v)(dv)^\alpha.$$

Applying the Laplace-Elzaki duality relation (8.2.7), we find the desired result. \qed

8.3 Applications

**Example 8.3.1.** Suppose $T_\alpha(v)f$ fractional Elzaki’s transform of $(t)$. Then the solution of fractional differential equation

$$y^{(\alpha)} + \lambda y = f(t), \quad y(0) = 0, \quad 0 < \alpha < 1, \quad (8.3.1)$$
8.3. APPLICATIONS

is given by

\[ y(t) = \frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+i\infty} \frac{v^\alpha}{\lambda + v^\alpha} E_\alpha(v^\alpha t^\alpha) T_\alpha \left( \frac{1}{v} \right)_f (dv)^\alpha, \] (8.3.2)

where \( \lambda \) is a constant.

Proof. Taking fractional Elzaki’s transform on the both side of the (8.3.1), we get

\[ \frac{1}{v^\alpha} T_\alpha(v)_y + \lambda T_\alpha(v)_y = T_\alpha(v)_f \]

\[ \Rightarrow T_\alpha(v)_y = \frac{v^\alpha}{1 + \lambda v^\alpha} T_\alpha(v)_f. \]

Now, applying the inversion formula of fractional Elzaki’s transform, yields the required result.

Consider the following problem on current and charge in a simple electric circuit:

**Example 8.3.2.** The current in a circuit (Fig. 8.1) containing inductance \( L \), resistance \( R \), and capacitance \( C \) with an applied voltage \( E(t) \) is governed by the equation [19]

\[ L \frac{dI}{dt} + RI + \frac{1}{C} \int_0^t I dt = E(t), \] (8.3.3)

where \( L, R \) and \( C \) are constants and \( I(t) \) is the current that is related to the accumulated charge \( Q \) on the condenser at time \( t \) given by

\[ Q(t) = \int_0^t I(t)dt \] so that \( I(t) = \frac{dQ}{dt}. \) (8.3.4)

We have to consider the fractional current equation

\[ I^{(\alpha)}(t) + RI + \frac{1}{C} \int_0^t I(t)(dt)^\alpha = E(t), \] (8.3.5)

with \( I(0) = 0, Q(0) = 0, \) at \( t = 0 \) and \( 0 < \alpha < 1. \)

Proof. Application of factional Elzaki’s transform into equation (8.3.5), with (8.2.19) gives

\[ T_\alpha(v)_I = \frac{v^\alpha}{v^{\alpha\Gamma(\alpha+1)} + Rv^\alpha + L} T_\alpha(v)_E, \] (8.3.6)
8.3. APPLICATIONS

Figure 8.1: Simple electric circuit.

where $\mathcal{T}_\alpha(v)_I$ and $\mathcal{T}_\alpha(v)_E$ are the transformed functions of $I(t)$ and $E(t)$ respectively. Then

$$
\mathcal{T}_\alpha \left( \frac{1}{v} \right)_I = \frac{1}{L} \left\{ \frac{v^\alpha}{v^{2\alpha} + \frac{R}{L} v^\alpha + \frac{\Gamma(\alpha+1)}{CL}} \right\} \mathcal{T}_\alpha \left( \frac{1}{v} \right)_E. \quad (8.3.7)
$$

The characteristic equation is given by

$$
v^{2\alpha} + \frac{R}{L} v^\alpha + \frac{\Gamma(\alpha+1)}{CL} = 0.
$$

The roots of the above characteristic equation is complex with $v^\alpha = -k \pm in$, where $k = \frac{R}{CL}$ and $n^2 = \frac{\Gamma(\alpha+1)}{CL} - \frac{R^2}{4L^2}$, with the negative real part. So, the system is stable.

Taking complex inversion formula for the fractional Elzaki’s Transform on (8.3.6), is given by

$$
I(t) = \frac{1}{L(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} \frac{v^{2\alpha} E_\alpha(v^{\alpha} t^\alpha)}{v^{2\alpha} + \frac{R}{L} v^\alpha + \frac{\Gamma(\alpha+1)}{CL}} \mathcal{T}_\alpha \left( \frac{1}{v} \right)_E \, (dv)^\alpha.
$$

Example 8.3.3. Consider the following fractional integro-differential system

$$
y_1^{(\alpha)}(t) = f(t) + \int_0^t \left[ a y_1(t) + b y_2(t) \right] \, (dt)^\alpha \quad (8.3.8)
$$
8.3. APPLICATIONS

\[ y_2^{(\alpha)}(t) = g(t) + \int_0^t [cy_2(t) - dy_1(t)] (dt)^{\alpha} \]  \hspace{1cm} (8.3.9)

subject to the initial conditions

\[ y_1(0) = p, \quad y_2(0) = q, \]  \hspace{1cm} (8.3.10)

where \(a, b, c, d, p, q\) are constants and \(0 < \alpha < 1\).

**Proof.** Taking the factional Elzaki’s transform into (8.3.8), changes to

\[ \frac{\mathcal{T}_\alpha(v)y_1}{v^\alpha} - v^\alpha \Gamma(\alpha + 1)p = \mathcal{T}_\alpha(v)f + v^\alpha \Gamma(\alpha + 1) [a\mathcal{T}_\alpha(v)y_1 + b\mathcal{T}_\alpha(v)y_2] \]

\[ \Rightarrow [1 - av^2\alpha \Gamma(\alpha + 1)] \mathcal{T}_\alpha(v)y_1 - bv^2\alpha \Gamma(\alpha + 1) \mathcal{T}_\alpha(v)y_2 = v^{2\alpha} \Gamma(\alpha + 1)p + v^\alpha \mathcal{T}_\alpha(v)f. \]  \hspace{1cm} (8.3.11)

Similarly, (8.3.9) becomes

\[ dv^{2\alpha} \Gamma(\alpha + 1) \mathcal{T}_\alpha(v)y_1 + [1 - cv^2\alpha \Gamma(\alpha + 1)] \mathcal{T}_\alpha(v)y_2 = v^{2\alpha} \Gamma(\alpha + 1)q + v^\alpha \mathcal{T}_\alpha(v)g. \]  \hspace{1cm} (8.3.12)

where \(\mathcal{T}_\alpha(v)y_1, \mathcal{T}_\alpha(v)y_2, \mathcal{T}_\alpha(v)f\) and \(\mathcal{T}_\alpha(v)g\) are the fractional Elzaki’s transform of \(y_1(t), y_2(t), f(t)\) and \(g(t)\) respectively.

Solving equation (8.3.11) and (8.3.12) to find \(\mathcal{T}_\alpha(v)y_1\) as

\[ \{bdv^{4\alpha} \Gamma^2(\alpha + 1) + (1 - av^2\alpha \Gamma(\alpha + 1)) (1 - cv^2\alpha \Gamma(\alpha + 1)) \} \mathcal{T}_\alpha(v)y_1 \]

\[ = (1 - cv^2\alpha \Gamma(\alpha + 1)) \{v^{2\alpha} \Gamma(\alpha + 1)p + v^\alpha \mathcal{T}_\alpha(v)f\} \]

\[ + bv^{2\alpha} \Gamma(\alpha + 1) \{v^{2\alpha} \Gamma(\alpha + 1)q + v^\alpha \mathcal{T}_\alpha(v)g\} , \]

or

\[ \mathcal{T}_\alpha(v)y_1 = \mathcal{P}(v), \quad \text{(say).} \]  \hspace{1cm} (8.3.13)

Taking complex inverse Elzaki’s transform, we have

\[ y_1(t) = \frac{1}{(M_\alpha)^{\alpha}} \int_{-i\infty}^{+i\infty} v^\alpha E_\alpha(v^\alpha t^\alpha) \mathcal{P} \left( \frac{1}{v} \right) (dv)^{\alpha}. \]
Similarly by solving for $\mathcal{T}_\alpha(v)_{y_2}$ one can obtain
\[
y_2(t) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{+i\infty} v^\alpha E_\alpha(v^\alpha t^\alpha) Q \left( \frac{1}{v} \right) (dv)^\alpha,
\]
where
\[
Q(v) = \frac{(1 - av^{2\alpha} \Gamma(\alpha + 1)) \{v^{2\alpha} \Gamma(\alpha + 1)q + v^\alpha \mathcal{T}_\alpha(v)g\}}{\{1 - av^{2\alpha} \Gamma(\alpha + 1)\} (1 - cv^{2\alpha} \Gamma(\alpha + 1)) - bdv^{4\alpha} \Gamma^2(\alpha + 1)}
\]
\[
+ \frac{dv^{2\alpha} \Gamma(\alpha + 1) \{v^{2\alpha} \Gamma(\alpha + 1)p + v^\alpha \mathcal{T}_\alpha(v)f\}}{\{(1 - av^{2\alpha} \Gamma(\alpha + 1))(1 - cv^{2\alpha} \Gamma(\alpha + 1)) - bdv^{4\alpha} \Gamma^2(\alpha + 1)\}}.
\]

Example 8.3.4. Consider the fractional one dimensional homogeneous heat like equation
\[
D_t^\alpha U(t,x) = D_x^{2\beta} U(t,x) - 3U(x,t), \tag{8.3.14}
\]
subject to the initial conditions
\[
U(0,x) = f(x), \quad U(t,0) = g(t) \quad \text{and} \quad U^{(\beta)}(t,0) = h(t) \tag{8.3.15}
\]
where $0 < \alpha, \beta < 1$.

Proof. Applying the fractional double Elzaki's transform on the both sides of (8.3.14), one gets
\[
\left\{ \frac{1}{u^\alpha} - \frac{1}{v^{2\alpha}} - 3 \right\} \mathcal{T}_{\alpha,\beta}(u,v) = u^\alpha \Gamma(\alpha + 1)f(x) - \Gamma(\beta + 1)g(t) - v^\beta \Gamma(\beta + 1)h(t),
\]
which gives
\[
\mathcal{T}_{\alpha,\beta}(u,v) = \frac{u^\alpha v^{2\beta} \{u^\alpha \Gamma(\alpha + 1)f(x) - \Gamma(\beta + 1)g(t) - v^\beta \Gamma(\beta + 1)h(t)\}}{v^{2\beta} - 3u^\alpha v^{2\beta} - u^\alpha}.
\]

Now, taking the double inversion of fractional Elzaki's transform, we will find the required result.
\[\square\]
8.4 Conclusion

In this chapter, we proposed a new definition of a fractional order Elzaki’s transform for the fractional differentiable function. We also established various duality relations among fractional Laplace and Sumudu transform. Further, the new definition is applied to solve fractional differential equations.