CHAPTER 4

CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

This chapter is devoted to study the existence and uniqueness of solutions to the Cauchy type problems for the fractional differential equation on the interval \([0, 1]\) in the space of continuous function. The analysis is done for Caputo type modification of the Saigo fractional derivative \([102]\). This chapter is divided into two parts. In the first part, we analyse for a class of single term Cauchy type problem in the space of continuous functions. In the second part, a class of one dimensional two term nonlinear Cauchy type problem is discussed.

4.1 Introduction

In this section, we discuss certain definitions and properties of generalized fractional operators for our mathematical analysis.

\(^1\)Contents of this chapter has appeared as a paper entitled:
“On existence and uniqueness solutions of a class of fractional differential equation”, in Proc. Int. Conf. on Special Functions & their Applications, Jodhpur, India, July 28–30, 2011.
Saigo [111–113] initially defined and studied a generalized fractional operators in connection with certain boundary value problems associated with the Euler-Darboux equation, where the boundary conditions on two characteristics involved generalized fractional operators.

**Definition 4.1.1.** Let $\alpha > 0$, $\beta$ and $\gamma$ be real numbers. Then the left-sided Saigo’s fractional integral operator of order $\alpha$ is defined as

$$
\left( I_{0+}^{\alpha,\beta,\gamma} f \right)(t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \binom{2}{\alpha+\beta,-\gamma;\alpha;1-\frac{\tau}{t}} f(\tau) d\tau,
$$

(4.1.1)

and the right-sided fractional integral in the half-axis $\mathbb{R}^+$ is defined as

$$
\left( I_{-}^{\alpha,\beta,\gamma} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (\tau-t)^{\alpha-1} \tau^{-\alpha-\beta} \binom{2}{\alpha+\beta,-\gamma;\alpha;1-\frac{t}{\tau}} f(\tau) d\tau,
$$

(4.1.2)

where $f(t)$ is a real valued continuous function on the open interval $(0, \infty)$ with the order

$$
f(t) = O\left(t^\lambda\right), \quad t \to 0,
$$

and $\lambda > \max\{0, \beta - \gamma\} - 1$.

Following semi group properties are satisfied by the operator (4.1.1)

$$
I_{0+}^{\alpha,\beta,\gamma} I_{0+}^{\sigma,\delta,\alpha+\gamma} f(t) = I_{0+}^{\alpha+\sigma,\beta+\delta,\gamma} f(t) \quad (4.1.3)
$$

$$
I_{0+}^{\alpha,\beta,\gamma} I_{0+}^{\sigma,\delta,\gamma-\beta-\sigma-\delta} f(t) = I_{0+}^{\alpha+\sigma,\beta+\delta,\gamma-\sigma-\delta} f(t) \quad (4.1.4)
$$

$$
I_{0+}^{\alpha,\beta,\gamma} I_{0+}^{\sigma,\delta,\nu} f(t) = I_{0+}^{\sigma,\delta,\nu} I_{0+}^{\alpha,\beta,\gamma} f(t) \quad (4.1.5)
$$

**Remark 4.1.2.** Erdélyi-Kober and Riemann-Liouville fractional integral operators $E_{0+}^{\alpha,\eta}$ and $I_{0+}^{\alpha}$ respectively are obtained by using the following relations:

$$
I_{0+}^{\alpha,0,\eta} f(t) = E_{0+}^{\alpha,\eta} f(t) \quad \text{and} \quad I_{0+}^{\alpha,-\alpha,\eta} f(t) = I_{0+}^{\alpha} f(t).
$$
Definition 4.1.3. Let $\alpha > 0$, $\beta$ and $\gamma$ be real numbers. Then the left-sided Gauss hypergeometric fractional derivative of order $\alpha$ with $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$ is defined as
\begin{equation}
(LD_{0+}^{\alpha,\beta,\gamma} f)(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha,\beta-n,\alpha+\gamma-n} f(t),
\end{equation}
and the right sided fractional derivative on the half-axis $\mathbb{R}^+$ is defined as
\begin{equation}
(LD_{-}^{\alpha,\beta,\gamma} f)(t) = \left(-\frac{d}{dt}\right)^n I_{-}^{n-\alpha,\beta-n,\alpha+\gamma-n} f(t).
\end{equation}

Lemma 4.1.4 ([139]). Let $\alpha > 0$, $\beta$ and $\gamma$ be real. Then, for $\mu > \max\{0, -(\gamma - \beta)\} - 1$,
\begin{equation}
I_{0+}^{\alpha,\beta,\gamma}(t^\mu) = \frac{\Gamma(\mu + 1) \Gamma(\mu + \gamma - \beta + 1)}{\Gamma(\mu - \beta + 1) \Gamma(\mu + \gamma + 1)} t^{\mu - \beta},
\end{equation}
and for $\mu > \max\{0, -(\alpha + \beta + \gamma)\} - 1$,
\begin{equation}
L_D^{\alpha,\beta,\gamma}_{0+}(t^\mu) = \frac{\Gamma(\mu + 1) \Gamma(\mu + \alpha + \beta + \gamma + 1)}{\Gamma(\mu + \beta + 1) \Gamma(\mu + \gamma + 1)} t^{\mu + \beta}.
\end{equation}

Rao et al. [102] introduced and studied the Caputo type modification of the Saigo fractional derivative (4.1.1) and (4.1.2) in the following way:

Definition 4.1.5. Let $\alpha > 0$, $\beta$ and $\gamma$ be real numbers such that $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$. Then the Caputo type modification of the Saigo fractional derivative of order $\alpha$ is defined as
\begin{equation}
C_D^{\alpha,\beta,\gamma}_{0+} f(t) = I_{0+}^{n-\alpha,\beta-n,\alpha+\gamma-n} D^n f(t),
\end{equation}
and the right-hand sided Caputo type fractional derivative of order $\alpha$ in the half-axis $\mathbb{R}^+$ is defined as
\begin{equation}
C_D^{\alpha,\beta,\gamma}_{-} f(t) = (-1)^n I_{-}^{n-\alpha,\beta-n,\alpha+\gamma-n} D^n f(t),
\end{equation}
where $I_{0+}^{\alpha,\beta,\gamma}$ and $I_{-}^{\alpha,\beta,\gamma}$ are Saigo’s fractional integral operators defined in (4.1.1) and (4.1.2) respectively.
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For \( f(t) \in C^{n}_{\lambda+n} \), the mappings
\[
C^{\alpha,\beta,\gamma}_{0+}: C^{n}_{\lambda+n} \rightarrow C^{n}_{\lambda+\beta} \quad \text{and} \quad I^{\alpha,\beta,\gamma}_{0+}: C^{n}_{\lambda+\beta} \rightarrow C^{n}_{\lambda}
\]
are linear [102].

4.2 Single term fractional differential equation

In this section, we consider the fractional differential equation involving Caputo type modification of Saigo fractional derivative of the form
\[
C^D_{0+}^{\alpha,\beta,\gamma} u(t) = \lambda g(t, u(t)), \quad 0 < t < 1, \quad (4.2.1a)
\]
\[
u(0) = 0, \quad (4.2.1b)
\]
where \( \lambda \in (0, 1) \) and \( 0 < \alpha < 1, \beta > -1, \gamma > -1 \).

4.2.1 Required results

In this section, we introduce some lemmas, which will be used for our study.

**Lemma 4.2.1** (cf. [13, 51]). Let \( c > b > 0 \) and \( t < 1, \ t \neq 0 \). Then
\[
_{2}F_{1}(a, b; c; t) < \begin{cases} J & a < -1, \\ \min\{H, J_1\} & a \in (-1, 0), \end{cases} \quad (4.2.2)
\]
where
\[
J := \left(1 - \frac{b}{c}\right) + \frac{b}{c}(1 - t)^{-a}, \\
H := \left(1 - \frac{bt}{c}\right)^{-a}, \\
J_1 := \left(\frac{b}{c}\right)(1 - t)^{c-a-b} + \left(1 - \frac{b}{c}\right)(1 - t)^{-b}.
\]
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If $a \leq c - 1$, then $\min\{H, J_1\} = H$.

The hypergeometric term in the Saigo operator’s integrand is strictly positive [13,51]

$$\, _2F_1 (\alpha + \beta, -\gamma; \alpha; t) > 0, \quad t \in (0, 1],$$

(4.2.3)

where $\alpha > \beta > 0$.

The following composition relations hold for the operator $C^D_{0+}^{\alpha,\beta,\gamma}$ and $T_{0+}^{\alpha,\beta,\gamma}$ [102]:

**Lemma 4.2.2.** Let $\lambda \geq \max \{0, \beta, - (\alpha + \beta + \gamma)\} - 1$ and $\beta, \gamma$ be real numbers, then the Caputo-type fractional derivative $C^D_{0+}^{\alpha,\beta,\gamma}$ is a left-inverse operator to the fractional integral $T_{0+}^{\alpha,\beta,\gamma}$ for the functions on the space $C_{\lambda+n}$

$$C^D_{0+}^{\alpha,\beta,\gamma} T_{0+}^{\alpha,\beta,\gamma} f(t) = f(t), \quad \text{for} \ f(t) \in C_{\lambda+n}.$$

(4.2.4)

**Lemma 4.2.3.** Let $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$ and $\beta, \gamma$ be real numbers such that $\lambda \geq \max \{0, - (\alpha + \beta + \gamma)\} - 1$, then for $f(t) \in C_{\lambda+n}$ the following relation for composition holds true

$$T_{0+}^{\alpha,\beta,\gamma} C^D_{0+}^{\alpha,\beta,\gamma} f(t) = f(t) - \sum_{i=0}^{n-1} c_i t^i,$$

(4.2.5)

where

$$c_i = \lim_{x \to 0} \frac{1}{i!} D^i f(t).$$

(4.2.6)

**Lemma 4.2.4** (cf. [35]). If $\Re(\gamma) > 0$, $\Re(\sigma) > 0$, $\Re(\gamma + \sigma - \alpha - \beta) > 0$, then

$$\int_0^1 x^{\gamma-1} (1-x)^{\sigma-1} \, _2F_1 (\alpha, \beta; \gamma; x) \, dx = \frac{\Gamma(\gamma) \Gamma(\sigma) \Gamma(\gamma + \sigma - \alpha - \beta)}{\Gamma(\gamma + \sigma - \alpha) \Gamma(\gamma + \sigma - \beta)}.$$  

(4.2.7)

4.2.2 Existence and uniqueness theorems

Let $X = C[0,1]$ be the Banach space of all continuous function endowed with the sup norm, and $K$ be the nonempty closed subset of $X$ defined as

$$K = \{ u \in X : \|u(t)\| \leq L, \ L \geq 0 \}.$$
Consider the following:

(H1) \( g(t, u) = t^\beta f(t, u), \beta > \lambda; \)

(H2) \( \Omega := \frac{\Gamma(\beta + 1)\Gamma(\gamma + 1)}{\Gamma(\alpha + \beta + \gamma + 1)}; \)

(H3) \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a continuous function.

(H4) \( \|f(t, u) - f(t, v)\| < \mathcal{L} \|u - v\|, \) where \( \Omega \mathcal{L} < 1. \)

Using Lemma 4.2.3 on FIVP (4.2.1a)–(4.2.1b), we can obtain the following integral equation

\[
 u(t) = \lambda t^{-\alpha - \beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha - 1} 2F_1 \left( \alpha + \beta, -\gamma; \alpha; 1 - \frac{\zeta}{t} \right) g(\zeta, u(\zeta)) d\zeta, \tag{4.2.8}
\]

Replacing the variable \( \zeta \) by \( ts \) of integral equation (4.2.8) and using (H1), we have

\[
 u(t) = \lambda t^{-\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} 2F_1 \left( \alpha + \beta, -\gamma; \alpha; 1 - s \right) g(ts, u(ts)) ds,
\]

\[
 = \lambda \int_0^1 \frac{(1 - s)^{\alpha - 1}s^{\beta}}{\Gamma(\alpha)} 2F_1 \left( \alpha + \beta, -\gamma; \alpha; 1 - s \right) f(ts, u(ts)) ds. \tag{4.2.9}
\]

Let \( T_\lambda : K \to K \) be the operator defined by

\[
 T_\lambda u(t) := \lambda \int_0^1 \frac{(1 - s)^{\alpha - 1}s^{\beta}}{\Gamma(\alpha)} 2F_1 \left( \alpha + \beta, -\gamma; \alpha; 1 - s \right) f(ts, u(ts)) ds. \tag{4.2.10}
\]

**Lemma 4.2.5.** Assume that (H2) and (H3) hold. Then, for \( 0 < t < 1 \), the operator \( T_\lambda \) is a completely continuous.

**Proof.** For \( u \in K \), we set \( L = \max_{0 \leq t \leq 1, \|u(t)\| \leq t} |f(t, u(t))| + 1. \)

Applying Lemma 4.2.4, we obtain

\[
 |T_\lambda| = \left| \lambda \int_0^1 \frac{(1 - s)^{\alpha - 1}s^{\beta}}{\Gamma(\alpha)} 2F_1 \left( \alpha + \beta, -\gamma; \alpha; 1 - s \right) f(ts, u(ts)) ds \right|,
\]

\[
 \leq L \frac{\Omega}{\Omega} \int_0^1 \frac{(1 - s)^{\alpha - 1}s^{\beta}}{\Gamma(\alpha)} 2F_1 \left( \alpha + \beta, -\gamma; \alpha; 1 - s \right) ds
\]

\[
 = L.
\]
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So, $T_\lambda$ is a bounded operator.

Next, we prove that $T_\lambda$ is continuous.

Since $f$ is a continuous function in compact set $[0, 1] \times \mathbb{R}$, so it is uniformly continuous. Thus, for given $\epsilon > 0$ there exist a $\delta > 0$ such that $\|f(t, u) - f(t, v)\| < \frac{\epsilon}{\Omega}$ for $\|u - v\| < \delta$.

Then we have

$$|T_\lambda u(t) - T_\lambda v(t)| = \left| \lambda \int_0^1 (1 - s)^{\alpha-1} s^\beta \frac{2F_1(\alpha + \beta, -\gamma; \alpha; 1 - s)}{\Gamma(\alpha)} f(ts, u(ts)) ds \right|$$

$$- \lambda \int_0^1 (1 - s)^{\alpha-1} s^\beta \frac{2F_1(\alpha + \beta, -\gamma; \alpha; 1 - s)}{\Gamma(\alpha)} f(ts, v(ts)) ds$$

$$\leq \int_0^1 (1 - s)^{\alpha-1} s^\beta \frac{2F_1(\alpha + \beta, -\gamma; \alpha; 1 - s)}{\Gamma(\alpha)} \times\|f(ts, u(ts)) - f(ts, v(ts))\| ds$$

$$< \epsilon.$$

Next, we prove the equicontinuity of $T_\lambda$.

For $u \in K$ and $t_1, t_2 \in (0, 1)$ we have,

$$|T_\lambda u(t_1) - T_\lambda u(t_2)| \leq \int_0^1 (1 - s)^{\alpha-1} s^\beta \frac{2F_1(\alpha + \beta, -\gamma; \alpha; 1 - s)}{\Gamma(\alpha)}$$

$$\times |f(ts, u(ts)) - f(ts, u(t_2s))| ds$$

$$\leq 2L.$$

Thus $T_\lambda$ is equicontinuous. By Arzelà-Ascoli theorem, we conclude that $T_\lambda$ is completely continuous operator.

Theorem 4.2.6. Assume that conditions (H1), (H2) and (H3) hold. Then for $\lambda \in (0, 1)$, the FIVP (4.2.1a)–(4.2.1b) has at least one solution.

Proof. Consider the operator $T_\lambda$ defined in (4.2.10). Then according to Lemma 4.2.5 the operator $T_\lambda$ is completely continuous and so it is compact in $K$. Hence, by the Theorem 2.2.3, the FIVP (4.2.1a)–(4.2.1b) has at least one solution.
(H5) For \( t \in [0, 1] \) and \( u \in \mathbb{R} \), there exist two constants \( 0 < c_1 < \frac{1}{2\Omega} \), \( c_2 > 0 \), \( 0 < \nu \leq 1 \) such that \( |f(t, u(t))| \leq c_1|u|^{\nu} + c_2 \).

**Theorem 4.2.7.** Assume that the conditions (H1)–(H3) and (H5) hold. Then the FIVP (4.2.1a)–(4.2.1b) has a solution.

*Proof.* Consider the operator \( T_\lambda \) defined in (4.2.10). Let \( U = \{ u \in X : \|u\| < M \} \) be an open subset of \( X \), where \( M > \{1, 2c_2\Omega\} \). Then in view of Lemma 4.2.5, the operator \( T_\lambda : U \to U \) is completely continuous.

Consider there is a point \( u \in \partial U \) and \( \lambda \in (0, 1) \) such that \( u = T_\lambda u \). Then

\[
|T_\lambda u(t)| = \left| \lambda \int_0^1 \frac{(1-s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1-s) f(ts, u(ts)) ds \right|
\]

\[
< \int_0^1 \frac{(1-s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1-s) |f(ts, u(ts))| ds
\]

\[
\leq \int_0^1 \frac{(1-s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1-s) (c_1|u(ts)|^\nu + c_2) ds
\]

\[
\leq \int_0^1 \frac{(1-s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1-s) (c_1 \|u(ts)\|^\nu + c_2) ds
\]

\[
< \int_0^1 \frac{(1-s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1-s) (c_1M^\nu + c_2) ds.
\]

Applying Lemma 4.2.4, we get

\[
|T_\lambda u(t)| < \Omega(c_1M^\nu + c_2)
\]

\[
\leq \Omega(c_1M + c_2)
\]

\[
< \frac{M}{2} + \frac{M}{2} = M,
\]

which implies that \( \|T_\lambda u\| \neq M = \|u\| \) is a contraction. Hence \( u \notin \partial U \). Then by the consequence of Theorem 2.2.4, the operator \( T_\lambda \) has a fixed point \( u \in \overline{U} \). So, the FIVP (4.2.1a)–(4.2.1b) has a solution \( u \in \overline{U} \). 

Following theorem is based on the uniqueness solution of FIVP (4.2.1a)–(4.2.1b).

**Theorem 4.2.8.** Assume that the conditions (H2), (H3) and (H4) hold. Then for \( \lambda \in (0, 1) \), the FIVP (4.2.1a)–(4.2.1b) has a unique solution in \( K \).
Proof. Let \( u, v \in K \). Then
\[
|T_\lambda u(t) - T_\lambda v(t)| < \int_0^1 \frac{(1 - s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1 - s) \times |f(ts, u(ts)) - f(ts, v(ts))|ds
\]
\[
\leq \|f(ts, u(ts)) - f(ts, v(ts))\|
\int_0^1 \frac{(1 - s)^{\alpha-1}s^\beta}{\Gamma(\alpha)} 2F_1(\alpha + \beta, -\gamma; \alpha; 1 - s) ds.
\]

Using Lemma 4.2.4, yields
\[
|T_\lambda u(t) - T_\lambda v(t)| \leq \Omega \|f(ts, u(ts)) - f(ts, v(ts))\|
< \Omega L \|u - v\|.
\]
Thus \( \|T_\lambda u(t) - T_\lambda v(t)\| < \Omega L \|u - v\| \).

Therefore, the operator \( T_\lambda \) is a contraction. Hence, by Theorem 2.2.2, the operator \( T_\lambda \) has a unique fixed point which correspond to the unique solution of FIVP (4.2.1a)–(4.2.1b).

Illustrative Example

Example 4.2.9. Consider the problem
\[
\begin{align*}
C_{D^\frac{1}{2}}^{\frac{1}{2}} u(t) &= 9u^\frac{1}{2}(t) + t^3, \quad 0 < t < 1, \quad (4.2.11a) \\
u(0) &= 0. \quad (4.2.11b)
\end{align*}
\]
Here \( g(t, u) = 9u^\frac{1}{2} + t^3 \).

Then \( f(t, u) = 9t^{-\frac{1}{2}}u^\frac{1}{2} + t^{\frac{5}{2}} \). Set \( 0 < \lambda < \frac{1}{2} \).

Clearly, \( \Omega = 0.0675 \) and \( f(t, u) \) satisfy (H3) for \( 0 < t < 1 \). Also,
\[
|f(t, u) - f(t, v)| = 9 \left| t^{-\frac{1}{2}} \left( u^\frac{1}{2} - v^\frac{1}{2} \right) \right|
\leq 9 \left| t^{-\frac{1}{2}} \right| \left| u^\frac{1}{2} - v^\frac{1}{2} \right|.
\]
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Applying well known equality \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \), we have

\[
|f(t, u) - f(t, v)| \leq 9 \frac{\|u - v\|}{\|u^\frac{2}{3} + u^\frac{1}{3}v^\frac{1}{3} + v^\frac{2}{3}\|}
\leq 9 \|u - v\|.
\]

So, \( f(t, u) \) satisfies \( \|f(t, u) - f(t, v)\| \leq \mathcal{L} \|u - v\| \) and \( \mathcal{L} \Omega = 0.677 < 1 \).

Then, by Theorem 4.2.6 and Theorem 4.2.8, we conclude that FIVP (4.2.11a)–(4.2.11b) has a solution.

4.3 Two term fractional differential equations

This section deals with the existence and uniqueness of solutions for the following nonlinear fractional initial value problem

\[
C D_{0+}^{\alpha,\beta,\gamma} u(t) = f \left( t, u(t), C D_{0+}^{\sigma,\delta,\gamma} u(t) \right), \quad t \in (0,1],
\]

with

\[
u^{(k)}(0) = c_k,
\]

where \( C D_{0+}^{\alpha,\beta,\gamma} \) is the Caputo type modification of Saigo fractional derivative (4.1.10), \( \alpha \in (m - 1, m) \), \( \sigma \in (n - 1, n) \), \( \alpha > \sigma \), \( m, n \in \mathbb{N} \) and \( c_k \geq 0 \); \( k = 0, 1, 2, \ldots, m - 1 \).

4.3.1 Required results

In this section, we mention some results, which are used in the later part of our discussion.

Let \( X = C [0,1] \) denote a Banach space of all continuous function endowed with the sup norm

\[
\|v\|_* = \sup_{t \in [0,1]} \|v(t)\|.
\]
We consider $B$ to be the non empty closed subspace of $X$ defined as
\[ B = \{ v \in X : \|v\|_* < M, M > 0 \} \].  

(4.3.3)

**Lemma 4.3.1.** Assume that $f(t) \in C^m_{\lambda+m}$ and $m - 1 < \sigma < \alpha < m$. Then for $k = 1, \cdots, m - 1$,

\[ C^{\alpha,\beta,\gamma} f_t = C^0_{(\alpha-m+k),(\beta+m-k),(\gamma+m-k)} f^{m-k} (t) \],

and

\[ C^{\alpha,\beta,\gamma} f_t = C^0_{(\alpha-\sigma-\delta,\beta+\gamma+\sigma)} C^{\sigma,\delta,\gamma} f_t \].

(4.3.4)

(4.3.5)

**Proof.** Let $f(t) \in C^m_{\lambda+m}$. For $k = 1, \cdots, m - 1$; $\alpha - m + k \in (k - 1, k)$. So, using Definition 4.1.5, we get

\[ C^{\alpha,\beta,\gamma} f_t = I_{(m-\alpha, -\beta-m, \alpha+\gamma-m)} f(m) (t) \]

\[ = I_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \]

\[ = C^0_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \].

Similarly, for $m - 1 < \sigma < \alpha < m$ and by using the Definition 4.1.5, we have

\[ C^{\alpha,\beta,\gamma} f_t = I_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \]

\[ = I_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \].

Now simplifying the above result by using the semi-group properties (4.1.3)–(4.1.5), we arrive at the following:

\[ C^{\alpha,\beta,\gamma} f_t = I_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \]

\[ = I_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \]

\[ = C^0_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \]

\[ = C^0_{(m-\sigma, -\delta-m, \sigma+\gamma-m)} f(m) (t) \].
First we prove that the solution of initial value problem (4.3.1a)–(4.3.1b) is equivalent to the solution of an integral equation.

**Lemma 4.3.2.** Let \( m-1 < \sigma < \alpha < m \), \( \lambda \geq \max\{\delta, \delta - (\alpha + \beta + \gamma), -(\sigma + \delta + \gamma), 0\} - 1 \) and \( f \) be a continuous function defined on \([0, 1]\). Then \( u \in C^m_{\lambda+m} [0, 1] \) is the solution of the initial value problem (4.3.1a)–(4.3.1b) if and only if

\[
u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \frac{1}{\Gamma(\delta)} f(s, u(s), v(s)) \, ds,
\]

(4.3.6)

where \( v \in C^m_{\lambda+m} [0, 1] \) is a solution of

\[
v(t) = \frac{t^{\sigma+\delta-\alpha-\beta}}{\Gamma(\alpha - \sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\gamma)} \frac{1}{\Gamma(\sigma)} \frac{1}{\Gamma(1-s)} f(s, u(s), v(s)) \, ds.
\]

(4.3.7)

**Proof.** Assume that \( u \in C^m_{\lambda+m} [0, 1] \) is a solution of the initial value problem (4.3.1a)–(4.3.1b), then using Lemma 4.3.1, we have

\[
C^\sigma_{\alpha, \beta, \gamma} u(t) = f(t, u(t), C^\sigma_{\alpha, \beta, \gamma} u(t)).
\]

Replacing \( C^\sigma_{\alpha, \beta, \gamma} u(t) \) by the function \( v(t) \), it reduces to

\[
C^\sigma_{\alpha, \beta, \gamma} v(t) = f(t, u(t), v(t)).
\]

It follows from Lemma 4.2.3 and from the initial conditions (4.3.1b) that

\[
v(t) = \frac{t^{\sigma+\delta-\alpha-\beta}}{\Gamma(\alpha - \sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\gamma)} \frac{1}{\Gamma(\sigma)} \frac{1}{\Gamma(1-s)} f(s, u(s), v(s)) \, ds.
\]

(4.3.7)

Applying operator \( T^\sigma_{\alpha, \beta, \gamma} \) on the expression \( v(t) = C^\sigma_{\alpha, \beta, \gamma} u(t) \), applying Lemma 4.2.3 and the initial conditions (4.3.1b), we obtain

\[
u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \frac{1}{\Gamma(\delta)} f(s, u(s), v(s)) \, ds.
\]

(4.3.6)
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Conversely, let \( v \in C_{\lambda+m}[0,1] \) be the solution of (4.3.7). Since, \( f \) is a continuous function, by using of Lemma 4.2.2–4.2.3, it reduces to

\[
C^\sigma D_{0+}^{\alpha-\beta-\delta,\gamma+n} v(t) = f(t, u(t), v(t)) \quad \text{for} \ t \in (0,1].
\]

Again, since \( u \in C_{\lambda+m}^m[0,1] \). Then by applying Lemma 4.2.2–4.2.3 and the initial conditions (4.3.1b) on (4.3.6), we get

\[
v(t) = C^\sigma D_{0+}^{\alpha,\beta,\gamma} u(t).
\]

Hence, we arrive (4.3.1a).

**Lemma 4.3.3.** Let \( n-1 < \sigma < n \leq m-1 < \alpha < m, \lambda \geq \max\{\delta, -(\alpha + \beta + n), 0\} - 1 \) and \( f \) be a continuous function on \([0,1]\). Then \( u \in C_{\lambda+m}^m[0,1] \) is the solution of the initial value problem (4.3.1a)–(4.3.1b) if and only if

\[
u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} c_{n+k} + \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} v(s) \, ds,
\]

where \( v \in C_{\lambda+m}^n[0,1] \) is a solution of

\[
v(t) = \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \frac{t^{-\alpha-\beta}}{\Gamma(n)} \int_0^t (t-s)^{\alpha-n-1} \, _2F_1\left(\alpha + \beta, -\gamma - n; \alpha - n; 1 - \frac{s}{t}\right)
\]

\[
\times f(s, u(s), \chi(s)) \, ds
\]

and

\[
\chi(t) = \frac{t^\sigma}{\Gamma(n-\sigma)} \int_0^t (t-s)^{n-\sigma-1} \, _2F_1\left(-\sigma - \delta, n - \gamma - \sigma; n - \sigma; 1 - \frac{s}{t}\right) v(s) \, ds.
\]

**Proof.** Let \( u \in C_{\lambda+m}^m[0,1] \) be the solution of initial value problem (4.3.1a)–(4.3.1b), then applying the Lemma 4.3.1, we have

\[
C^\sigma D_{0+}^{\alpha,\beta,\gamma} u(t) = f(t, u(t), C^\sigma D_{0+}^{\alpha,\beta,\gamma} u(t))
\]

\[
\Rightarrow C^\sigma D_{0+}^{\alpha-m+(m-n),\beta-m-(m-n),\gamma+m-(m-n)} u(m-(m-n))(t) = f(t, u(t), C^\sigma D_{0+}^{\alpha,\beta,\gamma} u(t))
\]
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\[ \Rightarrow \mathcal{D}_0^{\alpha-n,\beta+n,\gamma+n} u^{(n)}(t) = f\left(t, u(t), \mathcal{T}_0^{n-\sigma,-\delta-n,\gamma+\sigma-n} u^{(n)}(t)\right). \]

Substitute \( D^n u(t) \) by the function \( v(t) \), to get

\[ C\mathcal{D}_0^{\alpha-n,\beta+n,\gamma+n} v(t) = f(t, u(t), \chi(t)), \]

where \( \chi(t) \) is defined in (4.3.10).

Operating \( \mathcal{T}_0^{\alpha-n,\beta+n,\gamma+n} \) on both sides and applying Lemma 4.2.3, we obtain

\[
v(t) = \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_k + \mathcal{T}_0^{\alpha-n,\beta+n,\gamma+n} f(t, u(t), \chi(t)) \]

\[
= \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_k + \frac{t^{-\alpha-\beta}}{\Gamma(\alpha-n)} \int_0^t (t-s)^{\alpha-n-1} \left( \begin{array}{c} \alpha+\beta, -\gamma-n; \alpha-n; 1 - \frac{s}{t} \end{array} \right)
\times f(s, u(s), \chi(s)).
\]

Again in view of well known result of Kilbas et al. [65] for \( v(t) = \mathcal{D}^n u(t) \), we get

\[ u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} v(s) ds. \]

Conversely, let \( v \in \mathcal{C}_{\lambda+m}[0, 1] \) be the solution of (4.3.9). Then

\[ u^{(n)}(t) = v(t) \]

\[
= \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \frac{t^{-\alpha-\beta}}{\Gamma(\alpha-n)} \int_0^t (t-s)^{\alpha-n-1} \left( \begin{array}{c} \alpha+\beta, -\gamma-n; \alpha-n; 1 - \frac{s}{t} \end{array} \right)
\times f(s, u(s), \chi(s)) ds \]

\[
= \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \mathcal{T}_0^{\alpha-n,\beta+n,\gamma+n} f(t, u(t), \mathcal{T}_0^{n-\sigma,-\delta-n,\sigma+\gamma-n} u^{(n)}(t)) \]

\[
= \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \mathcal{T}_0^{\alpha-n,\beta+n,\gamma+n} f(t, u(t), C\mathcal{D}^{\sigma,\delta,\gamma}_0 u(t)).
\]

Since \( f \) is continuous and \( m-n-1 < \alpha-n \leq m-n \), operate \( C\mathcal{D}_0^{\alpha-n,\beta+n,\gamma+n} \) on the both sides of above equality to obtain
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\[ C^\mathcal{D}_{0+}^{\alpha-n,\beta+n,\gamma+n}u^{(n)}(t) = C^\mathcal{D}_{0+}^{\alpha-n,\beta+n,\gamma+n} \left\{ \sum_{k=0}^{m-n-1} \frac{c_{n+k}}{k!} t^k \right. \\
+ \left. I_{0+}^{\alpha-n,\beta+n,\gamma+n} f(t, u(t), C^\mathcal{D}_{0+}^{\sigma,\delta,\gamma}u(t)) \right\} \]

Now, using Lemma 4.2.2 and Lemma 4.2.3, to find

\[ C^\mathcal{D}_{0+}^{\alpha-n,\beta+n,\gamma+n}u^{(m)}(t) = \sum_{k=0}^{m-n-1} \frac{c_{n+k}}{k!} C^\mathcal{D}_{0+}^{\alpha-n,\beta+n,\gamma+n}t^k + f(t, u(t), C^\mathcal{D}_{0+}^{\sigma,\delta,\gamma}u(t)). \]

This implies

\[ C^\mathcal{D}_{0+}^{\alpha,\beta,\gamma}u(t) = f(t, u(t), C^\mathcal{D}_{0+}^{\sigma,\delta,\gamma}u(t)). \]

Obviously, using (4.3.8) it can be easily shown that \( v^{(m-n)} = u^{(m)} \in C_{\lambda+m}[0,1] \). This proves that \( u(t) \) is the solution of the initial value problem (4.3.1a)–(4.3.1b).

4.3.2 Existence and uniqueness theorems

In this section, we investigate the solution of initial value problem (4.3.1a)–(4.3.1b) for two cases: \( m - 1 < \sigma < \alpha < m \) and \( n - 1 < \sigma < n \leq m - 1 < \alpha < m \).

Case I: When \( m - 1 < \sigma < \alpha < m \)

Throughout this section, we suppose that \( \beta, \gamma \) and \( \delta \) are real numbers such that \( \gamma > 0, \alpha > \beta, \sigma > \delta, \gamma > \delta - 1 \) and \( \beta > \delta - 1 \).

To facilitate our discussion, let us first state the following assumptions:

(H1) \( g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function and there exist three non-negative functions \( L_1(t), L_2(t) \) and \( L_3(t) \) in \( C[0,1] \) such that

(i) \( g(0,0,0) = 0 \) and \( g(t,0,0) \equiv L_1(t) \neq 0 \) uniformly continuous on compact subinterval \( (0,1) \).

(ii) \( \| g(t,x,y) - g(t,\bar{x},\bar{y}) \| \leq L_2(t) \| x - \bar{x} \| + L_3(t) \| y - \bar{y} \|. \)
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(H2) \( g(t, x, y) = t^{\delta-\beta} f(t, x, y) \).

(H3) \( 0 < p < \infty, 0 < q < \infty, r < 1 \) such that \( M := \frac{p + q}{1 - r} > 0 \).

(H4) \( \mathfrak{U} := \frac{\Gamma (\beta - \delta + 1) \Gamma (\gamma + \sigma + 1)}{\Gamma (\alpha + \beta + \gamma - \delta + 1)} \).

For the notational convenience, we denote the following:

\[
p := \sup_{t \in [0,1]} \frac{1}{\Gamma (\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \times \tfrac{1}{2} F_1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right) L_1(tx) dx,
\]

\[
q := \sup_{t \in [0,1]} \sum_{k=0}^{m-1} \frac{c_k t^k}{\Gamma (k + 1) \Gamma (\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta + k} \times \tfrac{1}{2} F_1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right) L_2(tx) dx,
\]

and

\[
r := \sup_{t \in [0,1]} \frac{1}{\Gamma (\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \times \frac{t^{-\delta} x^{-\delta} L_2 (tx) \Gamma (\gamma - \delta + 1)}{\Gamma (1 - \delta) \Gamma (\gamma + \sigma + 1)} + L_3 (tx) \} dx.
\]

Let \( v \in B \). Consider the mapping \( \Upsilon \) defined by

\[
\Upsilon v (t) := \frac{t^{\sigma + \delta - \alpha - \beta}}{\Gamma (\alpha - \sigma)} \int_0^t (t - s)^{\alpha - \sigma - 1} \tfrac{1}{2} F_1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - \tfrac{s}{t} \right) \times f \left( s, u (s), v (s) \right) ds,
\]

\[
:= \frac{t^{\delta - \beta}}{\Gamma (\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} \tfrac{1}{2} F_1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right) \times f \left( tx, u (tx), v (tx) \right) dx,
\]

\[
:= \frac{1}{\Gamma (\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \tfrac{1}{2} F_1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right) \times g \left( tx, u (tx), v (tx) \right) dx,
\]

where \( u(t) \) is from (4.3.6).
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Let \( \lambda \geq \max\{\delta - (\alpha + \beta + \gamma), - (\sigma + \delta + \gamma), 0\} - 1 \). Then, in view of Lemma 4.3.2 and by assumption (H1), the initial value problem (4.3.1a)–(4.3.1b) is equivalent to that of the operator \( \Upsilon \), which has a fixed point in \( B \). First we shall state and prove the uniqueness solution of the initial value problem (4.3.1a)–(4.3.1b).

**Theorem 4.3.4.** Let the assumption (H1), (H2), (H3) hold and \( \gamma \geq \max\{0, \delta\} - 1 \). Then the initial value problem (4.3.1a)–(4.3.1b) has a unique solution on \([0, 1]\).

**Proof.** By Lemma 4.3.2 the initial value problem (4.3.1a)–(4.3.1b) is transformed to the integral equation (4.3.6) and (4.3.7). Let \( v \in \bar{B} \), then

\[
|u| \leq \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{t^{-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{-1} x^{\sigma-1} \binom{F_1}{\alpha+\beta-\delta, \sigma-\gamma; \alpha-\sigma; 1-x} |v(s)| ds.
\]

Since, \( c_k \geq 0; k = 1, 2, \cdots, m - 1 \) and replacing \( s/t \) by \( x \), we get

\[
|u| \leq \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{t^{-\delta}}{\Gamma(\sigma)} \int_0^1 x^{\sigma-1} \binom{F_1}{\alpha+\beta-\delta, \sigma-\gamma; \alpha-\sigma; x} dx.
\]

Using the formula (4.2.7), we have

\[
\|u\| \leq \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{\Gamma(\gamma - \delta + 1)}{\Gamma(1-\delta) \Gamma(\gamma + 1)} \|v\| t^{-\delta}.
\]

Consider the operator \( \Upsilon \) defined in (4.3.11), then

\[
\|\Upsilon v\|_* = \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times 2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) |g(tx, u(tx), v(tx))| dx \\
\leq \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} 2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) \times \{|g(tx, u(tx), v(tx)) - g(tx, 0, 0)| + |g(tx, 0, 0)|\} dx \\
\leq \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} 2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) \times \{L_1(tx) + L_2(tx) \|u\| + L_3(tx) \|v\|\} dx \\
\leq p + q + rM \leq M.
\]

(4.3.12)
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This proves that \( \Upsilon \) maps \( \overline{B} \) into itself.

Next we prove that \( \Upsilon \) is a contraction map. Let \( v_1, v_2 \in \overline{B} \). Then for \( t \in [0, 1] \), we have

\[
|u_1 - u_2| = \left| \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} 2F1 \left( \sigma + \delta, -\gamma; \sigma; 1 - \frac{s}{t} \right) (v_1 - v_2) ds \right| 
\leq \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} 2F1 \left( \sigma + \delta, -\gamma; \sigma; 1 - \frac{s}{t} \right) \|v_1 - v_2\| ds.
\]

Changing the variable of the integral \( s/t \) by \( x \), to get

\[
|u_1 - u_2| \leq \frac{t^{-\delta}}{\Gamma(\sigma)} \|v_1 - v_2\| \int_0^1 x^{\sigma-1} 2F1 \left( \sigma + \delta, -\gamma; \sigma; x \right) dx.
\]

It follows from (4.2.7) that

\[
\|u_1 - u_2\| \leq \frac{\Gamma(\gamma - \delta + 1)}{\Gamma(1 - \delta) \Gamma(\gamma + \sigma + 1)} t^{-\delta} \|v_1 - v_2\|. \quad (4.3.13)
\]

Hence,

\[
\|\Upsilon v_1(t) - \Upsilon v_2(t)\| \leq \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha-\sigma-1} x^{\beta-\delta}
\times 2F1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right)
\times \|g(tx, u_1(tx) - g(tx, u_2(tx), v_1(tx)) - g(tx, u_2(tx), v_2(tx))\| dx.
\]

Using the inequality (4.3.13) and assumption (H1), to find

\[
\|\Upsilon v_1(t) - \Upsilon v_2(t)\| \leq \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha-\sigma-1} x^{\beta-\delta}
\times 2F1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right)
\times \left\{ L_2(tx) \|u_1 - u_2\| + L_3(tx) \|v_1 - v_2\| \right\} dx
\leq \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha-\sigma-1} x^{\beta-\delta}
\times 2F1 \left( \alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x \right)
\times \left\{ L_2(tx) \left( \frac{(tx)^{-\delta} \Gamma(\gamma - \delta + 1)}{\Gamma(1 - \delta) \Gamma(\gamma + \sigma + 1)} + L_3(tx) \right) \right\} \|v_1 - v_2\| dx
\leq r\|v_1 - v_2\|_*.
\]
Thus \( \| \mathcal{Y} v_1 (t) - \mathcal{Y} v_2 (t) \|_* \leq r \| v_1 - v_2 \|_* \). Since, \( r < 1 \), this implies that the operator \( \mathcal{Y} \) is a contraction in \( \bar{B} \). As a consequence of Theorem 2.2.2 the operator \( \mathcal{Y} \) has a unique fixed point. Thus, this fixed point is a solution of the initial value problem (4.3.1a)–(4.3.1b), which completes the proof. \( \square \)

Next we state and prove the existence of the solution for the initial value problem (4.3.1a)–(4.3.1b), by using the Schauder’s fixed point Theorem 2.2.3.

**Theorem 4.3.5.** Let the assumption (H1)–(H4) hold. Then the initial value problem (4.3.1a)–(4.3.1b) has at least one solution in the space \( B \).

**Proof.** Consider the operator defined in (4.3.11) and \( E = \{ v \in B : \| v \|_* \leq \kappa \} \), where

\[
\| \mathcal{Y}v \|_* \leq p + q + r M := \kappa.
\]

In the view of the proof of Theorem 4.3.4, the operator \( \mathcal{Y} \) maps \( E \) into itself. Next, we prove that the operator \( \mathcal{Y} \) is compact and continuous. We shall divide the proof in following steps:

**Step 1:** \( \mathcal{Y} \) is continuous.

Let \( \{ v_n \} \) be a sequence in \( E \) such that \( v_n \to v \) as \( n \to \infty \). Then, for \( t \in [0, 1] \) it can be easily establish

\[
\| u_n - u \| \leq \frac{\Gamma (\gamma - \delta + 1)}{\Gamma (1 - \delta) \Gamma (\gamma + \sigma + 1)} t^{-\delta} \| v_n - v \|. \tag{4.3.14}
\]

Then

\[
\| \mathcal{Y} v_n (t) - \mathcal{Y} v (t) \| \leq \frac{1}{\Gamma (\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \times \left( \begin{array}{c}
2F_1 (\alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x) \\
\| g (tx, u_n (tx), v_n (tx)) - g (tx, u (tx), v (tx)) \|
\end{array} \right) dx.
\]

Using the inequality (4.3.14) and assumption (H1), we obtain
\[ \| \Upsilon v_n(t) - \Upsilon v(t) \| \leq \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \]
\[ \times 2F_1 (\alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x) \]
\[ \times \left\{ L_2(tx) \frac{(tx)^{-\delta} \Gamma(\gamma - \delta + 1)}{\Gamma(1 - \delta) \Gamma(\gamma + \sigma + 1)} + L_3(tx) \right\} \]
\[ \times \| v_n - v \| \, dx \]
\[ \leq r \| v_n - v \|_* . \]

This implies that
\[ \| \Upsilon v_n(t) - \Upsilon v(t) \|_* \leq r \| v_n - v \|_* . \]

Since \( v_n \to v \) as \( n \to \infty \), hence \( \| \Upsilon v_n(t) - \Upsilon v(t) \|_* \to 0 \) as \( n \to \infty \).

Step 2: The operator \( \Upsilon \) is bounded in \( E \) into itself. The proof is identical to the proof of Theorem 4.3.4.

Step 3: The operator \( \Upsilon \) is equicontinuous on \( E \).

Let \( v \in E \) and \( t_1, t_2 \in [0, 1] \) such that \( t_1 < t_2 \). Then
\[ \| \Upsilon v(t_2) - \Upsilon v(t_1) \| \leq \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \]
\[ \times 2F_1 (\alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x) \]
\[ \times \left\{ \| g(t_2x, u(t_2x), v(t_2x)) - g(t_2x, 0, 0) \| \right\} \]
\[ + \| g(t_2x, 0, 0) - g(t_1x, 0, 0) \| \]
\[ + \| g(t_1x, 0, 0) - g(t_1x, u(t_1x), v(t_1x)) \| \} \, dx . \]

Using assumption (H1), we find
\[ \| \Upsilon v(t_2) - \Upsilon v(t_1) \| \leq \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \]
\[ \times 2F_1 (\alpha + \beta - \sigma - \delta, -\sigma - \gamma; \alpha - \sigma; 1 - x) \]
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\[ \times \{ ||L_1(t_2 x) - L_1(t_1 x)|| + (L_2(t_1 x) + L_2(t_2 x)) \|u\| \\
+ (L_3(t_1 x) + L_3(t_2 x)) \|v\| \} \, dx. \]

Since by the assumption (H1), \( L_1(t) \) is uniformly continuous on the \([0,1]\). So for given \( \varepsilon > 0 \), we find \( \rho > 0 \) such that \( ||L_1(t_2) - L_1(t_1)|| < \varepsilon = \frac{\rho}{\varepsilon} \) for \( \| t_2 - t_1 \| < \rho \). Then

\[ \| \Upsilon v(t_2) - \Upsilon v(t_1) \|_* \leq \rho + 2q + 2r\kappa, \]

which is independent of \( v \).

Hence, the operator \( \Upsilon \) is relatively compact. As a consequence of the Arzelà-Ascoli theorem, the operator \( \Upsilon \) is compact and continuous. By the Theorem 2.2.3, we conclude that the operator \( \Upsilon \) has at least one solution of the initial value problem (4.3.1a)–(4.3.1b). This completes the proof.

\[ \Box \]

**Theorem 4.3.6.** Let the assumption \((H1)–(H3)\) hold. Then the initial value problem (4.3.1a)–(4.3.1b) has a solution.

**Proof.** Let \( \mathcal{U} = \{ v \in B : \|v\|_* < R \} \) with \( R = \frac{p + q}{1 - r} > 0 \).

Consider the operator \( \Upsilon \) defined in (4.3.11). Then by (H1) and Arzelà-Ascoli theorem, it can be easily shown that the operator \( \Upsilon : \mathcal{U} \to \mathcal{U} \) is compact and continuous.

Next we show that \( \mathcal{U} \) is a priori bounds.

If possible assume that there is a solution \( v \in \partial \mathcal{U} \) such that

\[ v = \lambda \Upsilon v \quad \text{with} \quad \lambda \in (0,1). \quad (4.3.15) \]

By the assumption that the \( v \) is a solution for \( \lambda \in (0,1) \), one can obtain

\[ \|v\|_* = \sup_{t \in [0,1]} \left| \frac{\lambda}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \right. \\
\left. \times \sum_{\alpha + \beta - \sigma - \delta, \sigma - \gamma; \alpha - \sigma; 1 - x} g(\alpha, \beta - \sigma - \delta, \sigma - \gamma; \alpha - \sigma; 1 - x) \right| \, dx \]

\[ < \sup_{t \in [0,1]} \frac{\lambda}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\beta - \delta} \]
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\[ \times {}_2 F_1 (\alpha + \beta - \sigma - \delta, \sigma - \gamma; \alpha - \sigma; 1 - x) \mid g (tx, u (tx), v (tx)) \mid dx \]

\[ \leq p + q + r \|v\|_{\ast} . \]

Therefore \( v \notin \partial U \). By the Theorem 2.2.4, \( \Upsilon \) has a fixed point in \( \overline{U} \), which is a solution of FIVP (4.3.1a)–(4.3.1b). This completes the proof.

Suppose that according to Theorem 4.3.6, \( v_0 \) is the fixed point. So by Lemma 4.3.2 for \( t \in [0, 1] \) one can obtain

\[ u_0 (t) = \sum_{k=0}^{m-1} \frac{t^k C_k}{k!} + \frac{t^{\sigma - \delta}}{\Gamma (\sigma)} \int_0^t (t - s)^{\sigma - 1} {}_2 F_1 \left( \sigma + \delta, -\gamma; \sigma; 1 - \frac{s}{t} \right) v_0 (s) ds. \quad (4.3.16) \]

Illustrative Example

Two examples are presented in this section in order to confirm the effectiveness of the proposed theorem.

Example 4.3.7. Consider the fractional differential equation

\[ {}_0^c D_{1.7}^{1.1.5} u (t) = \frac{(|u(t)| + |D_{0+}^{1.5,1.3,1.5} u(t)|)}{120 \sqrt{\pi} (1 + |u(t)| + |D_{0+}^{1.5,1.3,1.5} u(t)|)} + 14t^6, \quad t \in [0, 1], \]

\[ (4.3.17a) \]

subjected to the initial condition

\[ u(0) = 2, \quad u'(0) = 5. \quad (4.3.17b) \]

Set \( g(t, u, v) := \frac{t^{0.3} (u + v)}{120 \sqrt{\pi} (1 + u + v)} + 14t^{0.3} \).

Let \( t \in [0, 1] \) and \( u, v, \bar{u}, \bar{v} \in \mathbb{R}_+ \), then

\[ |g(t, u, v) - g(t, \bar{u}, \bar{v})| = \frac{t^{0.3}}{120 \sqrt{\pi}} \left( \frac{|u + v|}{1 + u + v} - \frac{|\bar{u} + \bar{v}|}{1 + \bar{u} + \bar{v}} \right) \]

\[ \leq \frac{t^{0.3}}{120 \sqrt{\pi}} \left( \frac{|u - \bar{u}| + |v - \bar{v}|}{(1 + u + v)(1 + \bar{u} + \bar{v})} \right) \]

\[ \leq \frac{t^{0.3}}{120 \sqrt{\pi}} \left( |u - \bar{u}| + |v - \bar{v}| \right). \]
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\[ \|g(t,u,v) - g(t,\bar{u},\bar{v})\| \leq L_2(t)\|u - \bar{u}\| + L_3(t)\|v - \bar{v}\|, \]

where
\[ L_2(t) = \frac{t^{0.3}}{120\sqrt{\pi}} \quad \text{and} \quad L_3(t) = \frac{t^{0.3}}{120\sqrt{\pi}}. \]

Also, \( g(t,0,0) := L_1(t) = 14t^{0.3}. \)

Hence, the FIVP (4.3.17a)–(4.3.17b) satisfies (H1). Then

\[
p = \sup_{t \in [0,1]} \frac{14}{\Gamma(0.2)} \int_0^1 (1-x)^{(0.2-1)x} F_1(-0.1, -3; 0.2; 1-x) (tx)^{0.3} dx
\]
\[
= \sup_{t \in [0,1]} \frac{14t^{0.3}}{\Gamma(0.2)} \int_0^1 \tau^{0.2-1}(1-\tau)^{7-1} F_1(-0.1, -3; 0.2; \tau) d\tau \quad (x \to 1 - \tau)
\]
\[
\leq \frac{14\Gamma(7)\Gamma(10.3)}{\Gamma(7.3)\Gamma(10.2)} \quad \text{(Using (4.2.7))}
\]
\[
= 9.9561 < \infty,
\]

\[
q = \sup_{t \in [0,1]} \sum_{k=0}^1 \frac{c_k t^{k+0.3}}{120\sqrt{\pi} \Gamma(k+1)\Gamma(0.2)} \int_0^1 (1-x)^{(0.2-1)x} F_1(-0.1, -3; 0.2; 1-x) dx,
\]
\[
= \sup_{t \in [0,1]} \frac{2t^{0.3}}{120\sqrt{\pi} \Gamma(0.2)} \int_0^1 \tau^{0.2-1} F_1(-0.1, -3; 0.2; \tau) d\tau
\]
\[
+ \sup_{t \in [0,1]} \frac{5t^{k+0.3}}{120\sqrt{\pi} \Gamma(0.2)} \int_0^1 \tau^{0.2-1} F_1(-0.1, -3; 0.2; \tau) d\tau \quad (x \to 1 - \tau)
\]
\[
\leq \frac{2\Gamma(4.3)}{120\sqrt{\pi} \Gamma(1.3)\Gamma(4.2)} + \frac{5\Gamma(5.3)}{120\sqrt{\pi} \Gamma(2.3)\Gamma(5.2)} \quad \text{(Using (4.2.7))}
\]
\[
= 0.012 + 0.0236
\]
\[
= 0.0356 < \infty,
\]

and
\[
r = \sup_{t \in [0,1]} \frac{1}{\Gamma(0.2)} \int_0^1 (1-x)^{(0.2-1)x} F_1(-0.1, -3; 0.2; 1-x)
\]
\[
\times \left\{ (xt)^{-1.2} \Gamma(1.2) \frac{\Gamma(0.2)\Gamma(4)}{120\sqrt{\pi}} + (xt)^{0.3} \right\} dx
\]
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\[
\frac{\Gamma(-0.5) \Gamma(2.8)}{720 \sqrt{\pi} \Gamma(-0.2) \Gamma(2.7)} + \frac{\Gamma(4.3)}{120 \sqrt{\pi} \Gamma(1.3) \Gamma(4.2)} = 5.1790 \times 10^{-4} + 0.006 \\
\approx 0.0065 < 1.
\]

Choose \( M > 10.0571. \)
Hence, by Theorem 4.3.4 and Theorem 4.3.5, the FIVP (4.3.17a)–(4.3.17b) has at least one solution defined in \([0,1]\).

In next example, we consider the classical Caputo’s fractional derivative.

**Example 4.3.8.** Consider the fractional differential equation

\[
\frac{CD^\alpha_0+ u(t)}{} - \frac{\Gamma(\alpha - \sigma + \rho + 1)}{4\Gamma(\rho + 1)} t^{\rho} \frac{CD^\sigma_0+ u(t)}{} - \frac{1}{2} u(t) = t^\tau, \quad t \in [0,1], \quad (4.3.18a)
\]

and

\[
\frac{u^{(k)}(0)}{} = c_k, \quad (4.3.18b)
\]

where \( \alpha, \sigma \in (m-1,m), \rho > -1, \tau > -1 \) and \( c_k \in \mathbb{R}_+, \) \( k = 0, 1, \ldots, m-1. \)

The above equation (4.3.18a) can be written as

\[
\frac{CD^{\alpha,-\alpha,\gamma}_0+ u(t)}{} = t^\tau + \frac{1}{2} u(t) + \frac{\Gamma(\alpha - \sigma + \rho + 1)}{4\Gamma(\rho + 1)} t^{\rho} \frac{CD^{\sigma,-\sigma,\gamma}_0+ u(t)}{} \quad (\gamma > 0). \quad (4.3.19)
\]

Setting

\[
g(\tau, u, v) \equiv t^{\alpha-\sigma+\tau} + \frac{1}{2} t^{\alpha-\sigma} u + \frac{\Gamma(\alpha - \sigma + \rho + 1)}{4\Gamma(\rho + 1)} t^{\alpha-\sigma+\rho} v.
\]

Clearly, for \( t \in [0,1], \ L_1(t) = t^{\alpha-\sigma+\tau} \) satisfies the condition of (H1).
Also, for \( t \in [0,1] \) and \( u, \bar{u}, v, \bar{v} \in \mathbb{R}_+, \) we have

\[
\| g(t, u, v) - g(t, \bar{u}, \bar{v}) \| \leq L_2(t) \| u - \bar{u} \| + L_3(t) \| v - \bar{v} \|,
\]

where

\[
L_2(t) = \frac{1}{2} t^{\alpha-\sigma} \quad \text{and} \quad L_3(t) = \frac{\Gamma(\alpha - \sigma + \rho + 1)}{4\Gamma(\rho + 1)} t^{\alpha-\sigma+\rho}.
\]
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Then by using (4.2.7), we have

\[
p = \sup_{t \in [0, 1]} \frac{1}{\Gamma(\alpha - \sigma)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\tau} \binom{0}{-\sigma - \gamma; \alpha - \sigma; 1 - x} dx
\]

\[
= \sup_{t \in [0, 1]} \frac{\Gamma(\tau + 1)}{\Gamma(\alpha - \sigma + \tau + 1)} t^{\alpha - \sigma + \tau}
\]

\[
\leq \frac{\Gamma(\tau + 1)}{\Gamma(\alpha - \sigma + \tau + 1)},
\]

\[
q = \sup_{t \in [0, 1]} \sum_{k=0}^{m-1} \frac{c_k t^{\alpha - \sigma + \rho + k} \Gamma(\alpha - \sigma + \rho + 1)}{4 \Gamma(k + 1) \Gamma(\alpha - \sigma)} \Gamma(\rho + 1) \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\rho + k} \binom{0}{-\sigma - \gamma; \alpha - \sigma; 1 - x} dx
\]

\[
\leq \sum_{k=1}^{m-1} \frac{c_k \Gamma(\alpha - \sigma + \rho + 1) \Gamma(\rho + 1) \Gamma(\alpha - \sigma + \rho + k + 1)}{4 \Gamma(k + 1) \Gamma(\rho + 1) \Gamma(\alpha - \sigma + \rho + k + 1)}
\]

\[
= \frac{\Gamma(\alpha - \sigma + \rho + 1)}{4 \Gamma(\rho + 1) \Gamma(\alpha - \sigma + \rho + k + 1)} \sum_{k=1}^{m-1} \frac{c_k \Gamma(\rho + k + 1)}{\Gamma(k + 1)}.
\]

Obviously, under the conditions, \( p \) and \( q \) are finite. Again by using (4.2.7), we obtain

\[
r = \sup_{t \in [0, 1]} \frac{t^\alpha}{2 \Gamma(\alpha - \sigma) \Gamma(\sigma + 1)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\sigma} \binom{0}{-\sigma - \gamma; \alpha - \sigma; 1 - x} dx
\]

\[
+ \sup_{t \in [0, 1]} \frac{t^{\alpha - \sigma + \rho} \Gamma(\alpha - \sigma + \rho + 1)}{4 \Gamma(\alpha - \sigma) \Gamma(\rho + 1)} \int_0^1 (1 - x)^{\alpha - \sigma - 1} x^{\rho} \binom{0}{-\sigma - \gamma; \alpha - \sigma; 1 - x} dx
\]

\[
\leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1.
\]

Setting

\[
\frac{4 \Gamma(\tau + 1)}{\Gamma(\alpha - \sigma + \tau + 1)} + \frac{\Gamma(\alpha - \sigma + \rho + 1)}{2 \Gamma(\rho + 1) \Gamma(\alpha - \sigma + \rho + k + 1)} \sum_{k=1}^{m-1} \frac{c_k \Gamma(\rho + k + 1)}{\Gamma(k + 1)} < M.
\]

Hence, as a consequence of the Theorem 4.3.4 and Theorem 4.3.5, the FIVP (4.3.18a)-(4.3.18b) has at least one solution defined in \([0, 1]\).
Case II: When \( n - 1 < \sigma < n \leq m - 1 < \alpha < m \)

Throughout this section, we suppose that \( \beta, \gamma \) and \( \delta \) are real numbers such that \( \gamma > 0, \alpha > \beta, \sigma > \delta, \gamma + \delta > -1 \) and \( n + \beta > -1 \).

Consider the following:

(H5) \( g(t, u, v) := t^{-n - \beta} f(t, u, v) \).

(H6) \( 0 < p^* < \infty, 0 < q^* < \infty, r^* < 1 \) such that \( M := \frac{p^* + q^*}{1 - r^*} > 0 \).

(H7) \( \Xi := \frac{\Gamma(n + \beta + 1) \Gamma(n + \gamma + 1)}{\Gamma(\alpha + \beta + \gamma + n + 1)}, n \in \mathbb{N} \).

We denote

\[
p^* = \sup_{t \in [0,1]} |\Phi(t)| + \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - n)} \int_0^1 (1 - x)^{\alpha - n - 1} x^{n + \beta} \times \text{2F1} (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) L_1(\text{xt}) \, dx,
\]

\[
q^* = \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - n)} \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| \int_0^1 (1 - x)^{\alpha - n - 1} x^{n + \beta} \times \text{2F1} (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) L_2(\text{xt}) \, dx
\]

and

\[
r^* = \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - n)} \int_0^1 (1 - x)^{\alpha - n - 1} x^{n + \beta} \text{2F1} (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x)
\times \left( \frac{(xt)^n}{\Gamma(n + 1)} L_2(\text{xt}) + \frac{\Gamma(\sigma + \delta + \gamma + 1)(xt)^{n + \delta}}{\Gamma(n + \delta + 1) \Gamma(\gamma + 1)} L_3(\text{xt}) \right) \, dx.
\]

Let \( v \in \overline{B} \). Define the mapping \( T \) by

\[
T v(t) := \sum_{k=0}^{m - n - 1} \frac{t^k}{k!} c_{\alpha + k} + \frac{t^{-\alpha - \beta}}{\Gamma(\alpha - n)} \int_0^t (t - x)^{\alpha - n - 1} \text{2F1} \left( \alpha + \beta, -\gamma - n; \alpha - n; 1 - \frac{x}{t} \right)
\times f(x, u(x), \chi(x)) \, dx
\]

\[
:= \Phi(t) + \frac{1}{\Gamma(\alpha - n)} \int_0^1 (1 - x)^{\alpha - n - 1} x^{n + \beta} \text{2F1} (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x)
\times g(t x, u(t x), \chi(t x)) \, dx,
\]

\[
(4.3.20)
\]
where $\Phi(t) := \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k}$; and $u(t)$ and $\chi(t)$ are from (4.3.8) and (4.3.10) respectively.

Let $\lambda \geq \max\{0, -(n+\alpha+\beta+\gamma)\} - 1$. Then, by Lemma 4.3.3 and (H1), the initial value problem (4.3.1a)–(4.3.1b) is equivalent to that of the operator $T$ which has a fixed point in $B$. The theorem for uniqueness solution of the FIVP (4.3.1a)–(4.3.1b) is as under.

**Theorem 4.3.9.** Let the assumption (H1), (H5), (H6) hold. Then the initial value problem (4.3.1a)–(4.3.1b) has a unique solution on $[0, 1]$.

**Proof.** Consider the operator $T$ defined in (4.3.20). Here, we shall make use of Banach contraction principle to prove that $T$ has a fixed point. First we show $TB \subset B$.

Let $v \in B$, then

$$
\|u(t)\| \leq \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| + \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} \|v(s)\| ds
$$

$$
\leq \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| + \frac{\|v\| t^n}{\Gamma(n)} \int_0^t (t-s)^{n-1} ds.
$$

By the definition of Beta function, we have

$$
\|u(t)\| \leq \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| + \frac{\|v\| t^n}{\Gamma(n+1)}. \tag{4.3.21}
$$

Again $\sigma + \delta > -1$, by applying Lemma 4.2.4, we obtain

$$
\|\chi(t)\| \leq \frac{t^{n+\delta}}{\Gamma(n-\sigma)} \int_0^1 (1-s)^{n-\sigma-1} \binom{\frac{2s}{1-s} \alpha}{\alpha} \binom{\frac{2s}{1-s} \beta}{-\gamma-n} \binom{\frac{2s}{1-s} \gamma}{\alpha-n} (1-s) \|v(s)\| ds
$$

$$
\leq \frac{\Gamma(\sigma + \delta + \gamma + 1) \|v\| t^{n+\delta}}{\Gamma(n+\delta+1) \Gamma(\gamma+1)}. \tag{4.3.22}
$$

Hence, by the assumption (H1), we have

$$
\|Tv\| \leq \sup_{t \in [0,1]} |\Phi(t)| + \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta}
$$

$$
\times \binom{\alpha+\beta}{\alpha-n} \binom{-\gamma-n}{\alpha-n} (1-x) \|g(tx, u(tx), \chi(tx))\| dx
$$
It follows from (4.3.21) and (4.3.22) that

\[
\|Tv\|_{\ast} \leq \sup_{t \in [0,1]} |\Phi(t)| + \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - n)} \int_{0}^{1} (1 - x)^{\alpha - n - 1} x^{n + \beta} \\
\times 2F_1 (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \\
\times \{ \| g (tx, u(tx), \chi(tx)) - g (tx, 0, 0) \| \} \, dx.
\]

It follows from (4.3.21) and (4.3.22) that

\[
\|Tv\|_{\ast} \leq \sup_{t \in [0,1]} |\Phi(t)| + \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha - n)} \int_{0}^{1} (1 - x)^{\alpha - n - 1} x^{n + \beta} \\
\times 2F_1 (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \\
\times \left\{ L_1 (xt) + \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} |c_k| L_2 (xt) \\
+ \left( \frac{\Gamma(\sigma + \delta + \gamma + 1)}{\Gamma(n + 1)} \frac{xt^{n+\delta}}{\Gamma(n + \delta + 1) \Gamma(\gamma + 1)} L_3 (xt) \right) \|v\| \right\} \, dx
\]

\[
= p^* + q^* \|v\| \\
\leq M. \tag{4.3.23}
\]

Next, to prove $T$ is a contraction map, let $v_1, v_2 \in B$. Then for $t \in [0,1]$, we obtain

\[
\|u_1(t) - u_2(t)\| \leq \frac{\|v_1 - v_2\|^n}{\Gamma(n + 1)} , \\
\|\chi_1(t) - \chi_2(t)\| \leq \frac{\Gamma(\sigma + \delta + \gamma + 1) \|v_1 - v_2\|^n \Gamma(\gamma + 1)}{\Gamma(n + \delta + 1) \Gamma(\gamma + 1)}
\]

and

\[
\|Tv_1(t) - Tv_2(t)\| \leq \frac{1}{\Gamma(\alpha - n)} \int_{0}^{1} (1 - x)^{\alpha - n - 1} x^{n + \beta} 2F_1 (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \\
\times \| g (tx, u_1 (tx), \chi_1 (tx)) - g (tx, u_2 (tx), \chi_2 (tx)) \| \, dx
\]

\[
\leq \frac{1}{\Gamma(\alpha - n)} \int_{0}^{1} (1 - x)^{\alpha - n - 1} x^{n + \beta} 2F_1 (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \\
\times \{ L_2 (tx) \|u_1 - u_2\| + L_3 (tx) \|\chi_1 - \chi_2\| \} \, dx
\]

\[
\leq \frac{1}{\Gamma(\alpha - n)} \int_{0}^{1} (1 - x)^{\alpha - n - 1} x^{n + \beta} 2F_1 (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \\
\times \left( \frac{(xt)^n}{\Gamma(n + 1)} L_2 (xt) + \frac{\Gamma(\sigma + \delta + \gamma + 1) (xt)^{n+\delta}}{\Gamma(n + \delta + 1) \Gamma(\gamma + 1)} L_3 (xt) \right)
\]

\[
= M.
\]
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\[ \times \| v_1 - v_2 \| \, dx \]
\[ \leq r^* \| v_1 - v_2 \|_s, \]

where \( \chi_1 \) and \( \chi_2 \) are defined in (4.3.10).

Thus \( \| Tv_1(t) - Tv_1(t) \|_s \leq r^* \| v_1 - v_2 \|_s. \)

By assumption \((H6), r^* < 1; \) therefore the operator \( T \) is a contraction in \( B. \) Hence, by Theorem 2.2.2, the operator \( T \) has a unique fixed point, which corresponds to the unique solution of the initial value problem (4.3.1a)–(4.3.1b).

Next theorems are based on the existence of the solution for the initial value problem (4.3.1a)–(4.3.1b).

**Theorem 4.3.10.** Let us assume that the assumption \((H1), (H5), (H6) \) and \((H7) \) hold. Then the initial value problem (4.3.1a)–(4.3.1b) has at least one solution in the space \( B. \)

*Proof.* Consider the operator defined in (4.3.20) and \( F = \{ v \in B : \| v \|_s \leq \mathcal{K} \}, \) where

\[ \| Tv \|_s \leq p^* + q^* + r^* R := \mathcal{K}. \]

In the view of the proof of Theorem 4.3.9, it can be easily shown that the operator \( T \) maps \( F \) into itself. To prove that the operator \( T \) is compact and continuous, we shall divide the proof in following steps:

**Step 1:** \( T \) is continuous.

Let \( \{ v_n \} \) be a sequence in \( F \) such that \( v_n \to v \) as \( n \to \infty. \) Clearly for \( t \in [0, 1], \) by using Lemma 4.2.4, to find

\[ \| u_n(t) - u(t) \| \leq \frac{\| v_n - v \| t^n}{\Gamma(n+1)}, \]
\[ \| \chi_n(t) - \chi(t) \| \leq \frac{\Gamma(\sigma + \delta + \gamma + 1)\| v_n - v \| t^{n+\delta}}{\Gamma(n+\delta+1)\Gamma(\gamma+1)}. \]
Then by the assumption (H1), we have
\[
\|Tv_n(t) - Tv(t)\| \leq \frac{1}{\Gamma(\alpha - n)} \int_0^1 (1 - x)^{\alpha - n - 1} x^{n+\beta} \times \left( \frac{(xt)^n}{\Gamma(n+1)} L_2(xt) + \frac{\Gamma(\sigma + \delta + \gamma + 1)(xt)^{n+\delta}}{\Gamma(n + \delta + 1) \Gamma(\gamma + 1)} L_3(xt) \right)
\times \|v_n - v\| \, dx
\leq r^* \|v_n - v\|_*.
\]
This implies that \(\|Tv_n(t) - Tv(t)\|_* \leq r^* \|v_n - v\|_*\).

Thus, \(\|v_n - v\|_* \to 0 \Rightarrow \|Tv_n(t) - Tv(t)\|_* \to 0\) as \(n \to \infty\).

Step 2: The operator \(T\) is bounded in \(F\) into itself. The proof is similar to the proof of Theorem 4.3.9.

Step 3: The operator \(T\) is equicontinuous on \(F\).

Let \(v \in F\) and \(t_1, t_2 \in [0, 1]\) such that \(t_1 < t_2\). Then
\[
\|Tv(t_2) - Tv(t_1)\| \leq \frac{1}{\Gamma(\alpha - n)} \int_0^1 (1 - x)^{\alpha - n - 1} x^{n+\beta} \times 2F_1(\alpha + \beta, -\gamma - n; \alpha - n; 1 - x)
\times \left( \|g(t_2x, u(t_2x), \chi(t_2x)) - g(t_2x, 0, 0)\| + \|g(t_1x, 0, 0) - g(t_1x, 0, 0)\|
+ \|g(t_1x, 0, 0) - g(t_1x, u(t_1x), \chi(t_1x))\| \right) \, dx
\leq \frac{1}{\Gamma(\alpha - n)} \int_0^1 (1 - x)^{\alpha - n - 1} x^{n+\beta}
\]
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\[ \times \binom{2}{1} (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \]
\[ \times \{ \| L_1(t_2 x) - L_1(t_1 x) \| + (L_2(t_1 x) + L_2(t_2 x)) \| u \| \]
\[ + (L_3(t_1 x) + L_3(t_2 x)) \| \chi \| \} dx. \]

By the assumption (H1), \( L_1(t) \) is uniformly continuous in \([0,1]\). So, for given \( \varepsilon > 0 \), we find \( \rho > 0 \) such that \( \| t_2 - t_1 \| < \rho \), then \( \| L_1(t_2) - L_1(t_1) \| < \varepsilon = \frac{\rho}{\Xi} \). Hence

\[ \| T v(t_2) - T v(t_1) \|_* \leq \rho + 2q^* + 2r^* K, \]

which is independent of \( v \).

Thus, the operator \( T \) is relatively compact. Hence, as a consequence of Arzelà-Ascoli theorem, the operator \( T \) is compact and continuous. Using the Theorem 2.2.3, we conclude that the operator \( T \) has at least one solution for the initial value problem (4.3.1a)–(4.3.1b).

**Theorem 4.3.11.** Let the assumption (H1), (H5), (H6) and (H7) hold. Then the initial value problem (4.3.1a)–(4.3.1b) has a solution.

**Proof.** Consider the operator \( T \) defined in (4.3.20) and \( H = \{ v \in B : \| v \|_* < P \} \).

Then, by (H1) and Arzelà-Ascoli theorem, it can be easily shown that the operator \( T : \overline{H} \to \overline{H} \) is compact and continuous.

Next, we show that \( H \) is a priori bounds. If possible, suppose that there is a solution \( v \in \partial H \) such that

\[ v = \lambda T v \quad \text{with } \lambda \in (0,1). \tag{4.3.26} \]

Then for \( \lambda \in (0,1) \), to obtain

\[ \| v \|_* \leq \sup_{t \in [0,1]} \lambda | \Phi(t) | + \sup_{t \in [0,1]} \frac{\lambda}{\Gamma(\alpha - n)} \int_{0}^{1} (1 - x)^{\alpha - n - 1} x^{n+\beta} \]
\[ \times \binom{2}{1} (\alpha + \beta, -\gamma - n; \alpha - n; 1 - x) \| g(t x, u(t x), \chi(t x)) \| dx \]
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\[
\leq \sup_{t \in [0,1]} \lambda |\Phi(t)| + \sup_{t \in [0,1]} \frac{\lambda}{\Gamma(\alpha - n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
\times _2 F_1 (\alpha + \beta, -\gamma - n; \alpha - n; 1-x) \left\{ L_1(xt) + \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} |c_k| L_2(xt) \right\} dx \\
+ \left( \frac{(xt)^n}{\Gamma(n+1)} L_2(xt) + \frac{\Gamma(\sigma + \delta + \gamma + 1)(xt)^{n+\delta}}{\Gamma(n + \delta + 1) \Gamma(\gamma + 1)} L_3(xt) \right) \|v\| \right\} dx \\
= \lambda (p^* + q^* + r^* \|v\|) \\
< \mathcal{P}.
\]

Therefore \( v \notin \partial \mathcal{H} \). Hence, by Theorem 2.2.4, \( T \) has a fixed point in \( \mathcal{H} \), which is a solution of initial value problem (4.3.1a)–(4.3.1b).

Illustrative Example

One example is presented in this section in order to confirm the effectiveness of the proposed theorem.

Example 4.3.12. Consider the fractional differential equation

\[
\mathcal{C} \mathcal{D}_{0+}^{3.5,2.5,2.5} u(t) - t^{4.5} \mathcal{C} \mathcal{D}_{0+}^{1.5,0.5,2.5} u(t) - u(t) = t^{6.5}, \quad t \in [0,1],
\]

and

\[
u(0) = 2.5, \quad u'(0) = 2, \quad u''(0) = 6, \quad u'''(0) = 7.05.
\]

The above equation (4.3.27a) can be written as

\[
\mathcal{C} \mathcal{D}_{0+}^{3.5,2.5,2.5} u(t) = t^{6.5} + u(t) + t^{4.5} \mathcal{C} \mathcal{D}_{0+}^{1.5,0.5,2.5} u(t).
\]

Here, \( 3 < \alpha < 4 \) and \( 1 < \sigma < 2 \).

Set

\[
g(t, u, v) \equiv t^2 + t^{-4.5} u(t) + v(t).
\]
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Also, for each \( u, \bar{u}, v, \bar{v} \in \mathbb{R} \), we have

\[
\|g(t, u, v) - g(t, \bar{u}, \bar{v})\| \leq L_2(t) \|u - \bar{u}\| + L_3(t) \|v - \bar{v}\|
\]

Clearly \( L_1(t) = t^2 \), \( L_2(t) = t^{-4.5} \) and \( L_3(t) = 1 \) satisfied the condition (H1).

Again

\[
p^* \leq \sup_{t \in [0,1]} |\phi(t)| + \sup_{t \in [0,1]} \frac{1}{\Gamma(1.5)} \int_0^1 (1 - x)^{1.5-1} x^{0.5} F_1(6, -4.5, 1.5, 1 - x) dx
\]

\[
\leq 6 + 7.05 + \sup_{t \in [0,1]} \frac{t^2}{\Gamma(1.5)} \int_0^1 x^{1.5}(1 - x)^{6.5} F_1(6, -4.5, 1.5, x) dx
\]

\[
\leq 13.05 + \frac{\Gamma(7.5)\Gamma(7.5)}{\Gamma(3)\Gamma(13.5)}
\]

\[
\approx 13.05 + 0.0010
\]

\[
= 13.051,
\]

\[
q^* = \sup_{t \in [0,1]} \frac{2.5t^{-4.5}}{\Gamma(1.5)} \int_0^1 (1 - x)^{1.5-1} F_1(6, -4.5, 1.5, 1 - x) dx + \sup_{t \in [0,1]} \frac{2t^{-3.5}}{\Gamma(1.5)} \int_0^1 (1 - x)^{1.5-1} x F_1(6, -4.5, 1.5, 1 - x) dx
\]

\[
\leq \frac{2.5}{\Gamma(-3.5)\Gamma(7)} + \frac{2}{\Gamma(-2.5)\Gamma(8)}
\]

\[
= 0.0129 - 4.1978 \times 10^{-4}
\]

\[
\approx 0.0133.
\]

and

\[
r^* = \sup_{t \in [0,1]} \frac{t^{-2.5}}{\Gamma(3)\Gamma(1.5)} \int_0^1 x^{0.5}(1 - x)^{2} F_1(6, -4.5, 1.5, x) dx + \sup_{t \in [0,1]} \frac{t^{2.5}\Gamma(5.5)}{\Gamma(3.5)\Gamma(3.5)\Gamma(1.5)} \int_0^1 x^{0.5}(1 - x)^{7} F_1(6, -4.5, 1.5, x) dx
\]

\[
\leq 2.0989 \times 10^{-5} + 0.0058
\]

\[
= 0.0058 \quad \text{(approx.)}
\]

\[
< 1.
\]
and similarly one can find $p^* \leq 13.0510$, $q^* \leq 0.0124$.

Take $M > 2.2523 \times 10^3$. As a consequence of Theorem 4.3.9 and Theorem 4.3.10, the FIVP (4.3.27a)–(4.3.27b) has at least one solution defined in $[0, 1]$.

### 4.4 Conclusion

In this chapter, we have studied initial value problems for fractional differential equations involving the Caputo type modification of Saigo fractional derivative. Some sufficient conditions for some existence and uniqueness of solutions are established by virtue of generalized fractional calculus and fixed point method. However, the theorems on the existence and uniqueness of solutions and their proof imply that they can be extended to multi-term fractional differential equations involving the Caputo type modification of Saigo fractional derivative. Further, it is noted that the approaches discussed in this chapter seem appropriate to study and solve the fractional differential equation on other function spaces.