CHAPTER 3

STABILITY OF FRACTIONAL DIFFERENTIAL EQUATIONS

In Chapter 2, we have discussed the existence and uniqueness of the solutions for integro-differential equations of fractional order. In this chapter†, we focuses on the Hyers-Ulam stability of Caputo fractional differential equations.

3.1 Introduction

A functional equation is stable if, for every approximate solution, there exists an exact solution near it. In 1940, Ulam (cf. [145–147]) stated the following problem concerning the stability of functional equations:

We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot, \cdot)$. For a given $\varepsilon > 0$, 

†Contents of this chapter has appeared as papers entitled:
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does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$\rho(f(xy), f(x) f(y)) < \delta,$$

(3.1.1)

for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \varepsilon$, for all $x \in G$?

The problem for the case of the approximately additive mappings was solved by Hyers [45] under the assumption that $G$ and $G'$ are Banach space. Indeed, Hyers proved that each solution of the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$

(3.1.2)

for all $x$ and $y$, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation,

$$f(x + y) = f(x) + f(y),$$

(3.1.3)

is said to have the Hyers-Ulam stability.

Hyers result was further generalized by Rassias [103], who attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p),$$

(3.1.4)

and proved the Hyers theorem. Thus, Rassias proved the Hyers-Ulam-Rassias stability of the Cauchy additive functional equation.

The terminologies, Hyers-Ulam-Rassias stability and Hyers-Ulam stability can also be applied to other cases of functional equations, differential equations, and of various integral equations. The following example illustrates the difference between Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

For a given continuous function $f$ and a fixed real number $a$, the integral equation

$$y(t) = \int_a^t f(\tau, y(\tau)) d\tau,$$

(3.1.5)
is called a Volterra integral equation of the second kind. If for each function $y(t)$ satisfying
\[ |y(t) - \int_a^t f(\tau, y(\tau))d\tau| \leq \psi(t), \quad (3.1.6) \]
where $\psi(t) \geq 0$ for all $t$, there exists a solution $y_0(t)$ of the Volterra integral equation (3.1.5) and a constant $C > 0$ with
\[ |y(t) - y_0(t)| \leq C\psi(t), \quad (3.1.7) \]
for all $t$, where $C$ is independent of $y(t)$ and $y_0(t)$, then we say that the integral equation (3.1.5) has the Hyers-Ulam-Rassias stability. If $\psi(t)$ is a constant function in the above inequalities, we say that the integral equation (3.1.5) has the Hyers-Ulam stability.

According to the theory and methods presented in this chapter, it is further divided into three sections. In Section 3.2, we first examined the sufficient conditions for the existence of solution for a class of fractional differential equation in the metric space. Then the successive approximation method is used to prove the Hyers-Ulam stability. One class of fractional differential equation, together with the initial conditions or boundary conditions are considered in Section 3.3. Here, the mean value theorem for integral is used to establish the Hyers-Ulam stability. Finally in section 3.4, we have discussed the Hyers-Ulam stability of nonhomogeneous fractional Legendre differential in the half line, by using the fractional power series solution method for fractional differential equation.

### 3.2 Stability in metric space

In this section, we first investigate the Hyers-Ulam stability for FIVP in complete metric space, by using the method of successive approximation.

Consider the fractional differential equations
\[ ^C D_{0+}^\alpha u(t) = f(t, u(t)), \quad t \in [0, 1], \quad (3.2.1a) \]
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with

\[ u^{(k)}(0) = c_k, \]  

(3.2.1b)

where \( \alpha \in (m - 1, m), m \in \mathbb{N} \) and \( c_k \in \mathbb{R}, k = 0, 1, 2, \ldots, m - 1. \)

3.2.1 Preliminaries

In 2007, Jung [59] investigated the Hyers-Ulam stability of Volterra integral equation on a compact interval and proved the following:

**Theorem 3.2.1** (cf. [59]). Given \( a \in \mathbb{R} \) and \( r > 0 \), let \( I(a; r) \) denote a closed interval \( \{ t \in \mathbb{R} \mid a - r \leq t \leq a + r \} \) and let \( f : I(a; r) \times \mathbb{C} \to \mathbb{C} \) be a continuous function which satisfies a Lipschitz condition \( |f(t, v_1) - f(t, v_2)| \leq L|v_1 - v_2| \) for all \( t \in I(a; r) \) and \( v_1, v_2 \in \mathbb{C} \), where \( L \) is a constant with \( 0 < Lr < 1 \). If a continuous function \( y : I(a; r) \to \mathbb{C} \) satisfies

\[ |y(t) - b - \int_a^t f(t, \tau, u(\tau))d\tau| \leq \varepsilon, \]

for all \( t \in I(a; r) \) and for some \( \varepsilon \geq 0 \), where \( b \in \mathbb{C} \), then there exists a unique continuous function \( u : I(a; r) \to \mathbb{C} \) such that

\[ y(t) = b + \int_a^t f(t, \tau, u(\tau))d\tau \quad \text{and} \quad |u(t) - y(t)| \leq \frac{\varepsilon}{1 - Lr}, \]

for all \( t \in I(a; r) \).

Further, Gachpazan and Baghani [31] discussed the Hyers-Ulam stability of the following nonhomogeneous nonlinear Volterra integral equation

\[ u(t) = f(t) + \varphi \left( \int_a^t F(t, \tau, u(\tau))d\tau \right), \]

(3.2.2)

where \( t \in I = [a, b], -\infty < a < b < \infty. \)

The authors [31] have used successive approximation method, to prove that (3.2.2) has the Hyers-Ulam stability in complete metric space \( (X := \mathcal{C}[a, b], \|\cdot\|_\infty) \) under the following assumptions:
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(i) \( f(t) \) is continuous on the interval \( I \); 

(ii) \( F(t,u,v) \) is continuous with respect to the three variables \( t, u \) and \( v \) on the domain \( D = \{(t,u,v) : t,u,v \in I \} \); 

(iii) \( F(t,u,v) \) satisfies Lipschitz condition with respect to \( v \); 

(iv) \( \varphi \) is bounded linear transformation on \( X \).

and proved the following:

**Theorem 3.2.2** (cf. [31,149]). The Volterra integral equation (3.2.2) has the Hyers-Ulam stability if there exists a constant \( k \geq 0 \) with the following property: for every \( \varepsilon > 0 \), \( y \in X \), if

\[
\|\varphi\| = \sup \left\{ \frac{\| \varphi t \|}{\| \varphi \|} : t \neq 0, t \in X \right\} < \infty,
\]  

(3.2.3)

and

\[
\left| y(t) - f(t) - \varphi \left( \int_a^t F(t,\tau,y(\tau))d\tau \right) \right| \leq \varepsilon,
\]  

(3.2.4)

then there exists some \( u \in X \) satisfying (3.2.2) such that

\[
|u(t) - y(t)| \leq Ke\varepsilon.
\]  

(3.2.5)

The constant \( K \) is called the Hyers-Ulam stability constant for the equation (3.2.2). The condition (3.2.3) implies that the operator \( \varphi \) is bounded.

3.2.2 Existence theorem

Consider the complete metric space \((X,d)\), \( X \subseteq C[0,1] \), where the metric \( d \) is defined by

\[
d(g,h) = \sup_{t \in [0,1]} |g(t) - h(t)|. 
\]  

(3.2.6)
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Consider the following:

(H1) \(|\phi(t)| < M, M > 0;\)

(H2) \(f(x, u)\) is continuous on \(X;\)

(H3) \(f(x, u)\) satisfies the Lipschitz condition under the second variable, i.e.,

\(|f(x, u) - f(x, v)| \leq L|u - v|,

such that \(0 < \frac{L}{\Gamma(\alpha + 1)} < 1.\)

First we prove that the solution of the FIVP (3.2.1a)–(3.2.1b) is equivalent to the solution of an integral equation.

**Lemma 3.2.3.** Let \(m - 1 < \alpha \leq m\) and (H1)–(H3) to be satisfied. Then for \(u \in C^m_\alpha [0, 1]\) the FIVP (3.2.1a)–(3.2.1b) is equivalent to the integral equation

\[u(t) = \phi(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, u(\tau))d\tau, \tag{3.2.7}\]

where \(\phi(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k.\)

**Proof.** Since \(f(u, v)\) is continuous. Operating \(\mathcal{I}_0^\alpha\) on the both sides of (3.2.1a) and applying Lemma 2.1.13, it reduces to

\[\mathcal{I}_0^\alpha \mathcal{D}_0^\alpha u(t) = \mathcal{I}_0^\alpha f(t, u(t))\]

\[\Rightarrow u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0) + \mathcal{I}_0^\alpha f(t, u(t)).\]

Using (3.2.1b), it can written as

\[u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, u(\tau))d\tau.\]

This completes the proof.
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Next, based on the above assumptions (H1)–(H3), we show that there exists a solution for the FIVP (3.2.1a)–(3.2.1b). The proof is done with the help of Banach contraction mapping principle (Theorem 2.2.2).

Consider the following iterative scheme:

\[ \Upsilon u_n \equiv u_{n+1}(t) = \phi(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_n(\tau)) d\tau, \]

where \( n = 1, 2, 3, \ldots \).

Then

\[ |u_3 - u_2| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau, u_2(\tau)) - f(\tau, u_1(\tau))| d\tau. \]

It follows from (H2) and (H3) that

\[ |u_3 - u_2| \leq \frac{L}{\Gamma(\alpha)} d(\Upsilon u_1, u_1) \int_0^t (t-\tau)^{\alpha-1} d\tau = \frac{Lt^\alpha}{\Gamma(\alpha + 1)} d(\Upsilon u_1, u_1). \]

Hence \( d(\Upsilon u_2, u_2) \leq \frac{Lt^\alpha}{\Gamma(\alpha + 1)} d(\Upsilon u_1, u_1). \)

Similarly, it can be easily proved that

\[ d(\Upsilon u_3, u_3) \leq \frac{L_2t^{2\alpha}}{\Gamma(2\alpha + 1)} d(\Upsilon u_1, u_1). \]

Suppose the result is true for \( k \), i.e.

\[ d(\Upsilon u_k, u_k) \leq \frac{L^{k-1}t^{(k-1)\alpha}}{\Gamma((k-1)\alpha + 1)} d(\Upsilon u_1, u_1). \] (3.2.9)

Now we verify that the result is also true for \( k + 1 \) by using (3.2.9). Then

\[ |u_{k+2} - u_{k+1}| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_{k+1}(\tau)) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_k(\tau)) d\tau \right| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau, u_{k+1}(\tau)) - f(\tau, u_k(\tau))| d\tau. \]
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It follows from (H2) and (H3) that

\[
|u_{k+2} - u_{k+1}| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} |u_{k+1}(\tau) - u_k(\tau)| d\tau \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} d(\Upsilon u_k, u_k) d\tau \\
\leq \frac{L_k}{\Gamma(\alpha)} d(\Upsilon u_1, u_1) \int_0^t (t - \tau)^{\alpha - 1} (k - 1)^{\alpha} d\tau \\
\leq \frac{L_k^{k\alpha}}{\Gamma(k\alpha + 1)} d(\Upsilon u_1, u_1). 
\]

(Using (3.2.9))

Thus

\[
d(\Upsilon u_{k+1}, u_{k+1}) \leq \frac{L_k^{k\alpha}}{\Gamma(k\alpha + 1)} d(\Upsilon u_1, u_1).
\]

Hence, by the principle of mathematical induction, we obtain

\[
d(\Upsilon u_n, u_n) \leq \frac{L_n^{(n-1)\alpha}}{\Gamma((n-1)\alpha + 1)} d(\Upsilon u_1, u_1), \tag{3.2.10}
\]

for all positive integer \( n \geq 1 \).

Again, clearly by iterative scheme (3.2.8)

\[
u_n(t) = u_1(t) + \sum_{k=1}^{n-1} [u_{k+1}(t) - u_k(t)]. \tag{3.2.11}
\]

Since \( \mathcal{X} \) is a complete metric space, for \( u_1 \in \mathcal{X} \) and by the Weierstrass M-test the series

\[
\sum_{n=1}^{\infty} [u_{n+1}(t) - u_n(t)]
\]

is absolutely and uniformly convergent.

Hence, by Theorem 2.2.2, there exists a unique solution \( u \in \mathcal{X} \) such that

\[
u(t) = \lim_{n \to \infty} u_{n+1}(t)
\]
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\[ \phi(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau = \Upsilon u. \]  

(3.2.12)

3.2.3 Hyers-Ulam stability

In this section, we prove that the fractional initial value problem (3.2.1a)–(3.2.1b) has the Hyers-Ulam stability.

**Theorem 3.2.4.** Suppose that the operator \( \Upsilon \) defined in (3.2.8). Then the equation \( \Upsilon u = u \) has the Hyers-Ulam stability.

**Proof.** Let \( v \in \mathcal{X} \) and \( d(\Upsilon v, v) < \varepsilon, \varepsilon > 0 \). Then, by (3.2.12), we have

\[ u(t) = \lim_{n \to \infty} \Upsilon^n v(t). \]

Hence, \( d(\Upsilon^n v, u) \leq \varepsilon \). Thus

\[ d(v, u) \leq d(v, \Upsilon^n v) + d(\Upsilon^n v, u) \]
\[ \leq d(v, \Upsilon v) + d(\Upsilon v, \Upsilon^2 v) + \cdots + d(v, \Upsilon^n v) + d(\Upsilon^n v, u). \]

Applying (3.2.10), the above inequality reduces to

\[ d(v, u) \leq d(v, \Upsilon v) \left( 1 + \frac{Lt}{\Gamma(\alpha+1)} + \cdots + \frac{L^{(n-1)}t^{(n-1)}}{\Gamma((n-1)\alpha+1)} \right) + \varepsilon \]
\[ \leq (1 + E_\alpha(Lt)) \varepsilon, \]

where \( E_\alpha(t) \) is the classical Mittag-Leffler function [95].

Hence, \( d(u, v) < K \varepsilon \), where \( K = (1+E_\alpha(Lt)), t \in [0, 1] \). This completes the proof. \( \Box \)

**Corollary 3.2.5.** If one applies the successive approximation method for solving FIVP (3.2.1a)–(3.2.1b) and find \( u_i(t) = u_{i+1}(t) \), for some \( i = 1, 2, \cdots \), then the \( u(t) = u_i(t) \), such that \( u(t) \) is the exact solution of FIVP (3.2.1a)–(3.2.1b).
Illustrative Example

In this section, we given an example to illustrate the Hyers-Ulam stability of a Cauchy type problem.

Example 3.2.6. Let us consider the fractional initial value problem of the form:

\[ C \mathcal{D}_{0+}^{\alpha} u(t) = \frac{t}{20} + \frac{t^2}{2(1 + t^2)} u(t), \quad (3.2.13a) \]

with

\[ u(0) = 0, \quad (3.2.13b) \]

where \( 0 < \alpha < 1 \) and \( t \in [0, 1] \).

Set

\[ f(t, u(t)) = \frac{t}{20} + \frac{t^2}{2(1 + t^2)} u(t) \]

and \( \phi(t) = 0 \).

Let \( u, v \in \mathbb{R}^+ \) and \( t \in [0, 1] \). Then clearly

\[ |f(t, u(t)) - f(t, v(t))| = \left| \frac{t^2}{2(1 + t^2)} u(t) - \frac{t^2}{2(1 + t^2)} v(t) \right| \]

\[ = \frac{t^2}{2(1 + t^2)} |u(t) - v(t)| \]

\[ \leq \frac{1}{2} |u(t) - v(t)| \]

and \( \phi(t) = 0 \).

Hence, FIVP (3.2.13a)–(3.2.13b) satisfies the required conditions of Theorem 3.2.4. Thus in consequence of Theorem 3.2.4, FIVP (3.2.13a)–(3.2.13b) has Hyers-Ulam stability.
3.3. Stability of linear fractional differential equation

In this section, we investigate the Hyers-Ulam stability for the following linear fractional differential equation

\[ C_{\alpha} \mathcal{D}_{a+} y(t) - \beta(t) y(t) = 0 \]  

(3.3.1)

with the boundary conditions

\[ y(a) = y(b) = 0 \]  

(3.3.2)

or with the initial conditions

\[ y(a) = y'(a) = 0, \]  

(3.3.3)

where \( 1 < \alpha \leq 2 \).

3.3.1 Preliminaries

Alsina and Ger [7] first observed the Hyers-Ulam stability of differential equation \( f'(t) = f(t) \). They proved that if a differentiable function \( f : \Omega \to \mathbb{R} \) satisfies \( |f'(t) - f(t)| \leq \varepsilon \) for all \( t \in \Omega \), then there exists a function \( g : I \to \mathbb{R} \) such that \( g'(t) = g(t) \) and \( |f(t) - g(t)| \leq 3\varepsilon \) for all \( t \in \Omega \), where \( \Omega \) is an open interval in \( \mathbb{R} \) and \( \varepsilon > 0 \).

Takahasi et al. [141] further generalized the result of Alsina and Ger [7] for the Banach space-valued differential equation \( y' = \lambda y \). Li [76] studied the Hyers-Ulam stability of differential equation of second order of the form \( y'' = \lambda^2 y \). Recently, Găvruţă et al. [36] discussed Hyers-Ulam stability of linear second-order differential equations of the form \( y'' + \beta(x)y = 0 \) with boundary conditions, and with initial conditions. They have shown through counter example that the Hyers-Ulam stability of the differential equation may not be stable for unbounded intervals.
lemma 3.3.1 (Mean-value Theorem for integrals [66]). Let \(u\) and \(v\) be continuous real-valued functions on an interval \([a, b]\) and suppose that \(v \geq 0\). Then there exists a point \(\xi\) in \([a, b]\) such that

\[
\int_a^b u(x)v(x)dx = u(\xi) \int_a^b v(x)dx. \tag{3.3.4}
\]

The following lemma is important for our discussion.

**Lemma 3.3.2.** Let \(f(t) \in C^n[a, b]\) and \(C_D^\alpha f \in C[a, b]\) for \(n-1 < \alpha \leq n\), then

\[
f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k + R_\alpha(t, a) \tag{3.3.5}
\]

and

\[
R_\alpha(t, a) = C_D^\alpha f(\xi) \frac{(t-a)^\alpha}{\Gamma(\alpha + 1)} \tag{3.3.6}
\]

where \(a \leq \xi \leq t, \forall t \in (a, b]\).

**Proof.** Let \(\alpha \notin \mathbb{N}\) and \(f(t) \in C^n[a, b]\). Then by using Definition 2.1.3, we get

\[
(I_\alpha^+ C_D^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} C_D^\alpha f(\tau)d\tau.
\]

Again \((t-\tau)^{\alpha-1} \geq 0\) for \(t \in (a, b]\). Then by applying integral mean value theorem (3.3.4), the above equality reduces to

\[
(I_\alpha^+ C_D^\alpha f)(t) = \frac{C_D^\alpha f(\xi)}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} d\tau,
\]

where \(a \leq \xi \leq t\).

Changing variable \(\frac{t-\tau}{t-a} \rightarrow \xi\), it follows

\[
(I_\alpha^+ C_D^\alpha f)(t) = \frac{C_D^\alpha f(\xi)}{\Gamma(\alpha)} \int_0^1 \xi^{\alpha-1} d\xi
\]

\[
= \frac{C_D^\alpha f(\xi)}{\Gamma(\alpha + 1)} (t-a)^\alpha.
\]

Since \(\alpha \notin \mathbb{N}\) and \(f(t) \in C^n[a, b]\), applying Lemma 2.1.13 on the left hand side, the above equation yields the result (3.3.5). \(\square\)
In this section, we investigate the Hyers-Ulam stability of equation (3.3.1).

Suppose that

(H1) \( \beta(t) \in C[a, b] \) such that \( \max_{t \in [a, b]} |\beta(t)| < \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} \);

(H2) \( M = \max\{|y(t)| : t \in [a, b]\} \);

(H3) There exists \( \epsilon > 0 \) such that the inequality \( |C^D_{\alpha+} y - \beta(t)y| \leq \varepsilon \), hold for \( t \in [a, b] \) and \( y \in C^2[a, b] \).

**Theorem 3.3.3.** Assume that the conditions (H1)–(H3) hold. Then the equation (3.3.1) has the Hyers-Ulam stability together with boundary conditions (3.3.2).

**Proof.** Let \( y \in C^2[a, b] \) be the solution of the inequality defined in (H3).

Since \( y(a) = y(b) = 0 \) and (H2) holds, then there exists \( t_0 \in (a, b) \) such that \( y(t_0) = M \). Then using Lemma 3.3.2, to obtain

\[
y(a) = \sum_{k=0}^{1} \frac{y^{(k)}(t_0)}{k!} (a - t_0)^k + C^D_{\alpha+} y(\xi) \frac{(a - t_0)^\alpha}{\Gamma(\alpha + 1)}
\]

and

\[
y(b) = \sum_{k=0}^{1} \frac{f^{(k)}(t_0)}{k!} (b - t_0)^k + C^D_{\alpha+} f(\eta) \frac{(b - t_0)^\alpha}{\Gamma(\alpha + 1)}
\]

where \( \xi \in [a, t_0] \) and \( \eta \in [t_0, b] \).

Thus \( t_0 \in \left( a, \frac{a + b}{2} \right) \)

\[
a < t_0 \leq \frac{a + b}{2}
\]

\[
\Rightarrow 0 < (t_0 - a) \leq \left( \frac{b - a}{2} \right)
\]

\[
\Rightarrow 0 > \frac{1}{(t_0 - a)^\alpha} \geq \left( \frac{2}{b - a} \right)^\alpha
\]

\[
\Rightarrow 0 > \frac{M\Gamma(\alpha + 1)}{(t_0 - a)^\alpha} \geq \frac{2^\alpha M\Gamma(\alpha + 1)}{(b - a)^\alpha}
\]
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Similarly, for \( t_0 \in \left[ \frac{a+b}{2}, b \right) \)

\[
\frac{a+b}{2} \leq t_0 < b
\]

\[
\Rightarrow (a+b) \leq 2t_0 < 2b
\]

\[
\Rightarrow \frac{2^\alpha M \Gamma(\alpha+1)}{(b-a)^\alpha} \leq \frac{M \Gamma(\alpha+1)}{(b-t_0)^\alpha} < 0.
\]

Since \( y \in C^2[a,b] \) and \( y(a) = y(b) = 0 \), it follows

\[
|^{C^D}_a y(\xi) = \left| \frac{\Gamma(\alpha+1)M}{(a - t_0)^\alpha} \right| = \frac{\Gamma(\alpha+1)M}{(t_0 - a)^\alpha}
\]

\[
\geq \frac{2^\alpha \Gamma(\alpha+1)M}{(b-a)^\alpha}, \quad t_0 \in \left( a, \frac{a+b}{2} \right] \]

and

\[
|^{C^D}_a y(\eta) | \geq |^{C^D}_{t_0} y(\eta) | = \frac{\Gamma(\alpha+1)M}{(b-t_0)^\alpha}
\]

\[
\geq \frac{2^\alpha \Gamma(\alpha+1)M}{(b-a)^\alpha}, \quad t_0 \in \left[ \frac{a+b}{2}, b \right). \]

Therefore,

\[
\max |y(t)| \leq \frac{(b-a)^\alpha}{2^\alpha \Gamma(\alpha+1)} \max \left| ^{C^D}_a y(t) \right|
\]

\[
\leq \frac{(b-a)^\alpha}{2^\alpha \Gamma(\alpha+1)} \left\{ \max \left| ^{C^D}_a y(t) - \beta(t) y(t) \right| + \max |\beta(t)| \max |y(t)| \right\}
\]

\[
\leq \frac{(b-a)^\alpha}{2^\alpha \Gamma(\alpha+1)} \left\{ \varepsilon + \max |\beta(t)| \max |y(t)| \right\}.
\]

From the above relations, it follows that there exists a constant

\[
K = \frac{(b-a)^\alpha}{2^\alpha \Gamma(\alpha+1)(1-\delta)} > 0,
\]

where \( \delta = \frac{(b-a)^\alpha \max |\beta(t)|}{2^\alpha \Gamma(\alpha+1)} \), which is independent of \( \varepsilon \) such that \( |y(t) - u_0(t)| \leq K \varepsilon \).

Hence, (3.3.1) is Hyers-Ulam stable under the boundary conditions (3.3.2).
Theorem 3.3.4. Assume that the conditions (H1)–(H3) hold. Then the equation (3.3.1) has Hyers-Ulam stability together with initial conditions (3.3.3).

Proof. Let \( y \in C^2[a, b] \) be the solution of the inequality defined in (H3) and \( y(a) = y'(a) = 0 \). Using Lemma 3.3.2, we have

\[
y(t) = \sum_{k=0}^{1} \frac{y^{(k)}(a)}{k!} (t - a)^k + CD^\alpha_{a+} y(\xi) \frac{(t - a)^\alpha}{\Gamma(\alpha + 1)} \\
= CD^\alpha_{a+} y(\xi) \frac{(t - a)^\alpha}{\Gamma(\alpha + 1)}, \quad \text{where} \quad a \leq \xi \leq x.
\]

Hence

\[
\max |y(t)| \leq \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)} \max |CD^\alpha_{a+} y(t)| \\
\leq \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)} \{ \varepsilon + \max |\beta(t)| \max |y(t)| \).
\]

Thus there exists a constant

\[
K = \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)(1 - \delta)} > 0,
\]

with \( \delta = \frac{(b - a)^\alpha \max |\beta(t)|}{\Gamma(\alpha + 1)} \), which is independent of \( \varepsilon \) such that \( |y(t) - u_0(t)| \leq K\varepsilon \). Hence (3.3.1) is Hyers-Ulam stability with (3.3.3).

3.4 Stability of nonhomogeneous fractional differential equation

In this section, we investigate the Hyers-Ulam stability for the nonhomogeneous sequential fractional differential equation, by using the fractional power series solution method.

We consider the nonhomogeneous generalized fractional Legendre differential equation of the form

\[
(1 - t^{2\alpha}) y^{(2\alpha)}(t) - 2t^\alpha y^{(\alpha)}(t) + p(\alpha + 1)y(t) = \sum_{k=0}^{\infty} v_k t^{k\alpha}, \quad \alpha \in (0, 1] \quad (3.4.1)
\]
where parameter $p$ is a real number, with positive radius of convergence $\rho$ for all $t \in U$ and the solution $y : U \to C$. The notation $y^{(k\alpha)} := C\mathcal{D}_{0+}^{k\alpha}$ denotes the Caputo’s sequential derivative

$$C\mathcal{D}_{0+}^{k\alpha} = \underbrace{C\mathcal{D}_{0+}^{\alpha} C\mathcal{D}_{0+}^{\alpha} \cdots C\mathcal{D}_{0+}^{\alpha}}_{k\text{-times}}.$$

### 3.4.1 Preliminaries

Let $X$ be a normed space over a scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and let $\Omega \subset \mathbb{R}$ be an open interval. Assume that $a_0, a_1, \cdots, a_n : \Omega \to X$ and $g : \Omega \to X$ are given continuous functions, and that $y : \Omega \to X$ is an $n$ times continuously differentiable function satisfying the inequality [17]

$$\|a_n(t)y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + g(t)\| \leq \varepsilon$$

for all $t \in \Omega$ and for a given $\varepsilon > 0$. If there exists an $n$ times continuously differentiable function $y_0 : \Omega \to X$ satisfying

$$a_n(t)y_0^{(n)}(t) + \cdots + a_1(t)y_0'(t) + a_0(t)y_0(t) + g(t) = 0$$

and $\|y(t) - y_0(t)\| \leq K(\varepsilon)$ for any $t \in \Omega$, where $K(\varepsilon)$ is an expression of $\varepsilon$ with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain $\Omega$ is not in the whole space $X$).

Consider the following notation to introduce a linear homogeneous fractional differential equation

$$[L_{n\alpha}(y)](t) := y^{(n\alpha)} + \sum_{k=0}^{n-1} p_k(t)y^{(k\alpha)}(t) = 0,$$  \hspace{1cm} (3.4.2)

where $y^{(k\alpha)}$ denotes the Caputo sequential derivative and the coefficients $p_k(t), k = 0, 1, \cdots, n-1$ are defined on the interval $[a, b]$. 
3.4. STABILITY OF NONHOMOGENEOUS FDE

The power series method for differential equations of generalized order was first suggested by Al-Bassam [4, 5]. Subsequently, Kilbas et al. [64] and Rivero et al. [105] studied the corresponding works by Al-Bassam and extended the result for $\alpha$-analytic solutions of linear fractional differential equations with variable coefficients around an $\alpha$-ordinary and $\alpha$-singular point using the power series method.

Before we proceed, we discuss a few definitions and elementary concepts on $\alpha$-analytic function [64, 105]:

**Definition 3.4.1.** Let $\alpha \in (0, 1]$, $f(t)$ be a real function defined on the interval $[a, b]$, and $t_0 \in [a, b]$. Then $f(t)$ is said to be $\alpha$-analytic at $t_0$ if there exists an interval $N(t_0)$ such that, for all $t \in N(t_0)$, $f(t)$ can be expressed as a series of natural powers of $(t - t_0)$. That is, $f(t)$ can be expressed as $\sum_{k=0}^{\infty} a_k (t - t_0)^{\alpha k}$, where $a_k \in \mathbb{R}$, this series being absolutely convergent for $|t - t_0| < \rho$, ($\rho > 0$) and the radius of convergence of the series is $\rho$.

The generalization of $\alpha$-singular point is defined as followings:

**Definition 3.4.2.** A point $t_0 \in [a, b]$ is said to be an $\alpha$-ordinary point of the equation (3.4.2), if the functions $p_k(t)$, $(k = 0, 1, \ldots, n - 1)$ are $\alpha$-analytic at $t_0$. A point $t_0 \in [a, b]$ which is not $\alpha$-ordinary is called $\alpha$-singular.

The $\alpha$-singular points can be regular $\alpha$-singular or an essential $\alpha$-singular, as introduced below:

**Definition 3.4.3.** Let $t_0 \in [a, b]$ be an $\alpha$-singular point of the equation (3.4.2). Then $t_0$ is said to be a regular $\alpha$-singular point of this equation (3.4.2) if the functions $(t - t_0)^{(n-k)\alpha} p_k(\alpha)$, $(k = 0, 1, \ldots, n - 1)$ are $\alpha$-analytic in $t_0$. Otherwise, $t_0$ is said to be an essential $\alpha$-singular point.

**Proposition 3.4.4** (cf. [64]). If $y_p(t)$ is a particular solution of the equation

\[ [L_{n\alpha}(y)](t) = f(t), \]
then the general solution to this equation is given by

\[ y_g(t) = y_h(t) + y_p(t), \]

where \( y_h(t) \) is the general solution of homogeneous equation \([L_{n\alpha}(y)](t) = 0\).

### 3.4.2 Fractional Legendre differential equation

We consider the solutions around an \( \alpha \)-ordinary point \( t_0 \in [a, b] \) of a fractional differential equation (3.4.2) of order \( 2\alpha \), i.e.,

\[ [L_{2\alpha}(y)](t) := y^{(2\alpha)}(t) + p(t)y^{(\alpha)}(t) + q(t)y(t) = 0, \quad \alpha \in (0, 1] \quad (3.4.3) \]

where \( y^{(2\alpha)} \) denotes the Caputo’s sequential derivative of \( y(t) \).

Since \( t_0 \) is an \( \alpha \)-ordinary point of (3.4.3), it follows from Definitions 3.4.1 and 3.4.3 that \( p(t) \) and \( q(t) \) can be expressed as

\[ p(t) = \sum_{k=0}^{\infty} p_k (t - t_0)^{k\alpha}, \quad t \in [t_0, t_0 + \rho_1]; \quad \rho_1 > 0 \]

and

\[ q(t) = \sum_{k=0}^{\infty} q_k (t - t_0)^{k\alpha}, \quad t \in [t_0, t_0 + \rho_2]; \quad \rho_2 > 0. \]

**Theorem 3.4.5** (cf. [64]). Let \( \alpha \in (0, 1] \), and \( a_0, a_1 \in \mathbb{R} \), and let \( t_0 \in [a, b] \) be an \( \alpha \)-ordinary point of the equation (3.4.3). Then there exists a unique solution \( y(t) \) of (3.4.3), for \( t \in (t_0, t_0 + \rho) \) with \( \rho = \min\{\rho_1, \rho_2\} \). This solution is an \( \alpha \)-analytic function in \( t_0 \) and satisfies the following initial conditions:

\[ \lim_{t \to t_0} y(t) = a_0 \quad \text{and} \quad \lim_{t \to t_0} y^{(\alpha)}(t) = a_1. \]

First, we apply the fractional power series solution method to find the solution of homogeneous fractional differential equation.
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Example 3.4.6. Consider the generalized fractional Legendre differential equation

\[ (1 - t^{2\alpha}) y^{(2\alpha)}(t) - 2t^{\alpha} y^{(\alpha)}(t) + p(p + 1)y(t) = 0 \]  (3.4.4)

where, \( y^{(2\alpha)} \) is the Caputo’s sequential derivative of \( y(t) \), \( 0 < \alpha \leq 1 \) and \( p \in \mathbb{R} \).

Since coefficients of (3.4.4) are \( \alpha \)-analytic at origin, so the power series solution can be expected to exist in \( |t| < 1 \). Assume that the solution \( y(t) \) can be written as

\[ y(t) = \sum_{k=0}^{\infty} a_k t^{k\alpha} \]  (3.4.5)

Substituting (3.4.5) and (2.1.14) into (3.4.4) yields

\[
\sum_{k=0}^{\infty} \left\{ a_{k+2} \frac{\Gamma((k + 2)\alpha + 1)}{\Gamma(k\alpha + 1)} + p(p + 1)a_k \right\} t^{k\alpha} - 2\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha + 1)}{\Gamma((k - 1)\alpha + 1)} t^{k\alpha}
- \sum_{k=2}^{\infty} a_k \frac{\Gamma(k\alpha + 1)}{\Gamma((k - 2)\alpha + 1)} t^{k\alpha} = 0
\]  (3.4.6)

Equating the coefficients in (3.4.6), to obtain

\[ a_2 = -\frac{p(p + 1)}{\Gamma(2\alpha + 1)} a_0, \quad a_3 = \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)} \left\{ 2\Gamma(\alpha + 1) - p(p + 1) \right\} a_1 \]

and

\[ a_{k+2} = \frac{\Gamma(k\alpha + 1)}{\Gamma((k + 2)\alpha + 1)} \left\{ 2 \frac{\Gamma(k\alpha + 1)}{\Gamma((k - 1)\alpha + 1)} + \frac{\Gamma(k\alpha + 1)}{\Gamma((k - 2)\alpha + 1)} - p(p + 1) \right\} a_k, \]

where \( k \geq 2 \). Hence,

\[
a_k = \frac{\Gamma((k + 2 - 2\left[\frac{k}{2}\right])\alpha + 1)}{\Gamma(k\alpha + 1)} a_{k+2-2\left[\frac{k}{2}\right]} \\
\times \prod_{j=1}^{\left[\frac{k}{2}\right]-1} \left\{ 2 \frac{\Gamma((k - 2j)\alpha + 1)}{\Gamma((k - 2j - 1)\alpha + 1)} + \frac{\Gamma((k - 2j)\alpha + 1)}{\Gamma((k - 2j - 2)\alpha + 1)} - p(p + 1) \right\},
\]  (3.4.8)

where \( k \geq 4 \).

The series (3.4.5) is convergent for \( |t| < 1 \), in accordance with the asymptotic representation of gamma function

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b} \left[ 1 + O\left(\frac{1}{z}\right) \right],
\]  (3.4.9)
and holds the following estimate

$$\lim_{k \to \infty} \left| \frac{a_{k+2}}{a_k} \right| = 1.$$  (3.4.10)

Supposing that $a_0$ and $a_1$ are arbitrary constants, then the general solution of (3.4.4) can be expressed as:

$$y(t) = \left( a_0 + \sum_{k=1}^{\infty} a_{2k} t^{2k\alpha} \right) + t^\alpha \left( a_1 + \sum_{k=1}^{\infty} a_{2k+1} t^{2k\alpha} \right)$$

$$= a_0 \left( 1 - p(p+1) \sum_{k=1}^{\infty} \frac{1}{\Gamma(2k\alpha + 1)} t^{2k\alpha} \right)$$

$$\times \prod_{j=1}^{k-1} \left\{ 2 \frac{\Gamma((2k-2j)\alpha + 1)}{\Gamma((2k-2j-1)\alpha + 1)} + \frac{\Gamma((2k-2j)\alpha + 1)}{\Gamma((2k-2j-2)\alpha + 1) - p(p+1)} \right\}$$

$$+ t^\alpha a_1 \left( 1 + \left\{ 2\Gamma(\alpha + 1) - p(p+1) \right\} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha + 1)}{\Gamma((2k+1)\alpha + 1)} t^{2k\alpha} \right)$$

$$\times \prod_{j=1}^{k-1} \left\{ 2 \frac{\Gamma((2k-2j+1)\alpha + 1)}{\Gamma((2k-2j)\alpha + 1)} + \frac{\Gamma((2k-2j+1)\alpha + 1)}{\Gamma((2k-2j-1)\alpha + 1) - p(p+1)} \right\}.$$  (3.4.11)

**Remark 3.4.7.** When $\alpha = 1$, (3.4.4) reduces to the classical Legendre differential equation.

### 3.4.3 Nonhomogeneous fractional Legendre differential equation

In this section, we investigate the solution of nonhomogeneous generalized fractional Legendre differential equation (3.4.1).

We define

$$c_k = \frac{1}{\Gamma(k\alpha + 1)} \sum_{i=1}^{[\frac{k}{2}]-1} \Gamma((k-2i)\alpha + 1) v_{k-2i}$$

$$\times \prod_{j=1}^{i-1} \left\{ 2 \frac{\Gamma((k-2j)\alpha + 1)}{\Gamma((k-2j-1)\alpha + 1)} + \frac{\Gamma((k-2j)\alpha + 1)}{\Gamma((k-2j-2)\alpha + 1) - p(p+1)} \right\},$$  (3.4.12)
for each $k \geq 4$, and $c_2 = \frac{v_0}{\Gamma(\alpha + 1)}$, $c_3 = \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)}$.

**Lemma 3.4.8.** Let the power series $\sum_{k=0}^{\infty} v_k t^{k\alpha}$ converges for all $t \in U = (-\rho, \rho)$ with $\rho \leq 1$. Suppose that for any positive $\rho_0 < \rho$ the power series

$$y_h(t) = \sum_{k=2}^{\infty} c_k t^{k\alpha}$$

satisfies the following inequality

$$|v_k| + \left\{ 2 \frac{\Gamma(k\alpha + 1)}{\Gamma((k-1)\alpha + 1)} + \frac{\Gamma(k\alpha + 1)}{\Gamma((k-2)\alpha + 1)} - p(p+1) \right\} |c_k| \leq \frac{\Gamma((k+2)\alpha + 1)}{\delta \Gamma(k\alpha + 1)} |c_k|$$

for $\delta > 0$ and values $c_k$’s given in (3.4.12). Then, the power series (3.4.13) converges for all $t \in U$ and satisfies the recurrence relation

$$c_{k+2} = v_k$$

for $k \geq 2$.

**Proof.** Consider that $0 < \rho_0 < \rho$. Using (3.4.9) and (3.4.15), we have

$$\lim_{k \to \infty} \left| \frac{c_{k+2}}{c_k} \right| = \lim_{k \to \infty} \left| \frac{v_k}{c_k} \right| \left( 2 \frac{\Gamma(k\alpha + 1)}{\Gamma((k-1)\alpha + 1)} + \frac{\Gamma(k\alpha + 1)}{\Gamma((k-2)\alpha + 1)} - p(p+1) \right) \leq \frac{1}{\delta},$$

$\delta > 0$.

Hence, $y_h(t)$ converges for all $t \in (-\rho_0, \rho_0)$.

Next, we prove that $y_h(t)$ satisfies (3.4.15).

Consider the case $k \geq 2$ is even. Suppose $k = 2m$, where $m \in \mathbb{N}$,

$$\frac{\Gamma((2m+2)\alpha)}{\Gamma(2m\alpha + 2)} c_{2m+2} = v_{2m} + \frac{1}{\Gamma(2m\alpha + 1)} \sum_{i=2}^{m} \Gamma((2m-2i)\alpha + 1) v_{2m-2i}$$
\[ \times \prod_{j=0}^{i-1} \left\{ 2 \frac{\Gamma((2m - 2j)\alpha + 1)}{\Gamma((2m - 1 - 2j)\alpha + 1)} + \frac{\Gamma((2m - 2j)\alpha + 1)}{\Gamma((2m - 2 - 2j)\alpha + 1)} - p(p + 1) \right\} \]

\[ = v_{2m} + \left\{ 2 \frac{\Gamma(2m\alpha + 1)}{\Gamma((2m - 1)\alpha + 1)} + \frac{\Gamma(2m\alpha + 1)}{\Gamma((2m - 2)\alpha + 1)} - p(p + 1) \right\} c_{2m} \]

Which shows that the result is true for even \( k \geq 2 \).

Similarly, we can verify that (3.4.15) is also true for odd \( k \geq 2 \). Hence, (3.4.15) is true for all \( k \geq 2 \).

Now, we prove that \( y_h(t) \) satisfies (3.4.1). Using (3.4.15) on the left side of (3.4.1), yields

\[ (1 - t^{2\alpha})y_h^{(2\alpha)}(t) - 2t^{\alpha}y_h^{(\alpha)}(t) + p(p + 1)y_h(t) \]

\[ = \Gamma(2\alpha + 1)c_2 + \frac{\Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)} c_3 + \sum_{k=2}^{\infty} \left[ \frac{\Gamma((k + 2)\alpha + 1)}{\Gamma(k\alpha + 1)} c_{k+2} \right. \]

\[ - \left\{ 2 \frac{\Gamma(k\alpha + 1)}{\Gamma((k - 1)\alpha + 1)} + \frac{\Gamma(k\alpha + 1)}{\Gamma((k - 2)\alpha + 1)} - p(p + 1) \right\} c_k \]

\[ = v_0 + v_1 t^{\alpha} + \sum_{k=2}^{\infty} v_k t^{k\alpha} \]

\[ = \sum_{k=0}^{\infty} v_k t^{k\alpha}. \]

As \( \rho_0 \) is arbitrarily close to \( \rho \), this means that (3.4.13) is convergent, for all \( t \in \mathbb{U}. \)

**Theorem 3.4.9.** Suppose that the nonhomogeneous generalized fractional Legendre differential equation (3.4.1) with positive radius of convergence \( \rho > 0 \) and that the coefficient \( v_k \)'s satisfies the condition (3.4.14). Then every solution \( y : (-\rho, \rho) \to \mathbb{C} \) of (3.4.1) can be expressed by

\[ y(t) = y_h(t) + y_p(t), \quad (3.4.16) \]

where \( y_p(t) \) is the solution of generalized fractional Legendre function defined in (3.4.11) with \( \rho = \min \{1, \rho_0\} \) and \( y_h(t) \) defined in (3.4.13).
3.4. **STABILITY OF NONHOMOGENEOUS FDE**

**Proof.** Since the coefficients of (3.4.1) are \( \alpha \)-analytic at the origin, so \( y(t) \) can be written as

\[
y(t) = \sum_{k=0}^{\infty} b_k t^{k\alpha}
\]

where \( b_k \) are to be determined. Substituting (3.4.17) into (3.4.1), yields

\[
\sum_{k=0}^{\infty} \left\{ b_{k+2} \frac{\Gamma((k+2)\alpha + 1)}{\Gamma((k+2)\alpha + 1)} + p(p+1)b_k - v_k \right\} t^{k\alpha} - 2 \sum_{k=1}^{\infty} b_k \frac{\Gamma((k+1)\alpha + 1)}{\Gamma((k+1)\alpha + 1)} t^{k\alpha} - \sum_{k=2}^{\infty} b_k \frac{\Gamma((k\alpha + 1)}{\Gamma((k-2)\alpha + 1)} t^{k\alpha} = 0
\]

for \( t \in (-\rho, \rho) \). Comparing the like powers on both sides we have,

\[
b_2 = \frac{1}{\Gamma(2\alpha + 1)} \left\{ v_0 - p(p+1)b_0 \right\},
\]

\[
b_3 = \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)} [v_1 + \{2\Gamma(\alpha + 1) - p(p+1)\} b_1]
\]

and

\[
b_{k+2} = \frac{\Gamma((k+2)\alpha + 1)}{\Gamma((k+2)\alpha + 1)} \left[ v_k + \left\{ \frac{2\Gamma((k\alpha + 1))}{\Gamma((k+1)\alpha + 1)} + \frac{\Gamma((k\alpha + 1))}{\Gamma((k-2)\alpha + 1)} - p(p+1) \right\} b_k \right],
\]

where \( k > 1 \). Consequently, it follows from (3.4.19) and (3.4.12) that

\[
y(t) = \left( b_0 + b_2 t^{2\alpha} + \sum_{k=2}^{\infty} b_{2k} t^{2k\alpha} \right) + \left( b_1 t^{\alpha} + b_3 t^{3\alpha} + \sum_{k=2}^{\infty} b_{2k+1} t^{(2k+1)\alpha} \right)
\]

\[
y_h(t) =\left( \sum_{k=1}^{\infty} c_{2k} t^{2k\alpha} + \sum_{k=1}^{\infty} c_{2k+1} t^{(2k+1)\alpha} \right)
\]

\[
y(t) = y_h(t) + \sum_{k=2}^{\infty} c_k t^{k\alpha}
\]

\[
y(t) = y_h(t) + y_p(t).
\]

Therefore, by Lemma 3.4.8, every solution \( y : (-\rho, \rho) \rightarrow \mathbb{C} \) of the nonhomogeneous Legendre differential equation (3.4.1) can be expressed as (3.4.16), where \( y_h(t) \) is defined in (3.4.11) and \( y_p(t) \) satisfies (3.4.1). \( \square \)
3.4. Stability of Nonhomogeneous FDE

3.4.4 Generalized Hyers-Ulam stability

First we establish the generalized Hyers-Ulam stability of power series with real or complex coefficients. For the generalized Hyers-Ulam stability of equation (3.4.1), we consider an analytical function $g$ satisfying the equation and suppose that $g$ has generalized Taylor’s series expansion of the form

$$g(t) = \sum_{k=0}^{\infty} a_k t^{k\alpha}. \quad (3.4.20)$$

Let us first give the definition of the generalized Hyers-Ulam stability.

**Definition 3.4.10.** Equation (3.4.20) has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the property:

for every $\varepsilon > 0$, $t \in [-1, 1]$, $p \in \mathbb{R}$ and if

$$\left| \sum_{k=0}^{\infty} a_k t^{k\alpha} \right| \leq \varepsilon \left( \sum_{k=0}^{\infty} |a_k|^p \frac{2^{k\alpha}}{2^{k\alpha}} \right), \quad (3.4.21)$$

then there exists some $v \in [-1, 1]$ satisfying (3.4.20) such that

$$|t - v| \leq K \varepsilon. \quad (3.4.22)$$

**Remark 3.4.11.** For the complex coefficients, we will take the domain as a closed unit disk $B = \{ z \in \mathbb{C}; |z| \leq 1 \}$.

**Remark 3.4.12.** For $t < 0$, we will use $(-1)^\alpha = e^{i\alpha\pi}$.

**Proposition 3.4.13.** If $\sum_{k=0, k \neq 1}^{\infty} |a_k| < |a_1|$ and $\lambda = \sum_{k=2}^{\infty} \frac{k|a_k|}{|a_1|} < 1$ then there exists a exact solution $v \in [-1, 1]$ of (3.4.20).

**Proof.** Let $X = [-1, 1]$. Then $(X, d)$ is complete metric space with $d(x, y) = |x - y|$. For $t \in [-1, 1]$, consider

$$g(t^\alpha) = -\frac{1}{a_1} \sum_{k=0, k \neq 1}^{\infty} a_k t^{k\alpha}. \quad (3.4.23)$$
Then,
\[
|g(t^\alpha)| = \frac{1}{|a_1|} \left| \sum_{k=0, k \neq 1}^{\infty} a_k t^{k\alpha} \right|
\leq \frac{1}{|a_1|} \left( \sum_{k=0, k \neq 1}^{\infty} |a_k| \right) \quad \text{(because } t \in [-1, 1])
< 1.
\]

Hence, \( g \) is well defined on \( X \). Now, we consider \( t_1, t_2 \in [-1, 1] \) such that \( t_1^\alpha \neq t_2^\alpha \).

Then
\[
d(g(t_1^\alpha), g(t_2^\alpha)) \leq \frac{1}{|a_1|} \left( \sum_{k=2}^{\infty} |a_k| |t_1^{k\alpha} - t_2^{k\alpha}| \right)
\leq \frac{1}{|a_1|} \left( |t_1^\alpha - t_2^\alpha| \sum_{k=2}^{\infty} |a_k| \left| \sum_{r=1}^{k} t_1^{(k-r)\alpha} t_2^{(r-1)\alpha} \right| \right)
= \lambda d(t_1, t_2).
\]

Since, \( \lambda < 1 \) by the assumption, it follows from Theorem 2.2.2 that there exist \( v \in X \) such that \( g(v^\alpha) = v^\alpha \).

Consider the function
\[ g(t) := (1 - t^{2\alpha}) z^{(2\alpha)}(t) - 2t^\alpha z^{(\alpha)}(t) + p(p + 1) z(t). \]

**Theorem 3.4.14.** If the conditions of Proposition 3.4.13 and the property (3.4.21) are satisfied then (3.4.20) has the generalized Hyers-Ulam stability.

**Proof.** For \( t \in [-1, 1] \), we consider
\[ g(t^\alpha) = -\frac{1}{a_1} \sum_{k=0, k \neq 1}^{\infty} a_k t^{k\alpha}. \quad (3.4.24) \]

Then, for \( t, v \in X \) we have
\[
|t^\alpha - v^\alpha| \leq |t^\alpha - g(t^\alpha)| + |g(t^\alpha) - v^\alpha|.
\]
Therefore,

\[ |t^\alpha - v^\alpha| \leq \frac{1}{(1-\lambda)|a_1|} \left( \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} \right) \]

Consider \(|a_1| = \max_{k \geq 0} |a_k|\). Then

\[ |t^\alpha - v^\alpha| \leq \frac{\varepsilon |a_1|^{p-1}}{(1-\lambda)2^{\alpha - 1}} \]

where \( K = \frac{2^\alpha |a_1|^{p-1}}{(1-\lambda)(2^\alpha - 1)} \) and \( 0 < \lambda < 1 \). This completes the proof.

For the complex power series we require the following theorem.

**Theorem 3.4.15.** If

\[ \sum_{k=0, k \neq 1}^{\infty} |a_k| < |a_1|, \quad a_1 \neq 0 \]

then there exists a unique solution of (3.4.20) in the unit disk \( \mathcal{B} \).

**Proof.** For \(|z| \leq 1\), by setting

\[ g(z^\alpha) = -\frac{1}{a_1} \left( \sum_{k=0, k \neq 1}^{\infty} a_k z^{k\alpha} \right) \]

Then,

\[ |g(z^\alpha)| = \frac{1}{|a_1|} \left| \sum_{k=0, k \neq 1}^{\infty} a_k z^{k\alpha} \right| \]
\[ \frac{1}{|a_1|} \left( \sum_{k=0, k \neq 1}^{\infty} |a_k||z^{k\alpha}| \right) \leq \frac{1}{|a_1|} \left( \sum_{k=0, k \neq 1}^{\infty} |a_k| \right) \quad : |z| \leq 1. \]

Using the (3.4.25), to obtain

\[ |g(z^\alpha)| < 1. \]

Since \(|g(z^\alpha)| < 1 \) for \(|z| = 1\), hence for \(|g(z^\alpha)| < | - z| > 1\). Then by Rouche’s Theorem, \(g(z^\alpha) - z^\alpha\) has exactly one zero in \(|z| < 1\). Thus, \(g\) has a unique fixed point in \(|z| = 1\). □

**Illustrative Examples**

In this section, our task is to show that there exists \(y(t)\) which satisfies all the conditions of Theorem 3.4.9. We present an example related to the Legendre differential equation (3.4.1).

**Example 3.4.16.** Let \(y : (-1, 1) \rightarrow \mathbb{R}\) be defined by

\[ y(t) = y_h(t) + qy_p(t), \quad y_p(t) = \frac{1}{1-t^\alpha} - 1 - t^\alpha \quad (3.4.27) \]

where, \(y_h(t)\) is the generalized fractional Legendre function (3.4.11) for

\[ a_0 = a_1 = 1. \]

Then (3.4.27) can be written as

\[ y(t) = y_h(t) + \sum_{k=2}^{\infty} q t^{k\alpha} = \left[ 1 + \sum_{k=1}^{\infty} t^{2k\alpha} \left( q - \frac{p(p+1)}{\Gamma(2k\alpha + 1)} \right) t^{2k\alpha} \times \prod_{j=1}^{k-1} \left\{ \frac{\Gamma((2k - 2j)\alpha + 1)}{\Gamma((2k - 2j - 1)\alpha + 1) + \frac{(2k - 2j)\alpha + 1}{\Gamma((2k - 2j - 2)\alpha + 1) - p(p+1)}} \right\} \right] \]
3.5. Conclusion

In this chapter, we proved the Hyers-Ulam stability and generalized Hyers-Ulam stability of fractional differential equations. We also studied the partial solution of fractional generalized fractional Legendre differential equation at \( \alpha \)-ordinary point. Furthermore, we showed that the solution of nonhomogeneous generalized fractional Legendre differential equation satisfies the generalized Hyers-Ulam stability for infinite series of fractional power.