Chapter - II

Structure of Semirings

Satisfying $ab + a = a$
2.1 INTRODUCTION:

In this chapter, we introduce the concept of semiring and study the properties of semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). We study whether the algebraic structure of \((S, \cdot)\) may determine the order structure of \((S, +)\) and vice-versa. Throughout this chapter unless otherwise mentioned \( S \) is a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). We also study the properties of semirings and ordered semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). We proved that in a totally ordered semiring \((S, +, \cdot)\), if \((S, +)\) is positively totally ordered (negatively totally ordered) and \((S, \cdot)\) is commutative, then \((S, \cdot)\) is negatively totally ordered (positively totally ordered).

In section 2.2, the required preliminaries (concepts and examples) are presented. In section 2.3, properties of semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \) are discussed. We also characterize zerosumfree semirings. Here, we have mainly proved that if \((S, +, \cdot)\) is a positive rational domain (PRD) semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \), then

\[(i) \quad a + ab = b + ab, \text{ for all } a, b \text{ in } S\]

\[(ii) \quad a + a^2 = a^2 \text{ and } a^2 + a = a, \text{ for all } \text{‘}a\text{’ } \text{in } S\]
Section 2.4 is concerned with ordered semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). We also discuss the examples.

2.2 PRELIMINARIES:

In this section, we mention some preliminary definitions and examples which will be used in this chapter and later in subsequent chapters.

**Definition 2.2.1:**

A system \((S, \ast)\), where \( S \) is a non-empty set and \( \ast \) is an associative binary operation on \( S \) is called a semigroup.

**Definition 2.2.2:**

A semigroup \((S, \ast)\) is said to be partially ordered if there exists a binary relation ‘\( \leq \)’ on \( S \) satisfying the following properties:

(i) Reflexivity: \( a \leq a \), for every \( a \) in \( S \),

(ii) Antisymmetry: \( a \leq b, b \leq a \) imply \( a = b \) for all \( a,b \) in \( S \),

(iii) Transitivity: \( a \leq b, b \leq c \) imply \( a \leq c \), for all \( a,b,c \) in \( S \),

(iv) Compatibility: \( a \leq b \) implies \( ac \leq bc \) and \( ca \leq cb \), for all \( a, b, c \) in \( S \).

Partially ordered semigroup may also be denoted by \((S, \ast, \leq)\).
Note: Sometimes we write \( a \geq b \) for \( b \leq a \). That is “\( \geq \)” is the dual relation of “\( \leq \)”.

Definition 2.2.3:

A partially ordered semigroup (p. o. s. g) in which any two elements are comparable is said to be a totally ordered semigroup (t. o. s. g.) or fully ordered semigroup (f. o. s. g.).

Definition 2.2.4:

A triple \((S, +, \cdot)\) is said to be a semiring if \( S \) is a non-empty set and “\(+, \cdot\)” are binary operations on \( S \) satisfying that

(i) \((S, +)\) is a semigroup

(ii) \((S, \cdot)\) is a semigroup

(iii) \(a(b + c) = ab + ac\) and \((b + c)a = ba + ca\), for all \(a, b, c\) in \(S\).

Examples of Semirings 2.2.5:

(i) The set of natural numbers under the usual addition, multiplication

(ii) Every distributive lattice \((L, \wedge, \vee)\).

(iii) Any ring \((R, +, \cdot)\).

(iv) If \((M, +)\) is a commutative monoid with identity element zero then the set \(\text{End}(M)\) of all endomorphism of \(M\) is a semiring.
under the operations of point wise addition and composition of functions.

(v) If \( b \) is a fixed element in a semigroup \( (S, +) \) such that \( b + b = b \) and the multiplication ‘\( \cdot \)’ is defined by \( x.y = b \), for all \( x, y \) in \( S \), then \( (S, +, \cdot) \) is a semiring.

(vi) Let \( S = \{a, b\} \) with the operations given by the following tables:

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Then \( (S, +, \cdot) \) is a semiring.

**Definition 2.2.6:**

A semiring \( (S, +, \cdot) \) is said to be totally ordered semiring (t.o.s.r.) if there exists a partially order ‘\( \leq \)’ on \( S \) such that

(i) \( (S, +) \) is a t. o. s. g.

(ii) \( (S, \cdot) \) is a t. o. s. g.

It is usually denoted by \( (S, +, \cdot, \leq) \).

**Definition: 2.2.7:**

An element ‘\( x \)’ in a totally ordered semiring (t. o. s. r.) \( (S, +, \cdot) \) is said to be
(i) an additive identity if $a + x = x + a = a$.

(ii) a multiplicative identity if $ax = xa = a$.

(iii) an additive zero if $a + x = x + a = x$ and

(iv) a multiplicative zero if $ax = xa = x$, for every ‘a’ in S.

Note:

(1) When we say a semiring with zero (or) a t.o.s.r. with zero mean multiplicative zero.

(2) Let ‘x’ is said to be left(right) additive identity if $x + a = a (a + x = a)$ for every ‘a’ in S.

(3) Let ‘x’ is said to be left (right) additive zero if $x + a = x (a + x = x)$ for every ‘a’ in S.

Definition 2.2.8:

A non-empty subset A of a semiring $(S, +, \bullet)$ is said to be a subsemiring of S if $(A, +, \bullet)$ is a semiring by itself.

[Note: A non-empty t.o. subset A of a totally ordered semiring $(S, +, \bullet, \leq)$ is a totally ordered subsemiring of S, if $(A, +, \bullet, \leq)$ is a totally ordered semiring by itself].
Examples of totally ordered semirings 2.2.9:

(i) Consider the set $S = \{0,1,2,3,\ldots\}$ with $m + n = \max(m, n)$ or $\min(m, n)$, $mn = m + n$. Where the addition in the multiplication is the usual addition, for all $m, n$ in $S$ and the order being the usual order relation.

Then $(S, +, \cdot, \leq)$ is a totally ordered semiring.

(ii) If $(X, \leq)$ is a totally ordered set and $+$ and $\cdot$ are min and max operations, then $(X, +, \cdot, \leq)$ is a t.o.s.r. If $S = X \cup \{z\}$ and $+, \cdot$ and $\leq$ are extended to $S$ by defining $s + z = z = z + s$, $s \cdot z = z = z \cdot s$ and $s \leq z$ for all $s$ in $S$, then $(S, +, \cdot, \leq)$ is a t.o.s.r.

(iii) The set of natural numbers under the usual addition and multiplication and ordering.

(iv) Consider the set $S = \{1, 2, 3, 4\}$ with the order $1 < 2 < 3 < 4$ and with the following addition and multiplication.

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Then $(S, +, \cdot)$ is a totally ordered semiring.
Definition 2.2.10: 

In a totally ordered semiring \((S, +, \cdot, \leq)\)

(i) \((S, +, \leq)\) is positively totally ordered (p.t.o.), if \(a + b \geq a, b\) for all \(a, b\) in \(S\) and

(ii) \((S, \cdot, \leq)\) is positively totally ordered (p.t.o.), if \(ab \geq a, b\) for all \(a, b\) in \(S\).

Definition 2.2.11:

A totally ordered semiring \((S, +, \cdot, \leq)\) is said to be a positively ordered in the strict sense if both \((S, +, \leq)\) and \((S, \cdot, \leq)\) are positively ordered in the strict sense.

Definition 2.2.12:

In a t. o. s. r. \((S, +, \cdot, \leq)\)

(i) \((S, +)\) is said to be right naturally totally ordered (r. n. t. o) if \((S, +)\) is positively ordered in the strict sense and if \(a < b\) implies \(b = a + c\) for some \(c\) in \(S\), and

(ii) \((S, \cdot)\) is said to be r. n. t. o. if \((S, \cdot)\) is positively ordered in the strict sense and if \(a < b\) implies \(b = ac\) for some \(c\) in \(S\).
Definition 2.2.13:

(S, +, ⋅, \leq) is said to be a weak partially ordered semiring (w.p.o.s.r.) if (S, +, ⋅) is a semiring and \(\leq\) is a partial order relation on S such that (S, +, ⋅, \leq) is a p.o.s.r. In case \(\leq\) is a total order relation (full order relation) the weak p.o.s.r. is said to be a weak t.o.s.r. (or weak f.o.s.r.).

Definition 2.2.14:

A semigroup (S, ⋅) with zero (usually we denote the zero element by the symbol ‘0’) is said to have no zero-divisors if xy = 0 implies x = 0 (or) y = 0 for all x, y in S.

Definition 2.2.15:

An element x in a semigroup (S, ⋅) is said to be multiplicative idempotent if \(x^2 = x\).

Note:

E (⋅) denotes the set of all multiplicative idempotents in (S, ⋅).

|E (⋅)| denotes the cardinal number of the set E [⋅].

Definition 2.2.16:

An element ‘x’ in a semigroup (S, +) is said to be an additive idempotent if x + x = x.
Note:

E (+) denotes the set of all additive idempotents in (S, +).

|E (+)| denotes the cardinal number of the set E [+].

**Definition 2.2.17:**

(i) A semigroup (S, •) is said to be a band if every element in S is an idempotent.

(ii) A commutative band is called a semilattice.

**Definition 2.2.18:**

An element x in a p.o.s.g (S, •, ≤) is non-negative (non-positive) if \( x^2 \geq x \) (\( x^2 \leq x \)).

**Definition 2.2.19:**

A p.o.s.g. (S, •, ≤) is non-negatively (non-positively) ordered if every element of S is non-negative (non-positive).

**Definition 2.2.20:**

An element x in a t.o.s.r is minimal (maximal) if \( x \leq a \) (\( x \geq a \)) for every a in S.

**Definition 2.2.21:**

In a semigroup (S, •), a non-empty subset A of S is called

(i) a left ideal, if \( sa \in A \), for every \( s \in S \) and for every \( a \in A \)
(ii) a right ideal, if as ∈ A, for every a ∈ A and for every s ∈ S

(iii) an ideal, if A is both a left ideal as well as a right ideal

(iv) a completely prime ideal, if it is an ideal and if ab ∈ A for any a, b in S, then either a ∈ A or b ∈ A and

(v) Completely semiprime, if it is an ideal and if a^2 ∈ A for any a in S, then a ∈ A.

**Definition 2.2.22:**

An element x in a semigroup (S, •) is said to be

(i) left cancellable, if xa = xb for any a, b in S implies a = b;

(ii) right cancellable if ax = bx for any a, b in S implies a = b;

(iii) cancellable if it is both left as well as right cancellable.

**Definition 2.2.23:**

A semigroup (S, •) with all of its elements are left (right) cancellable is said to be left (right) cancellative semigroup.

**Definition 2.2.24:**

A semigroup (S, •) with zero is called o-simple if

(i) {0} and S are the only ideals and

(ii) S^2 ≠ {0}
Definition 2.2.25:

A semigroup \((S, \bullet)\) is weakly commutative, if for any \(x, y \in S\), 
\((xy)^n \in ySx\) for some positive integer ‘\(n\)’.

Definition 2.2.26:

An element ‘\(a\)’ in a t. o. s. r. \((S, \bullet, \leq)\) is said to be

(i) left positive if \(ax \geq x\) for every \(x \in S\).

(ii) right positive if \(xa \geq x\) for every \(x \in S\).

(iii) positive if it is both left as well as right positive.

(iv) left negative if \(ax \leq x\) for every \(x \in S\).

(v) right negative if \(xa \leq x\) for every \(x \in S\).

(vi) negative if it is both left as well as right negative.

Note: ‘\(\geq\)’ is the dual of ‘\(\leq\).

Definition 2.2.27:

Two distinct elements \(a, b\) in a t.o.s.g \((S, \bullet, \leq)\) are said to form an
anomalous pair if \(a^n < b^{n+1}\) and \(b^n < a^{n+1}\) where \(a, b\) are positive (or)
\(a^n > b^{n+1}\) and \(b^n > a^{n+1}\) for all \(n > 0\) where \(a, b\) are negative.
**Definition 2.2.28:**

An element $x$ different from the identity in a non-negatively ordered semigroup $(S, \cdot, \leq)$ is said to be $o$-Archimedean if for every $y$ in $S$ there exists a natural number ‘$n$’ such that $x^n \geq y$.

**Definition 2.2.29:**

A non-negatively ordered semigroup $(S, \cdot, \leq)$ is said to be $o$-Archimedean if every one of its elements different from its identity (if exists) is $o$-Archimedean.

**Definition 2.2.30:**

A t.o.s.g $(S, \cdot, \leq)$ is $o$-isomorphic to a t.o.s.g $(T, \cdot, \leq_1)$ if there exists a mapping $f: S \to T$ such that

(i) $f$ is one-to-one and onto map

(ii) whenever $a \leq b$ for any $a, b$ in $S$, then $f(a) \leq_1 f(b)$ (that is, $f$ preserves the order) and

(iii) $f(ab) = f(a) f(b)$ (that is, $f$ preserves the multiplication)

**Definition 2.2.31:**

A t.o.s.g $(S, +, \cdot, \leq)$ is $o$-isomorphic to a t.o.s.r $(T, \oplus, \ominus, \leq_1)$ if there exists a mapping $f: S \to T$ such that

(i) $f$ is one-to-one and onto map
(ii) whenever \( a \leq b \) for any \( a, b \) in \( S \), then \( f(a) \leq f(b) \) (that is, \( f \) preserves the order)

(iii) \( f(a + b) = f(a) \oplus f(b) \) (that is, \( f \) preserves the addition) and

(iv) \( f(ab) = f(a) \odot f(b) \) (that is, \( f \) preserves the multiplication).

**Definition 2.2.32:**

Let \( a \in S \). The least element of the set \( \{ x \in N : (\text{there exists } y \in N) \ xa = ya, x \neq y \} \) is called the index of \( a \) and is denoted by \( m \), where \( N \) is the set of natural numbers.

**Definition 2.2.33:**

The least element of the set \( \{ x \in N : (m + x)a = ma \} \) is called the period of \( a \) and is denoted by \( r \).

**Definition 2.2.34:**

A semiring \((S, +, \cdot)\) is said to be Boolean semiring if \((S, \cdot)\) is a band.

i.e., \( a = a^2 \), for every ‘\( a \)’ in \( S \).
2.3 STRUCTURE OF SEMIRINGS SATISFYING THE IDENTITY \( ab + a = a \):

In this section, the structure of semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \) and its properties are studied.

**Definition 2.3.1:**

A semiring \((S, +, \cdot)\) is said to be a Positive Rational Domain (PRD) if and only if \((S, \cdot)\) is an abelian group.

**Definition 2.3.2:**

A semiring \(S\) is said to be mono semiring if \(a + b = a \cdot b\) for all \(a, b\) in \(S\).

**Definition 2.3.3:**

A semigroup \((S, \cdot)\) is said to satisfy quasi commutative if \(ab = b^m a\) for some integer \(m = 1\).

**Definition 2.3.4:**

A semigroup \((S, \cdot)\) is said to satisfy quasi separative if \(x^2 = xy = yx = y^2 \Rightarrow x = y\), for all \(x, y\) in \(S\).
**Definition 2.3.5:**

A semigroup $(S, +)$ is said to satisfy weakly separative if \( x + x = x + y = y + y \implies x = y \), for all \( x, y \) in \( S \).

**Definition 2.3.6:**

A semigroup $(S, +)$ is said to be a band if \( a + a = a \), for all ‘\( a \)’ in \( S \).

**Definition 2.3.7:**

A semigroup $(S, \cdot)$ is said to be a band if \( a^2 = a \), for all ‘\( a \)’ in \( S \).

**Definition 2.3.8:**

A semigroup $(S, +)$ satisfying the identity \( x = x + y + x \) is a rectangular band.

**Definition 2.3.9:**

A semigroup $(S, \cdot)$ satisfying the identity \( x = xyx \) is a rectangular band.

**Definition 2.3.10:**

A semigroup $(S, \cdot)$ is said to be left (right) regular if it satisfies the identity \( aba = ab \) (\( aba = ba \)) for all \( a, b \) in \( S \).
Definition 2.3.11:

A semigroup \((S, \cdot)\) is said to be left (right) singular if it satisfies the identity \(ab = a\) (\(ab = b\)) for all \(a, b \in S\).

Definition 2.3.12:

A semigroup \((S, +)\) is said to be left (right) singular if it satisfies the identity \(a + b = a\) (\(a + b = b\)) for all \(a, b \in S\).

Definition 2.3.13:

A semiring \((S, +, \cdot)\) with multiplicative zero is said to be zero-square semiring if \(x^2 = 0\), for all \(x \in S\).

Definition 2.3.14:

A semiring \((S, +, \cdot)\) with additive identity zero is said to be zerosumfree semiring if \(x + x = 0\), for all \(x \in S\).

Definition 2.3.15:

If \((S, \cdot)\) is non-negatively ordered, then an element \(x \in S\) is said to be 0-Archimedean if for every \(x\) and \(y\) in \(S\), there exists a natural number \(m\) such that \(x^m \geq y\).
**Theorem 2.3.16:** Let \((S, +, \cdot)\) be a semiring satisfying the identity 
\[ab + a = a, \text{ for all } a, b \text{ in } S.\] 
If \(S\) contains the multiplicative identity 1, then \((S, +)\) is a band.

**Proof:** Consider \(ab + a = a, \text{ for all } a, b \text{ in } S\)

Taking \(b = 1\)

\[a \cdot 1 + a = a\]

\[a + a = a, \text{ for all } ‘a’ \text{ in } S\]

\(\therefore (S, +)\) is a band.

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**Theorem 2.3.17:** Let \((S, +, \cdot)\) be a semiring satisfying the identity 
\[ab + a = a, \text{ for all } a, b \text{ in } S.\] Let \(S\) containing the multiplicative identity 1 and \((S, +)\) be commutative. Then \((S, \cdot)\) is commutative, if \((S, +)\) is not a rectangular band.

**Proof:** Suppose \((S, +)\) is a rectangular band

Consider \(ab + a = a, \text{ for all } a, b \text{ in } S\)

\[\Rightarrow ab + a + ab = a + ab\]

\[\Rightarrow a (b + 1 + b) = ab + a \quad (\because (S, +) \text{ is commutative})\]

\[\Rightarrow ab = ab + a \quad (\therefore (S, +) \text{ is a rectangular band})\]

\[\Rightarrow ab = a\]

Now \(ab + a = a\)

Taking \(a = 1\)
\[ \Rightarrow 1.b + 1 = 1 \]

\[ \Rightarrow b + 1 = 1, \text{ for all } b \in S \]

Also \( ba + b = b \), for all \( a, b \in S \)

\[ \Rightarrow ba + b + ba = b + ba \]

\[ \Rightarrow b (a + 1 + a) = ba + b \quad (\because (S,+) \text{ is commutative}) \]

\[ \Rightarrow ba = ba + b \quad (\because (S,+) \text{ is a rectangular band}) \]

\[ \Rightarrow ba = b \]

\[ \therefore ab \neq ba, \text{ which proves the result} \]

Also \( ab = a \)

\[ \Rightarrow ab + b = a + b \]

\[ \Rightarrow (a + 1) b = a + b \]

\[ \Rightarrow 1.b = a + b \quad (\because \text{from } b + 1 = 1) \]

\[ \Rightarrow b = a + b = b + a \]

This is evident from the following example

**Example:**

\[
\begin{array}{c|cccc}
+ & l & a & b \\
\hline
l & l & l & l \\
\hline
b & l & b & b \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 1 & a & b \\
\hline
1 & 1 & a & b \\
\hline
b & b & a & b \\
\end{array}
\]
**Theorem 2.3.18:** Let \((S, +, \cdot)\) be a semiring satisfying the identity 
\[ab + a = a,\] for all \(a, b \in S\). Then the following are true.

(i) If \((S, \cdot)\) is a band, then \((S, +)\) is a band

(ii) Converse is true if \((S, +)\) is right cancellative

**Proof:** Since \(ab + a = a\), for all \(a, b \in S\)

(i) Suppose \((S, \cdot)\) is a band

\[\Rightarrow a^2 = a, \text{ for all } a \text{ in } S\]

Given \(ab + a = a\), for all \(a, b \in S\)

Taking \(b = a\)

\[\Rightarrow a.a + a = a\]

\[\Rightarrow a^2 + a = a\]

\[\Rightarrow a + a = a \quad (\because (S, \cdot) \text{ is a band})\]

\[\therefore a + a = a, \text{ for all } a \text{ in } S\]

Hence \((S, +)\) is a band

(ii) To prove that \((S, \cdot)\) is a band

Consider \(ab + a = a\), for all \(a, b \in S\)

Clearly \(a.a + a = a\)

\[\Rightarrow a^2 + a = a\]

\[\Rightarrow a^2 + a = a + a \quad (\therefore (S, +) \text{ is a band})\]

\[\Rightarrow a^2 = a \quad (\because (S, +) \text{ is right cancellative})\]
\[ a^2 = a, \text{ for all } a \text{ in } S \]

Hence \( (S, \circ) \) is a band.

**Theorem 2.3.19:** Let \( (S, +, \circ) \) be a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). Let \( (S, +) \) be commutative and \( (S, \circ) \) is a rectangular band. Then the following are true.

(i) \( ab = a \) and \( ba = b \)

(ii) \( (S, +) \) is a band

**Proof:** Given that \( (S, +) \) is commutative and \( (S, \circ) \) is a rectangular band

(i) Consider \( ab + a = a \) and \( ba + b = b \) for all \( a, b \) in \( S \)

\[ \Rightarrow ab = a (ba + b) \]
\[ \Rightarrow ab = aba + ab \]
\[ \Rightarrow ab = a + ab \quad (\because (S, \circ) \text{ is a rectangular band}) \]
\[ \Rightarrow ab = ab + a \quad (\because (S, +) \text{ is commutative}) \]
\[ \Rightarrow ab = a \]

Also \( ba = b (ab + a) \)

\[ \Rightarrow ba = bab + ba \]
\[ \Rightarrow ba = b + ba \quad (\because (S, \circ) \text{ is a rectangular band}) \]
\[ \Rightarrow ba = ba + b \quad (\because (S, +) \text{ is commutative}) \]
\[ \Rightarrow ba = b \]
\( \therefore ab = a \) and \( ba = b \) for all \( a, b \) in \( S \)

(ii) Consider \( ab = a \), for all \( a, b \) in \( S \)

\[
\Rightarrow ab + a = a + a
\]

\[
\Rightarrow a = a + a
\]

\( \therefore (S, +) \) is a band.

**Theorem 2.3.20:** Let \( (S, +, \cdot) \) be a zerosumfree semiring with additive identity 0. Then \( ab + a = a \), for all \( a, b \) in \( S \) if and only if \( ab = 0 \).

**Proof:** Consider \( ab + a = a \), for all \( a, b \) in \( S \)

\[
\Rightarrow ab + a + a = a + a
\]

\[
\Rightarrow ab + 0 = 0 \quad (\because S \text{ is a zerosumfree semiring, } a + a = 0)
\]

\[
\Rightarrow ab = 0
\]

\( \therefore ab = 0 \)

Conversely,

\[
ab = 0, \text{ for all } a, b \text{ in } S
\]

\[
\Rightarrow ab + a = 0 + a
\]

\[
\Rightarrow ab + a = a, \text{ for all } a, b \text{ in } S
\]

\( \therefore ab + a = a, \text{ for all } a, b \text{ in } S. \)
**Theorem 2.3.21:** Let \((S, +, \cdot)\) be a zero square semiring, where 0 is the additive identity. If \(S\) satisfies the identity \(ab + a = a\), for all \(a, b\) in \(S\), then \(aba = 0\) and \(bab = 0\).

**Proof:** Consider \(ab + a = a\), for all \(a, b\) in \(S\)
\[\Rightarrow aba + a^2 = a^2\]
\[\Rightarrow aba + 0 = 0 \quad (\because S \text{ is zero square semiring, } a^2 = 0)\]
\[\Rightarrow aba = 0\]

Also \(ba + b = b\), for all \(a, b\) in \(S\)
\[\Rightarrow bab + b^2 = b^2\]
\[\Rightarrow bab = 0 \quad (\because S \text{ is zero square semiring, } b^2 = 0)\]

**Theorem 2.3.22:** Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). Let \(S\) containing the multiplicative identity 1 and \((S, \cdot)\) is left singular, then \((S, +)\) is a right singular semigroup.

**Proof:** By hypothesis, \(ab = a\), for all \(a, b\) in \(S\) \((\because (S, \cdot)\) is left singular)
\[\Rightarrow ab + b = a + b\]
\[\Rightarrow (a + 1) b = a + b\]
\[\Rightarrow 1.b = a + b \quad (\because \text{from theorem 2.3.17, } a + 1 = 1)\]
\[\Rightarrow b = a + b\]
Also \( ba = b \)

\[ \Rightarrow ba + a = b + a \]

\[ \Rightarrow (b + 1) a = b + a \]

\[ \Rightarrow 1.a = b + a \quad (\because \text{from theorem 2.3.17, } b + 1 = 1) \]

\[ \Rightarrow a = b + a \]

\[ \therefore a + b = b \text{ and } b + a = a, \text{ for all } a, b \text{ in } S \]

Hence \((S, +)\) is a right singular semigroup.

**Theorem 2.3.23:** Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b \in S\). If \((S, +)\) is a right singular semigroup, then \((S, +)\) is a rectangular band.

**Proof:** By hypothesis, \(a + b = b\), for all \(a, b \in S\)

\[ (\because (S, +) \text{ is right singular}) \]

\[ \Rightarrow a + b + a = b + a \]

\[ = a, \text{ for all } a, b \in S \quad (\because (S, +) \text{ is a right singular}) \]

\[ \therefore (S, +) \text{ is a rectangular band.} \]
Example: The following example satisfies the conditions of theorem 2.3.23.

\[
\begin{array}{cccc}
+ & l & a & b \\
l & l & a & b \\
a & l & a & b \\
b & l & a & b \\
\end{array}
\]

\[
\begin{array}{cccc}
\cdot & l & a & b \\
l & l & a & b \\
a & a & a & a \\
b & b & a & b \\
\end{array}
\]

Theorem 2.3.24: Let \((S, +, \cdot)\) be a PRD semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). Then \(1 + a = a\), for all \(a\) in \(S\).

Proof: Suppose \(ab + a = a\), for all \(a, b\) in \(S\)

\[\Rightarrow a a^{-1} + a = a, \text{ for all } a, a^{-1} \text{ in } S \]

\[\Rightarrow 1 + a = a \]

\[\therefore 1 + a = a, \text{ for all } 'a' \text{ in } S. \]

Theorem 2.3.25: Let \((S, +, \cdot)\) be a PRD semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). Then the following are true.

(i) \(a + ab = b + ab\), for all \(a, b\) in \(S\)

(ii) \(a + a^2 = a^2\) and \(a^2 + a = a\), for all \(a\) in \(S\)
**Proof:** By hypothesis, \( ab + a = a \), for all \( a, b \) in \( S \)

Since \( S \) is a PRD semiring

Using theorem 2.3.24, \( 1 + a = a \), for all \('a'\) in \( S \)

(i) \( ab = a \ (1 + b) \) \( \quad (\because 1 + b = b) \)

\[
= a + ab
\]

Also \( ab = (1 + a) \ b \) \( \quad (\because 1 + a = a) \)

\[
= b + ab
\]

\[
\therefore a + ab = b + ab
\]

(ii) \( 1 + a = a \), for all \( a \) in \( S \)

\[
\Rightarrow a(1 + a) = a.a
\]

\[
\Rightarrow a + a^2 = a^2
\]

Now \( ab + a = a \), for all \( a, b \) in \( S \)

Taking \( b = a \)

\[
\Rightarrow a.a + a = a, \text{ for all } 'a' \text{ in } S
\]

\[
\Rightarrow a^2 + a = a
\]

Hence \( a + a^2 = a^2 \) and \( a^2 + a = a \), for all \('a'\) in \( S \).

**Theorem 2.3.26:** Let \( (S, +, \bullet) \) be a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). If \( (S, \bullet) \) is a left regular semigroup, then \( (S, +) \) is \( E \)-inversive semigroup.
Proof: By hypothesis, \((S, \cdot)\) is a left regular semigroup

i.e., \(aba = ab\)

Consider \(ab + a = a\), for all \(a, b\) in \(S\)

\[\Rightarrow b (ab + a) = b.a\]

\[\Rightarrow bab + ba = ba\]

\[\Rightarrow ba + ba = ba, \text{ for all } ba \text{ in } E[+] \quad (\because \text{\((S, \cdot)\) is a left regular, } bab = ba)\]

Where \(E[+]\) is the set of all additive idempotents in \((S, +)\)

This means that there exists \(a\) in \(S\) such that \(ba + ba = ba\)

\[\Rightarrow b \text{ is an } E \text{ – inversive element}\]

Hence \((S, +)\) is an \(E\) – inversive semigroup.

Theorem 2.3.27: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). If \(S\) contains a multiplicative identity which is also an additive identity, then \((S, \cdot)\) is quasi separative.

Proof: Let `\(e\)` be the multiplicative identity is also an additive identity

By hypothesis, \(ab + a = a\), for all \(a, b\) in \(S\)

To prove that, \((S, \cdot)\) is quasi separative

i.e., \(a^2 = ab = ba = b^2 \Rightarrow a = b\), for all \(a, b\) in \(S\)

Let \(a^2 = ab\)

\[\Rightarrow a^2 = a.(b + e)\]
= ab + a

= a  (∴ ab + a = a)

⇒ a^2 = a

Similarly,

⇒ b^2 = ba

= b (a + e)

⇒ b^2 = ba + b

= b  (∴ ba + b = b)

⇒ b^2 = b

∴ a^2 = a and b^2 = b

If  a^2 = ab = ba = b^2

⇒ a = ab = ba = b

⇒ a = b

Hence, (S, •) is quasi separative.
2.4. STRUCTURE OF ORDERED SEMIRINGS SATISFYING
THE IDENTITY $ab + a = a$:

In this section, the structure of ordered semirings satisfying the identity $ab + a = a$, for all $a, b$ in $S$ are studied.

**Theorem 2.4.1:** Let $(S, +, \cdot)$ be a totally ordered semiring and satisfying the identity $ab + a = a$, for all $a, b$ in $S$. If $(S, +)$ is p.t.o (n.t.o.) and $(S, \cdot)$ is commutative, then $(S, \cdot)$ is n.t.o. (p.t.o.).

**Proof:** Let $ab + a = a$, for all $a, b$ in $S$

$\Rightarrow a = ab + a \geq ab$ \hspace{1cm} \(\because (S, +) \text{ is p.t.o.}\)

$\Rightarrow a \geq ab$

Suppose $ab > b$

$\Rightarrow ab + a \geq b + a$

$\Rightarrow a \geq b + a$ \hspace{1cm} \(\because ab + a = a\)

$\Rightarrow b + a \leq a$, which contradicts the hypothesis that $(S, +)$ is p.t.o.

$\Rightarrow ab \leq b$

$\therefore ab \leq a \ & \ ab \leq b$

Hence $(S, \cdot)$ is n.t.o.

Similarly we can prove that $(S, \cdot)$ is p.t.o., if $(S, +)$ is n.t.o.

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**Theorem 2.4.2:** Let \((S, +, \cdot)\) be a totally ordered semiring and satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is non – negatively ordered (non – positively ordered.), then \((S, \cdot)\) is non – positively ordered (non – negatively ordered).

**Proof:** By hypothesis, \(ab + a = a\), for all \(a, b\) in \(S\)

Taking \(b = a\)

\[\Rightarrow a^2 + a = a, \text{ for all ‘}a\text{’ in }S\]

Suppose \((S, +)\) is non – negatively ordered

\[\Rightarrow a = a^2 + a \geq a\text{ and }a^2\]

\[\Rightarrow a \geq a^2\]

\[\therefore (S, \cdot)\text{ is non – positively ordered}\]

Suppose \((S, +)\) is non – positively ordered

\[\Rightarrow a = a^2 + a \leq a\text{ and }a^2\]

\[\Rightarrow a \leq a^2\]

\[\therefore (S, \cdot)\text{ is non – negatively ordered}\]
Theorem 2.4.3: Let \((S, +, \cdot)\) be a totally ordered PRD semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is non–negatively ordered (non–positively ordered.), then \((S, \cdot)\) is non–negatively ordered (non–positively ordered).

Proof: Using theorem 2.3.25, \(a + a^2 = a^2\), for all ‘a’ in \(S\)

Since \((S, +)\) is non–negatively ordered, \(a^2 = a + a^2 \geq a\)

\[\Rightarrow a^2 \geq a, \text{ for all ‘a’ in } S\]

i.e., \((S, \cdot)\) is non–negatively ordered

Suppose \((S, +)\) is non–positively ordered, \(a^2 = a + a^2 \leq a\)

\[\Rightarrow a^2 \leq a, \text{ for all ‘a’ in } S\]

i.e., \((S, \cdot)\) is non–positively ordered

Theorem 2.4.4: Let \((S, +, \cdot)\) be a totally ordered PRD semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is p.t.o (n.t.o.), then 1 is minimum (maximum) element.

Proof: By theorem 2.3.24, \(1 + a = a\), for all \(a, 1\) in \(S\)

Suppose \((S, +)\) is p.t.o.

\[\Rightarrow a = 1 + a \geq a \text{ and } 1\]

\[\Rightarrow a \geq 1\]

\[\therefore 1 \text{ is the minimum element.}\]

Suppose \((S, +)\) is n.t.o.
\[ \Rightarrow a = 1 + a \leq a \text{ and } 1 \]

\[ \Rightarrow a \leq 1 \]

\[ \therefore 1 \text{ is the maximum element.} \]