Publications
PROPERTIES OF SEMIRINGS

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ABSTRACT

In this paper, we study the properties of semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). We characterize zerosum semirings.

Keywords: Rectangular band; Left (Right) singular; PRD; Mono semiring.

2000 Mathematics Subject Classification: 20M10, 16Y60.

I. INTRODUCTION:

A triple \((S, +, \cdot)\) is called a semiring if \((S, +)\) is a semigroup; \((S, \cdot)\) is semigroup; \(a(b + c) = ab + ac\) and \((b + c)a = ba + ca\) for every \( a, b, c \) in \( S \). A semiring \((S, +, \cdot)\) is said to be a totally ordered semiring if the additive semigroup \((S, +)\) and multiplicative semigroup \((S, \cdot)\) are totally ordered semigroups under the same total order relation. An element \( x \) in a totally ordered semigroup \((S, \cdot)\) is non-negative (non-positive) if \( x^2 \geq x \) (\( x^2 \leq x \)). A totally ordered semigroup \((S, \cdot)\) is said to be non-negatively (non-positively) ordered if every one of its elements is non-negative (non-positive). \((S, \cdot)\) is positively (negatively) ordered in strict sense if \( xy \geq x \) and \( xy \geq y \) (\( xy \leq x \) and \( xy \leq y \)) for every \( x \) and \( y \) in \( S \). A semigroup \((S, +)\) is said to be a band if \( a + a = a \) for all \( a \) in \( S \). A semigroup \((S, +)\) is said to be rectangular band if \( a + b + a = a \) for all \( a, b \) in \( S \). A semigroup \((S, \cdot)\) is said to be a band if \( a.a = a^2 = a \) for all \( a \) in \( S \). A semigroup \((S, \cdot)\) is said to be left (right) singular if \( ab = a \) (\( ab = b \)) for all \( a, b \) in \( S \). A semigroup \((S, +)\) is said to be left (right) singular if \( a + b = a \) (\( a + b = b \)) for all \( a, b \) in \( S \). A semiring \((S, +, \cdot)\) is said to be a rectangular band if \( a + b + a = a \). A semiring \((S, +, \cdot)\) is said to be monosemiring if \( a + b = ab \) for all \( a, b \) in \( S \). A semiring \((S, +, \cdot)\) with additive identity zero which is multiplicative zero is said to be zero square ring if \( x^2 = 0 \) for all \( x \in S \). A semiring \((S, +, \cdot)\) is said to be a Positive Rational Domain (PRD) if and only if \((S, \cdot)\) is an abelian group. A semiring \((S, +, \cdot)\) with additive identity zero is said to be zerosumfree semiring if \( x + x = 0 \) for all \( x \in S \).

Theorem 1: Let \((S, +, \cdot)\) be a semiring satisfying the identity \( ab + a = a \) for all \( a, b \) in \( S \). If \( S \) contains the multiplicative identity \( 1 \), then \((S, +)\) is a band.

Proof: Consider \( ab + a = a \) for all \( a, b \) in taking \( b = 1 \)

\[ a.1 + a = a \]
\[ a + a = a, \text{ for all } a \in S \]

\( \therefore \) \((S, +)\) is a band

Theorem 2: Let \((S, +, \cdot)\) be a semiring satisfying the identity \( ab + a = a \) for all \( a, b \) in \( S \). Let \( S \) contain the multiplicative identity \( 1 \) and \((S, +)\) be commutative. Then \((S, \cdot)\) is commutative if \((S, +)\) is not a rectangular band.

Proof: Suppose \((S, +)\) is a rectangular band

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Consider \( ab + a = a \), for all \( a, b \) in \( S \)

\[
\Rightarrow ab + a + ab = a + ab
\]

\[
\Rightarrow a (b + 1 + b) = ab + a \quad (\because (S,+) \text{ is commutative})
\]

\[
\Rightarrow ab = ab + a \quad (\because (S,+) \text{ is a rectangular band})
\]

\[
\Rightarrow ab = a
\]

Now \( ab + a = a \)

Taking \( a = 1 \)

\[
\Rightarrow 1 + b + 1 = 1
\]

\[
\Rightarrow b + 1 = 1, \text{ for all } b \text{ in } S
\]

Also \( ba + b = b \), for all \( a, b \) in \( S \)

\[
\Rightarrow ba + b + ba = b + ba
\]

\[
\Rightarrow b (a + 1 + a) = ba + b \quad (\because (S,+) \text{ is commutative})
\]

\[
\Rightarrow ba = ba + b \quad (\because (S,+) \text{ is a rectangular band})
\]

\[
\Rightarrow ba = b
\]

\[
\therefore ab \neq ba, \text{ which proves the result.}
\]

Also \( ab = a \)

\[
\Rightarrow ab + b = a + b
\]

\[
\Rightarrow (a + 1) b = a + b
\]

\[
\Rightarrow 1. b = a + b \quad (\because \text{from } b + 1 = 1)
\]

\[
\Rightarrow b = a + b = b + a
\]

This is evident from the following example

**Example:**

<table>
<thead>
<tr>
<th>+</th>
<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>a</td>
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<tr>
<td>b</td>
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<td>b</td>
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<table>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

**Theorem 3:** Let \((S, +, -)\) be a semiring satisfying the identity \( ab + a = a \) for all \( a, b \) in \( S \). Then the following are true.

i) If \((S, -)\) is a band, then \((S, +)\) is a band.

ii) Converse is true if \((S, +)\) is right cancellative.

**Proof:**

(i) Given \( ab + a = a \) for all \( a, b \) in \( S \)

Taking \( b = a \)

\[
\Rightarrow a.a + a = a
\]

\[
\Rightarrow a^2 + a = a
\]

\[
\Rightarrow a + a = a \quad (\because (S, -) \text{ is a band})
\]
\[ a + a = a, \text{ for all } a \in S \]

Hence \((S, +)\) is a band

\textbf{(ii)} To prove that \((S, \cdot)\) is a band

Consider \(ab + a = a\) for all \(a, b \in S\)

Clearly \(a.a + a = a\)

\[ a^2 + a = a \]  
\[ \Rightarrow a^2 + a = a + a \quad (\because \text{\((S, +)\) is a band}) \]
\[ \Rightarrow a^2 = a \quad (\because \text{\((S, +)\) is right cancellative}) \]

\[ \therefore a^2 = a, \text{ for all } a \in S \]

Hence \((S, \cdot)\) is a band.

\textbf{Theorem 4:} Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\) for all \(a, b \in S\). Let \((S, +)\) be commutative and \((S, \cdot)\) is rectangular band. Then the following are true.

\textbf{i)} \(ab = a\) and \(ba = b\)

\textbf{ii)} \((S, +)\) is a band.

\textbf{Proof:}

\textbf{(i)} Consider \(ab + a = a\) for all \(a, b \in S\) and \(ba + b = b\) for all \(b, a \in S\)

\[ \Rightarrow ab = a \ (ba + b) \]
\[ \Rightarrow ab = aba + ab \]
\[ \Rightarrow ab = a + ab \quad (\because \text{\((S, \cdot)\) is a rectangular band}) \]
\[ \Rightarrow ab = ab + a \quad (\because \text{\((S, +)\) is commutative}) \]
\[ \Rightarrow ab = a \]

Also \(ba = b \ (ab + a)\)

\[ \Rightarrow ba = bab + ba \]
\[ \Rightarrow ba = b + ba \quad (\because \text{\((S, \cdot)\) is a rectangular band}) \]
\[ \Rightarrow ba = ba + b \quad (\because \text{\((S, +)\) is commutative}) \]
\[ \Rightarrow ba = b \]

\[ \therefore ab = a \text{ and } ba = b \text{ for all } a, b \in S \]

\textbf{(ii)} Consider \(ab + a = a\) for all \(a, b \in S\)

\[ \Rightarrow ab + a = a + a \]
\[ \Rightarrow a = a + a \]

\[ \therefore (S, +) \text{ is a band} \]

\textbf{Theorem 5:} Let \((S, +, \cdot)\) be a zerosumfree semiring with additive identity 0. Then \(ab + a = a\) for all \(a, b \in S\) if and only if \(ab = 0\).

\textbf{Proof:} Consider \(ab + a = a\) for all \(a, b \in S\)
⇒ \( ab + a + a = a + a \)
⇒ \( ab + 0 = 0 \) (\( \because \) S is a zerosumfree semiring, \( a + a = 0 \))
⇒ \( ab = 0 \)
∴ \( ab = 0 \)

Conversely,
\( ab = 0 \), for all \( a, b \) in \( S \)
⇒ \( ab + a = 0 + a \)
⇒ \( ab + a = a \), for all \( a, b \) in \( S \)
∴ \( ab + a = a \), for all \( a, b \) in \( S \)

**Theorem 6:** Let \( (S, +, \cdot) \) be a zero square semiring, where \( 0 \) is the additive identity. If \( S \) satisfies the identity \( ab + a = a \) for all \( a, b \) in \( S \), then \( aba = 0 \) and \( bab = 0 \).

**Proof:** Consider \( ab + a = a \), for all \( a, b \) in \( S \)
⇒ \( aba + a^2 = a^2 \)
⇒ \( aba + 0 = 0 \) (\( \because \) S is zero square semiring, \( a^2 = 0 \))
⇒ \( aba = 0 \)

Also \( ba + b = b \), for all \( a, b \) in \( S \)
⇒ \( bab + b^2 = b^2 \)
⇒ \( bab + 0 = 0 \) (\( \because \) S is zero square semiring, \( a^2 = 0 \))
⇒ \( bab = 0 \)

**Theorem 7:** Let \( (S, +, \cdot) \) be a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). Let \( S \) contain the multiplicative identity \( 1 \) and \( (S, \cdot) \) be a left singular, then \( (S, +) \) is a right singular semigroup.

**Proof:** By hypothesis \( ab = a \), for all \( a, b \) in \( S \) (\( \because \) \( (S, \cdot) \) is left singular)
⇒ \( ab + b = a + b \)
⇒ \( (a + 1) b = a + b \)
⇒ \( 1. b = a + b \) (\( \because \) from theorem 2)
⇒ \( b = a + b \)

Also \( ba = b \)
⇒ \( ba + a = b + a \)
⇒ \( (b + 1) a = b + a \)
⇒ \( 1. a = b + a \) (\( \because \) from theorem 2)
⇒ \( a = b + a \)
∴ \( a + b = b \) and \( b + a = a \), for all \( a, b \) in \( S \)

Hence \( (S, +) \) is a right singular semigroup.
**Theorem 8:** Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is a right singular semigroup, then \((S, +)\) is a rectangular band.

**Proof:** By hypothesis \(a + b = b\), for all \(a, b\) in \(S\) \((\because (S, +)\) is right singular)

\[a + b + a = b + a\]

\[a + b + a = a\], for all \(a, b\) in \(S\), which proves the theorem. \((\because (S, +)\) is a right singular semigroup)

i.e., \((S, +)\) is a rectangular band.

The following is an example for theorem7.

**Example:**

<table>
<thead>
<tr>
<th>+</th>
<th>1</th>
<th>a</th>
<th>b</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
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<td>1</td>
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<tr>
<td>b</td>
<td>1</td>
<td>a</td>
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<table>
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<tr>
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<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
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<td>a</td>
<td>a</td>
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<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>b</td>
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</tbody>
</table>

**Theorem 9:** Let \((S, +, \cdot)\) be a PRD satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). Then \(1 + a = a\), for all \(a\) in \(S\).

**Proof:** Suppose \(ab + a = a\), for all \(a, b\) in \(S\)

\[a a^{-1} + a = a\], for all \(a, a^{-1}\) in \(S\)

\[1 + a = a\]

\[\therefore 1 + a = a\], for all \(a\) in \(S\)

**Theorem 10:** Let \((S, +, \cdot)\) be a PRD satisfying the identity \(ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is commutative, then the following are true.

(i) \((S, +, \cdot)\) is mono semiring
(ii) \((S, \cdot)\) is a band

**Proof:** Given that \((S, +)\) is commutative

(i) \(ab = (1 + a)(1 + b)\)

\[ab = 1 + b + a + ab\]

\[\Rightarrow ab = (1 + b) + (a + ab)\]

\[\Rightarrow ab = (1 + b) + (a + ab)\]

\[\Rightarrow ab = (1 + b) + (ab + a)\] \((\because (S, +)\) is commutative)

\[\Rightarrow ab = b + a\] \((\because 1 + b = b \& ab + a = a)\)

\[\Rightarrow ab = a + b\]

\[\therefore ab = a + b\], for all \(a, b\) in \(S\).

In particular \(1 = a a^{-1} = a + a^{-1}\) for all \(a, a^{-1}\) in \(S\)

Hence, \((S, +, \cdot)\) is mono semiring.

(ii) Using theorem 9, \(1 + a = a\), for all \(a\) in \(S\)

\[a (1 + a) = a a\]

\[a + a^2 = a^2\] (1)
Now \( ab + a = a \), for all \( a, b \) in \( S \)

taking \( b = a \),

\[ a.a + a = a, \text{ for all } a \text{ in } S \]

\[ a^2 + a = a \]

\[ a + a^2 = a \quad (\because (S, +) \text{ is commutative}) \quad (2) \]

From (1) and (2), \( a = a^2 \), for all \( a \) in \( S \)

\[ \therefore \quad (S, \cdot) \text{ is a band.} \]

**Theorem 11:** Let \((S, +, \cdot)\) be a totally ordered semiring and satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). If \((S, +)\) is p.t.o (n.t.o.) and \((S, \cdot)\) is commutative, then \((S, \cdot)\) is n.t.o. (p.t.o.).

**Proof:** Let \( ab + a = a \), for all \( a, b \) in \( S \)

\[ a = ab + a \geq ab \quad (\therefore (S, +) \text{ is p.t.o.}) \]

\[ a \geq ab \]

Suppose \( ab > b \)

\[ ab + a \geq b + a \]

\[ a \geq b + a \quad (\therefore ab + a = a) \]

\[ b + a \leq a \]

Which contradicts the hypothesis that \((S, +)\) is p.t.o.

\[ ab \leq b \]

\[ \therefore \quad ab \leq a \& \ ab \leq b \]

Hence \((S, \cdot)\) is n.t.o.

Similarly we can prove that \((S, \cdot)\) is p.t.o if \((S, +)\) is n.t.o.

**REFERENCES**


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STRUCTURE OF SEMIRINGS

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ABSTRACT

In this paper, we study some properties of semirings and ordered semirings satisfying the identity \( a + ab + a = a \), for all \( a, b \) in \( S \). We characterize zerosum Semirings and zero square Semirings.

Keywords: Rectangular band; Left (Right) singular; PRD; Mono semiring, E – inverse semigroup, Boolean semiring, C - semiring.

2000 Mathematics Subject Classification: 20M10, 16Y60.

1. INTRODUCTION:

A triple \((S, +, \cdot)\) is called a semiring if \((S, +)\) is a semigroup; \((S, \cdot)\) is semigroup; \(a (b + c) = ab + ac\) and \((b + c) a = ba + ca\) for every \( a, b, c \) in \( S \). A semiring \((S, +, \cdot)\) is said to be a totally ordered semiring if the additive semigroup \((S, +)\) and multiplicative semigroup \((S, \cdot)\) are totally ordered semigroups under the same total order relation. An element \( x \) in a totally ordered semigroup \((S, \cdot)\) is non-negative (non-positive) if \( x^2 \geq x \) (\( x^2 \leq x \)). A totally ordered semigroup \((S, \cdot)\) is said to be non-negatively (non-positively) ordered if every one of its elements is non-negative (non-positive). \((S, \cdot)\) is positively (negatively) ordered in strict sense if \( xy \geq x \) and \( xy \geq y \) (\( xy \leq x \) and \( xy \leq y \)) for every \( x \) and \( y \) in \( S \). A semigroup \((S, +)\) is said to be a band if \( a + a = a \) for all \( a \) in \( S \). A semigroup \((S, \cdot)\) is said to be rectangular band if \( a + b + a = a \) for all \( a, b \) in \( S \). A semigroup \((S, +)\) is said to be a band if \( a + a = a \) and \( (b + c) a = ba + ca \) for every \( a, b, c \) in \( S \). A semiring \((S, +, \cdot)\) is said to be a totally ordered semiring if the additive semigroup \((S, +)\) and multiplicative semigroup \((S, \cdot)\) are totally ordered semigroups under the same total order relation. A semiring \((S, +, \cdot)\) is said to be non-negativity (non-positivity) ordered if every one of its elements is non-negative (non-positive). \((S, \cdot)\) is positively (negatively) ordered in strict sense if \( xy \geq x \) and \( xy \geq y \) (\( xy \leq x \) and \( xy \leq y \)) for every \( x \) and \( y \) in \( S \). A semiring \((S, +)\) is said to be a band if \( a + a = a \) for all \( a \) in \( S \). A semiring \((S, +, \cdot)\) is said to be a totally ordered semiring if \( a + ab + a = a \), for all \( a, b \) in \( S \).

Theorem 1: Let \((S, +, \cdot)\) be a semiring. If \( S \) contains the multiplicative identity which is also an additive identity, then \((S, +)\) is left singular if and only if \((S, \cdot)\) is left (right) singular. If \( \alpha \neq 0 \) in \( S \), then \( \alpha \cdot \alpha = 0 \). If \( \alpha = 0 \) in \( S \), then \( \alpha \cdot \alpha = \alpha \).

Proof: Let \( 'e' \) be the multiplicative identity which is also an additive identity.

Assume that \( S \) satisfies the condition \( a + ab + a = a \), for all \( a, b \) in \( S \).

\[
\begin{align*}
\Rightarrow & a + ab + a = a \\
\Rightarrow & a + a = a \\
\Rightarrow & ab + a = a \\
\Rightarrow & a + ab = a \\
\Rightarrow & ab + a = a \\
\Rightarrow & (S, \cdot) \text{ is left singular.}
\end{align*}
\]

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Conversely, let \((S, \cdot)\) be a left singular semigroup.

Consider \(a + ab + a = a [e + b] + a\)
\[
= ab + a \\
= a [b + e] \\
= ab \\
= a
\]

Hence, \(S\) satisfies the identity \(a + ab + a = a\), for all \(a, b\) in \(S\).

**Definition:** An element \(a\) of a semigroup \(S\) is called an E-inverse if there is an element \(x\) in \(S\) such that \(ax + ax = ax\), i.e. \(ax \in E(S)\), where \(E(S)\) is the set of all idempotent elements of \(S\).

\(\rightarrow\) A Semigroup \(S\) is called an E-inverse Semigroup if every element of \(S\) is an E-inverse.

**Theorem 2:** Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \(S\) contains the multiplicative identity which is also an additive identity then \((S, +)\) is E-inverse semigroup.

**Proof:** Consider \(a + ab + a = a\), for all \(a, b\) in \(S\)
\[
\Rightarrow a + a [b + e] = a \\
\Rightarrow a + ab = a \\
\Rightarrow ab + ab^2 = ab \\
\Rightarrow ab + ab.b = ab \\
\Rightarrow ab + ab = ab \quad (\text{\textasciitilde using theorem 1,} (S, \cdot) \text{ is a left singular}) \\
\therefore (S, +) \text{ is E-inverse semigroup.}
\]

**Theorem 3:** Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \(S\) contains the multiplicative identity which is also an additive identity, then \((S, +, \cdot)\) is a monosemiring and \((S, +)\) is left singular semigroup.

**Proof:** Assume that \(S\) satisfies the condition \(a + ab + a = a\), for all \(a, b\) in \(S\)

Let \(e\) be the multiplicative identity which is also an additive identity

Given \(a + ab + a = a\), for all \(a, b\) in \(S\)
\[
\Rightarrow a + ab + a + b = a + b \\
\Rightarrow a [e + b] + a + b = a + b \\
\Rightarrow a b + a + b = a + b \\
\Rightarrow a [b + e] + b = a + b \\
\Rightarrow a b + b = a + b \\
\Rightarrow [a + e] b = a + b \\
\Rightarrow a b = a + b \\
\Rightarrow a = ab = a + b \\
\therefore (S, +, \cdot) \text{ is monosemiring and} (S, +) \text{ is left singular semigroup.}
\]

**Theorem 4:** Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \(S\) contains the multiplicative identity which is also an additive identity, then \((S, +)\) is a rectangular band.

**Proof:** Assume that \(S\) satisfies the condition \(a + ab + a = a\), for all \(a, b\) in \(S\)

Let \(e\) be the multiplicative identity which is also an additive identity

\[
i.e.\ a.e = e.a = a \& a + e = e + a = a.
\]

Given \(a + ab + a = a\), for all \(a, b\) in \(S\)
\[
\Rightarrow a + b + a + ab + a = a + b + a \\
\Rightarrow a + b + a [e + b] + a = a + b + a \\
\Rightarrow a + b + ab + a = a + b + a \\
\Rightarrow a + [e + a] b + a = a + b + a \\
\Rightarrow a + ab + a = a + b + a \\
\Rightarrow a = a + b + a \\
\therefore a + b + a = a \text{ \ for all } a, b \text{ in } S
\]

Hence \((S, +)\) is rectangular band.
Theorem 5: Let $(S, +, \cdot)$ be a semiring satisfying the identity $a + ab + a = a$, for all $a, b$ in $S$. If $S$ contains the multiplicative identity which is also an additive identity, then

(i) $(S, +)$ is a band

(ii) $(S, \cdot)$ is a band

Proof: Assume that $S$ satisfies the condition $a + ab + a = a$, for all $a, b$ in $S$.

Let $e$ be the multiplicative identity which is also an additive identity.

(i) Let $a + ab + a = a$, for all $a, b$ in $S$

$\Rightarrow a[e + b] + a = a$

$\Rightarrow ab + a = a$

$\Rightarrow a + ab + a = a + a$

$\Rightarrow a = a + a$

$\therefore a + a = a$, for all $a$, in $S$

Hence $(S, +)$ is band.

(ii) Consider $a + ab + a = a$, for all $a, b$ in $S$

$\Rightarrow a + ab + a = a$

Taking $b = a$

$\Rightarrow a + a^2 + a = a$

$\Rightarrow a[e + a] + a = a$

$\Rightarrow a^2 + a = a$

$\Rightarrow a[a + e] = a$

$\Rightarrow a^2 = a$

$\therefore a^2 = a$, for all $a$, in $S$

Hence $(S, \cdot)$ is band.

Theorem 6: Let $(S, +, \cdot)$ be a zero sum free semiring containing the multiplicative identity which is also an additive identity. Then $a + ab + a = a$, for all $a, b$ in $S$ if and only if $ab = 0$.

Proof: Consider $a + ab + a = a$ for all $a, b$ in $S$

$\Rightarrow a + ab + a + a = a + a$

$\Rightarrow a + ab + 0 = 0$ ($\because S$ is a zero sum free semiring, $a + a = 0$)

$\Rightarrow a + ab = 0$

$\Rightarrow a[e + b] = 0$

$\Rightarrow ab = 0$

$\therefore ab = 0$

Conversely,

$ab = 0$, for all $a, b$ in $S$

$\Rightarrow a + ab = a + 0$

$\Rightarrow a + ab = a$, for all $a, b$ in $S$

$\Rightarrow a[e + b] = a$

$\Rightarrow ab = a$

$\Rightarrow a + ab = a + a$

$\Rightarrow a + ab = 0$

$\Rightarrow a + ab + a = 0 + a$

$\Rightarrow a + ab + a = a$

$\therefore a + ab + a = a$, for all $a, b$ in $S$.

Theorem 7: Let $(S, +, \cdot)$ be a semiring satisfying the identity $a + ab + a = a$, for all $a, b$ in $S$. If $S$ contains the multiplicative identity which is also an additive identity and $(S, +)$ is a left cancellative, then $S$ is a zero square semiring.

Proof: Assume that $S$ satisfies the condition $a + ab + a = a$, for all $a, b$ in $S$.

Let $e$ be the multiplicative identity which is also an additive identity.
Given $a + ab + a = a$, for all $a, b$ in $S$

Taking $b = a$

\[ a + a^2 + a = a \]
\[ a + a [a + e] = a \]
\[ a + a^2 = a + 0 \]
\[ a^2 = 0 \quad (\because (S, +) \text{ is a left cancellative}) \]
\[ \therefore a^2 = 0, \text{ for all } a \text{ in } S \]

Hence $S$ is a zero square semiring.

**Theorem 8:** Let $(S, +, \cdot)$ be a zero square semiring, where $0$ is the additive identity. If $S$ satisfies the identity $a + ab + a = a$, for all $a, b$ in $S$, then $aba = 0$ and $bab = 0$.

**Proof:** Consider $a + ab + a = a$, for all $a, b$ in $S$

\[ (a + ab + a).a = a.a \]
\[ a^2 + aba + a^2 = a^2 \]
\[ 0 + aba + 0 = 0 \quad (\because S \text{ is zero square semiring, } a^2=0) \]
\[ \therefore aba = 0 \]

Also $b + ba + b = b$, for all $a, b$ in $S$

\[ (b + ba + b).b = b.b \]
\[ b^2 + bab + b^2 = b^2 \]
\[ 0 + bab + 0 = 0 \quad (\because S \text{ is zero square semiring, } b^2=0) \]
\[ \therefore bab = 0 \]

**Theorem 9:** Let $(S, +, \cdot)$ be a zero square semiring, where $0$ is the additive identity. If $S$ satisfies the identity $a + ab + a = a$, for all $a, b$ in $S$, then $(S, +)$ is a band.

**Proof:** Consider $a + ab + a = a$, for all $a, b$ in $S$.

Taking $b = a$

\[ a + a^2 + a = a \]
\[ a + 0 + a = a \quad (\because S \text{ is zero square semiring, } a^2=0) \]
\[ \therefore (S, +) \text{ is a band.} \]

**Theorem 10:** Let $(S, +, \cdot)$ be a semiring satisfying the identity $a + ab + a = a$, for all $a, b$ in $S$. If $(S, \cdot)$ is a right singular, then $(S, +)$ is a rectangular band.

**Proof:** By hypothesis $ab = b$, for all $a, b$ in $S$ (\because $(S, \cdot)$ is right singular)

Consider $a + ab + a = a$, for all $a, b$ in $S$

\[ a + b + a = a \quad (\because (S, \cdot) \text{ is right singular}) \]
\[ \therefore a + b + a = a, \text{ for all } a, b \text{ in } S \]

Hence $(S, +)$ is a rectangular band.

**Theorem 11:** Let $(S, +, \cdot)$ be a semiring containing the multiplicative identity ‘$1$’ and $1 + b = 1$, for all $b$ in $S$. Then $S$ satisfies the identity $a + ab + a = a$, for all $a, b$ in $S$ if and only if $(S, +)$ is a band.

**Proof:** Consider $a + ab + a = a$, for all $a, b$ in $S$

\[ a [1 + b] + a = a \]
\[ a + a = a \quad (\because 1 + b = 1) \]
\[ \therefore (S, +) \text{ is a band.} \]

Conversely, $1 + b = 1$, for all $b$ in $S$

\[ a + ab = a \]
\[ a + ab + a = a + a \]
\[ a + ab + a = a \quad (\because (S, +) \text{ is a band}) \]
\[ \therefore a + ab + a = a, \text{ for all } a, b \text{ in } S. \]

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**Theorem 12:** Let \((S, +, \cdot)\) be a PRD satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). Then \(a + 1 + a = a\), for all \(a\) in \(S\).

**Proof:** Suppose \(a + ab + a = a\), for all \(a, b\) in \(S\).
\[
\Rightarrow a + a a^{-1} + a = a, \quad \text{for all } a, a^{-1} \text{ in } S \\
\Rightarrow a + 1 + a = a \\
\therefore a + 1 + a = a, \quad \text{for all } a \text{ in } S.
\]

**Definition:** A Boolean semiring is a semiring in which \(a^2 = a\).

**Theorem 13:** Every Boolean semiring in which \(a + ab + a = a\), for all \(a, b\) in \(S\), then \(S = \{a, 2a\} \cup \{b, 2b\} \cup \ldots . \)

**Proof:** Suppose \(a + ab + a = a\), for all \(a, b\) in \(S\).
\[
\text{Taking } b = a \\
\Rightarrow a + a^2 + a = a \\
\Rightarrow 3a = a \\
\Rightarrow 3a + a = a + a \\
\Rightarrow 4a = 2a \\
\Rightarrow 4a + a = 2a + a \\
\Rightarrow 5a = 3a \text{ and so on.}
\]

<table>
<thead>
<tr>
<th>+</th>
<th>a</th>
<th>2a</th>
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</thead>
<tbody>
<tr>
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<td>2a</td>
<td>a</td>
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**Theorem 14:** Let \((S, +, \cdot)\) be a totally ordered semiring and satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is p.t.o (n.t.o.), then \((S, \cdot)\) is n.t.o. (p.t.o.).

**Proof:** Let \(a + ab + a = a\), for all \(a, b\) in \(S\).
\[
\Rightarrow a = a + ab + a \geq ab \quad (\therefore (S, +) \text{ is p.t.o.}) \\
\Rightarrow a \geq ab
\]
Suppose \(ab > b\)
\[
\Rightarrow a + ab + a \geq a + b + a \\
\Rightarrow a + ab + a \geq (a + b) + a \geq a + b \\
\Rightarrow a \geq a + b \\
\therefore a + ab + a = a
\]

which contradicts the hypothesis that \((S, +)\) is p.t.o.
\[
\Rightarrow ab \leq b \\
\therefore (S, \cdot) \text{ is n.t.o.}
\]

Similarly we can prove that \((S, \cdot)\) is p.t.o if \((S, +)\) is n.t.o.

**Theorem 15:** Let \((S, +, \cdot)\) be a totally ordered monosemiring and satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is p.t.o then \(a + b = a\).

**Proof:** Let \(a + ab + a = a\), for all \(a, b\) in \(S\).
\[
\Rightarrow a = a + ab + a \geq ab \quad (\therefore (S, +) \text{ is p.t.o.}) \\
\Rightarrow a \geq ab \\
\Rightarrow a \geq a + b \rightarrow (1) \\
\therefore (S, +, \cdot) \text{ is a mono semiring}
\]
\[
\therefore (S, +) \text{ is p.t.o.}, a + b \geq a \rightarrow (2)
\]

From (1) & (2), \(a + b = a\)
\[
\therefore a + b = a, \text{ for all } a, b \text{ in } S.
\]

**Definition:** A C – semiring is a semiring in which
(i) \((S, +)\) is a commutative monoid
(ii) \((S, \cdot)\) is a commutative monoid
(iii) \(a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca\), for every \(a, b, c\) in \(S\)
(iv) \(a.0 = 0.a = 0\)
(v) \((S, +)\) is a band and \(1\) is the absorbing element of \(+\).
Theorem 16: Let \((S, +, \cdot)\) be a totally ordered \(C\) - semiring and satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is p.t.o (n.t.o.), then \((S, \cdot)\) is n.t.o. (p.t.o.).

Proof: Let \(a + ab + a = a\), for all \(a, b\) in \(S\)
\[\Rightarrow a + a (b + b) + a = a\]
\[\Rightarrow ab + a + ab + a = a \quad (\because (S, +) \text{ is p.t.o.})\]
\[\Rightarrow ab + a = a \quad (\because a + ab + a = a)\]
\[\Rightarrow a = ab + a \geq ab\]
\[\Rightarrow a \geq ab\]

Suppose \(ab > b\)
\[\Rightarrow ab + a \geq b + a\]
\[\Rightarrow a \geq b + a \quad (\because ab + a = a)\]
\[\Rightarrow a \geq a + b\]
\[\Rightarrow a + b \leq a\]

Which contradicts the hypothesis that \((S, +)\) is p.t.o.

\[\Rightarrow ab \leq b\]
\[\therefore ab \leq a \text{ & } ab \leq b\]

Hence \((S, \cdot)\) is n.t.o.

Similarly we can prove that \((S, \cdot)\) is p.t.o if \((S, +)\) is n.t.o.

REFERENCES


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