Chapter - V

Some Special Classes of Semirings
5.1 INTRODUCTION

In Chapter 5, the complemented semirings are discussed and we established that if \((S, +, \cdot)\) is a complemented semiring containing the multiplicative identity 1 and if \(S\) contains additive identity zero, then (i) \((S, \cdot)\) is a band and (ii) \((S, +)\) is commutative and examples are given. We also discussed boolean like semirings. In section 5.3 and 5.4, we discuss the properties of complemented semirings and in section 5.5, we study the properties of boolean like semirings.

5.2 PRELIMINARIES:

In this section, we study the concepts (and results) which are not mentioned in the earlier chapters and which are needed for the study of main theorems of this chapter. We also discuss the properties of complemented semirings and boolean like semirings.

Definition 5.2.1:

An element `a’ of a semiring \(S\) is complemented if and only if there exists an element \(b\) of \(S\) satisfying \(a + b = 1\) and \(ab = ba = 0\).
Result 5.2.2: [Proposition 2.1(iii), 28]

If a positively ordered semigroup $S$ contains identity $1$, then $1$ is the minimum element.

Result 5.2.3: [Proposition 1, 31]

If a totally ordered semiring $(S, +, \cdot)$ contains $1$, then $(S, +)$ is non-negatively or non-positively ordered.

Result 5.2.4: [Proposition 2.1(iii), 28]

If a positively ordered semigroup $S$ contains additive identity $0$, then $0$ is the minimum element.
5.3 PROPERTIES OF COMPLEMENTED SEMIRINGS:

In this section, the properties of complemented semirings are studied. We proved that if `a' is a complemented element in a semiring, then \( a^n + b^n = 1 \), for all \( n \geq 1 \).

**Theorem 5.3.1:** Let \((S, +, \cdot)\) is a complemented semiring containing the multiplicative identity 1. If \( S \) contains additive identity zero, then

(i) \((S, \cdot)\) is a band

(ii) \((S, +)\) is commutative

**Proof:** Given that \((S, +, \cdot)\) is a complemented semiring

Therefore \( a + b = 1 \) and \( ab = ba = 0 \), for all \( a, b \) in \( S \)

(i) Consider \( a.1 = a.(a + b) \)

\[ \Rightarrow a = a^2 + ab \]

\[ \Rightarrow a = a^2 + 0 \quad (\therefore ab = 0) \]

\[ \Rightarrow a = a^2 \]

\[ \therefore (S, \cdot) \text{ is a band.} \]

(ii) Consider \( ab + b^2 + a^2 + ba = (a + b) b + (a + b) a \)

\[ = (a + b) (b + a) \]

\[ = a (b + a) + b (b + a) \]

\[ \Rightarrow ab + b^2 + a^2 + ba = ab + a^2 + b^2 + ba \]
\[ \Rightarrow 0 + b^2 + a^2 + 0 = 0 + a^2 + b^2 + 0 \quad (\because ab = 0 \text{ and } ba = 0) \]

\[ \Rightarrow b^2 + a^2 = a^2 + b^2 \]

\[ \Rightarrow b + a = a + b \quad (\because \text{from (i), } (S, \cdot) \text{ is a band, } a^2 = 0 \text{ and } b^2 = 0) \]

\[ \therefore (S, +) \text{ is commutative.} \]

This is evident from the following example

**Example:**

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**Theorem 5.3.2:** Let `a` is a complemented element in a semiring. Then

\[ a^n + b^n = 1, \text{ for all } n \geq 1. \]

**Proof:** Since `a` is a complemented element, there exists some \( b \in S \), such that \( a + b = 1 \)

\[ \Rightarrow (a + b)^2 = 1 \]

\[ \Rightarrow (a + b) (a + b) = 1 \]

\[ \Rightarrow a^2 + ab + ba + b^2 = 1 \quad (\because \text{using distributive laws}) \]
\[ a^2 + 0 + 0 + b^2 = 1 \quad (\because ab = 0 \text{ and } ba = 0) \]
\[ a^2 + b^2 = 1 \]

Now \( (a + b)^3 = 1 \)
\[ a^3 + a^2 b + b^2 a + b^3 = 1 \]
\[ a^3 + a \cdot a \cdot b + b \cdot b \cdot a + b^3 = 1 \]
\[ a^3 + 0 + 0 + b^3 = 1 \quad (\because ab = 0 \text{ and } ba = 0) \]
\[ a^3 + b^3 = 1 \]

Also \( (a + b)^4 = 1 \)
\[ (a^2 + b^2) (a^2 + b^2) = 1 \]
\[ a^4 + a^2 b^2 + b^2 a^2 + b^4 = 1 \]
\[ a^4 + a \cdot a \cdot b \cdot b + b \cdot b \cdot a \cdot b + b^4 = 1 \]
\[ a^4 + 0 + 0 + b^4 = 1 \quad (\because ab = 0 \text{ and } ba = 0) \]
\[ a^4 + b^4 = 1 \]

Continuing like this, we get \( a^n + b^n = 1 \), for all \( n \geq 1 \).
5.4 PROPERTIES OF ORDERED COMPLEMENTED SEMIRINGS:

In this section, the properties of ordered complemented semirings are studied. We proved that in a totally ordered complemented semiring \((S, +, \cdot)\), if \((S, +)\) is p.t.o., then \((S, \cdot)\) is n.t.o.

**Theorem 5.4.1:** Let \((S, +, \cdot)\) be a t.o. complemented semiring containing multiplicative identity 1. If \((S, \cdot)\) is p.t.o., then \((S, +)\) is n.t.o.

**Proof:** By hypothesis, \((S, +, \cdot)\) is a complemented semiring

We have \(a + b = 1\) and \(ab = ba = 0\), for all \(a, b\) in \(S\)

Since \((S, \cdot)\) is p.t.o., \(a.1 \geq a, 1\)

\[\Rightarrow a \geq 1\]

\[\Rightarrow 1\text{ is the minimum element}\]

\[\Rightarrow a + b = 1 \leq a, b\]

\[\Rightarrow a + b \leq a, b\]

\[\therefore (S, +)\text{ is n.t.o.}\]

**Theorem 5.4.2:** Let \((S, +, \cdot)\) be a t.o. complemented semiring. If \((S, +)\) is p.t.o., then \((S, \cdot)\) is n.t.o.
**Proof:** Given that \((S, +, \cdot)\) is a complemented semiring

\[ a + b = 1 \quad \text{and} \quad ab = ba = 0, \text{ for all } a, b \text{ in } S \]

Since \((S, +)\) is p.t.o., 0 is the minimum element

\[ \Rightarrow ab = 0 \leq a, b \]

\[ \Rightarrow ab \leq a, b \]

\[ \therefore (S, \cdot) \text{ is n.t.o.} \]

**Theorem 5.4.3:** Let \((S, +, \cdot)\) be a t.o. complemented semiring. If \((S, \cdot)\) is p.t.o., then \((S, +)\) is a band.

**Proof:** Since \((S, \cdot)\) is p.t.o., 1 is the minimum element

So \(1 + 1 \geq 1\)

\[ \Rightarrow x.(1 + 1) \geq x.1 \]

\[ \Rightarrow x + x \geq x \quad \rightarrow \quad (1) \]

Since \(x + y = 1 \leq x, y \quad (\therefore 1 \text{ is the minimum element})\)

\[ \Rightarrow x + x = 1 \leq x \quad \rightarrow \quad (2) \]

\[ \therefore \text{From (1) and (2), } x + x = x \]

Hence \((S, +)\) is a band.
This is evident from the following example

**Example:** $1 < 0$

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5.5 PROPERTIES OF BOOLEAN LIKE SEMIRINGS:

In this section, the properties of boolean like semirings are studied. We proved that in a boolean like semiring \((S, +, \cdot)\) if \((S, +)\) is regular, then \((S, \cdot)\) is a band.

**Definition 5.5.1:**

An element \(a\) of a multiplicative semigroup \(S\) is called an E – inverse if there is an element \(x\) in \(S\) such that \(ax (ax) = ax\), i.e. \(ax \in E (\cdot)\), where \(E (\cdot)\) is the set of all multiplicative idempotent elements of \(S\).

→ A Semigroup \(S\) is called an E – inverse Semigroup if every element of \(S\) is an E – inverse.

**Definition 5.5.2:**

A non – empty set \(S\) together with two binary operations \(\cdot + \) and \(\cdot\) satisfying the following conditions is called a boolean like semiring

(i) \((S, +)\) is a semigroup

(ii) \((S, \cdot)\) is a semigroup

(iii) \(a.(b + c) = a.b + a.c\) and \((b + c).a = b.a + c.a\)

(iv) \(ab (a + b + ab) = ab\), for all \(a,b\) in \(S\) and \(a.0 = 0.a = 0\)

(v) weak commutative: \(a.b.c = b.a.c\), for all \(a, b, c\) in \(S\)
**Theorem 5.5.3:** Let \((S, +, \cdot)\) be a boolean like semiring. If \(S\) contains the multiplicative identity 1 and \(a + 1 = 1\), then \((S, \cdot)\) is multiplicatively subidempotent.

**Proof:** Given that \((S, +, \cdot)\) is a boolean like semiring

We have \(ab (a + b + ab) = ab\), for all \(a, b\) in \(S\)

Taking \(b = 1\)

\[
\Rightarrow a (a + 1 + a) = a
\]

\[
\Rightarrow a (1 + a) = a \quad (\because a + 1 = 1)
\]

\[
\Rightarrow a + a^2 = a
\]

\[
\therefore (S, \cdot) \text{ is multiplicatively subidempotent.}
\]

**Theorem 5.5.4:** Let \((S, +, \cdot)\) be a boolean like semiring containing the multiplicative identity 1. If \((S, +)\) is regular, then \((S, \cdot)\) is a band.

**Proof:** Consider \(ab (a + b + ab) = ab\), for all \(a, b\) in \(S\)

Taking \(b = 1\)

\[
\Rightarrow a(a + 1 + a.1) = a.1
\]

\[
\Rightarrow a(a + 1 + a) = a
\]

\[
\Rightarrow a(a) = a \quad \Rightarrow a^2 = a \quad (\because (S, +) \text{ is regular, } a + 1 + a = a)
\]

\[
\therefore (S, \cdot) \text{ is a band.}
\]
**Theorem 5.5.5:** Let \((S, +, \cdot)\) be a boolean like semiring containing the multiplicative identity which is also an additive identity, then \((S, \cdot)\) is an E - inverse semigroup.

**Proof:** By hypothesis, \((S, +, \cdot)\) is a boolean like semiring

Let `e` be the multiplicative identity is also an additive identity

Consider \(ab (a + b + ab) = ab\), for all \(a, b \in S\)

\[
\Rightarrow ab (a + (e + a)b) = ab
\]

\[
\Rightarrow ab (a + ab) = ab
\]

\[
\Rightarrow ab (a (e + b)) = ab
\]

\[
\Rightarrow ab (ab) = ab
\]

\[
\therefore ab \in E[\cdot]
\]

Hence \((S, \cdot)\) is an E - inverse semigroup.

**Theorem 5.5.6:** Let \((S, +, \cdot)\) be a boolean like semiring containing the multiplicative identity 1, which is also an absorbing element w.r.to. `+`.

If \((S, \cdot)\) is left cancellative, then \(a + b = 1\), for all \(a, b \in S\).

**Proof:** Since \((S, +, \cdot)\) is a boolean like semiring

Let 1 be the multiplicative identity is also an absorbing element w.r.to. `+`

Consider \(ab (a + b + ab) = ab\), for all \(a, b \in S\)
\[ \Rightarrow ab (a + (1 + a) b) = ab.1 \]
\[ \Rightarrow ab (a + b) = ab.1 \quad (\because 1 + a = 1) \]
\[ \Rightarrow a + b = 1 \quad (\because (S, \cdot) \text{ is left cancellative}) \]
\[ \therefore a + b = 1, \text{ for all } a, b \text{ in } S. \]

**Theorem 5.5.7:** Let \((S, +, \cdot)\) be a boolean like semiring containing the multiplicative identity 1. If \((S, +)\) is left singular, then \((S, \cdot)\) is left regular.

**Proof:** Consider \(ab (a + b + ab) = ab\), for all \(a, b\) in \(S\)
\[ \Rightarrow ab (a + a b) = ab \quad (\because (S, +) \text{ is a left singular, } a + b = a) \]
\[ \Rightarrow ab (a [1 + b] ) = ab \]
\[ \Rightarrow ab (a.1) = ab \quad (\because (S, +) \text{ is a left singular, } 1 + b = 1) \]
\[ \Rightarrow aba = ab \]
\[ \therefore (S, \cdot) \text{ is left regular.} \]

**Theorem 5.5.8:** Let \((S, +, \cdot)\) be a boolean like semiring. Then the set \(X\) of all zero square elements is a multiplicative ideal of \(S\).

**Proof:** Let \(s \in S\) and \(x \in X\)
\[ \Rightarrow (xs)^2 = xs. xs \]
\[ = sxxs \quad (\because \text{by weak commutative : } xsx = sxx) \]
\[
\begin{align*}
&= s^2 x s \\
&= x^2 ss \quad (\therefore \text{by weak commutative}) \\
&= x^2 s^2 \\
&= 0.s^2 = 0 \quad (\therefore \text{\(x\) is a zero square element, \(x^2 = 0\)}) \\
\therefore \quad xs \in X \\
\
\text{Similarly,} \\
\Rightarrow \quad (sx)^2 = sx \cdot sx \\
&= xs sx \quad (\therefore \text{by weak commutative : \(sxs = xss\)}) \\
&= xs^2 x \\
&= s^2 x x \quad (\therefore \text{by weak commutative}) \\
&= s^2 x^2 \\
&= s^2 \cdot 0 = 0 \quad (\therefore \text{\(x\) is a zero square element, \(x^2 = 0\)}) \\
\therefore \quad sx \in X \\
\
\text{Hence \(X\) is a multiplicative ideal.}
\end{align*}
\]

**Theorem 5.5.9:** Let \((S, +, \cdot)\) be a t.o. boolean like semiring. If \((S, \cdot)\) is p.t.o. (n.t.o.), then 0 is the maximum (minimum) element.

**Proof:** Since \((S, +, \cdot)\) is a boolean like semiring

We have \(a.0 = 0.a = 0\), for all \(\text{``a'' in } S\)

Suppose \((S, \cdot)\) is p.t.o.
Then $a.0 \geq a$ and $0$

$\Rightarrow a.0 \geq a$

$\Rightarrow 0 \geq a \quad (\because a.0 = 0)$

$.\therefore 0$ is the maximum element.

Suppose $(S, \bullet)$ is n.t.o.

Then $a.0 \leq a$ and $0$

$\Rightarrow a.0 \leq a$

$\Rightarrow 0 \leq a \quad (\because a.0 = 0)$

$.\therefore 0$ is the minimum element.