CHAPTER 4

ORDERING POLICIES FOR WEIBULL DISTRIBUTED DETERIORATION WITH ASSOCIATED SALVAGE VALUE UNDER PROGRESSIVE CREDIT PERIODS
4.0 Introduction:
The term "progressive credit periods" offered by the supplier to the retailer for settling the account is defined as follows: If the retailer settles account by credit period M, then the supplier does not charge any interest. If the retailer pays after M but before N (N > M), then the supplier charges the retailer on an un-paid amount at the interest rate lc₁. If the retailer settles after credit period N (N > M), then he will have to pay an interest rate lc₂ (lc₂ > lc₁) for the un-paid amount. The objective is to minimize the retailer's total inventory cost. In this chapter, a mathematical model is developed when units in inventory deteriorate with respect to time and supplier offers two progressive credit periods. The salvage value is associated to deteriorated units. A flow - chart is given to find the optimal solution. The sensitivity analysis is carried out to analyze the effect of critical parameters on the optimal solution. In this chapter, we will discuss two models.

MODEL 4.1

Optimal Ordering Policy for time dependent Deterioration
with associated Salvage Value
when delay in payments is permissible

4.1.1 Assumptions and notations:
The following assumptions are used to develop aforesaid model:

- The system deals with single item.
- The demand rate of R-units per time unit is deterministic and constant.
- The replenishment rate is infinite.
• The lead time is zero and shortages are not allowed.

• The deterioration rate of units in inventory follows the Weibull distribution function given by
  \[ \theta(t) = \alpha \beta t^{\beta - 1}, \quad 0 \leq t \leq T \]  
where \( \alpha (0 < \alpha < 1) \) denotes scale parameter, \( \beta (\beta \geq 1) \) denotes shape parameter and \( t (t > 0) \) is time to deterioration.

• The deteriorated units can neither be repaired nor be replaced during the cycle time.

• During the fixed credit period, \( M \), the unit cost of generated sales revenue is deposited in an interest bearing account. The difference between sales price and unit cost is retained by the system to meet the day-to-day expenses of the system. At the end of the credit period the account is settled and interest charges are payable on the unpaid account.

• The salvage value, \( \gamma C (0 \leq \gamma < 1) \) is associated to deteriorated units during the cycle time. Here \( C \) is the purchase cost of an item.

The following notations are used in developing the model:

\( R \) : demand rate per unit of time.

\( \gamma \) : salvage parameter.

\( C \) : purchase cost per unit.

\( P \) : unit selling price (\( P > C \)).

\( h \) : inventory holding cost per unit per time unit excluding interest charges.

\( A \) : ordering cost per order.

\( I_e \) : interest earned per unit per annum.
lc: interest charged per unit in stock per annum by the supplier to the retailer. \( lc > l_e \)

\( T \): cycle time, (decision variable)

\( M \): allowable credit period offered by the supplier to the retailer for settling the accounts.

### 4.1.2 Mathematical Model:

Let \( Q(t) \) be on-hand inventory at any instant of time \( t (0 < t < T) \) of a cycle. The depletion of units in inventory is due to demand and deterioration of units. The instantaneous state of \( Q(t) \) at any instant of time is described by the differential equation.

\[
\frac{dQ(t)}{dt} + \theta(t)Q(t) = -R, \quad 0 \leq t \leq T \tag{4.1.2.1}
\]

with initial condition \( Q(0) = Q \) and boundary condition \( Q(T) = 0 \), where \( \theta(t) \) is given by equation (4.1.1.1).

Taking series expansion and ignoring second and higher power of \( \alpha \) (assuming \( \alpha \) to be very small), the solution of differential equation (4.1.2.1) using boundary condition \( Q(T) = 0 \) is given by

\[
Q(t) = R \left[ T - t + \frac{\alpha T}{\beta + 1} \left( T^\theta - (1 + \beta) t^\theta \right) + \frac{\alpha \beta t^{\theta+1}}{\beta + 1} \right] \tag{4.1.2.2}
\]

Using \( Q(0) = Q \), we get procurement quantity as

\[
Q = R \left[ T + \frac{\alpha T^{\theta+1}}{\beta + 1} \right] \tag{4.1.2.3}
\]

The number of units that deteriorate; \( D(T) \) during one cycle is given by
D(T) = Q - RT = \frac{\alpha RT^{\beta+1}}{\beta + 1} \tag{4.1.2.4}

Hence, cost due to deterioration (CD) is

\[ CD = \frac{\alpha CRT^{\beta+1}}{\beta + 1} \tag{4.1.2.5} \]

and salvage value of deteriorated units; SV is

\[ SV = \frac{\alpha \gamma CRT^{\beta+1}}{\beta + 1} \tag{4.1.2.6} \]

The inventory holding cost; IHC during the cycle is

\[ IHC = h \int_0^T Q(t) dt = hR \left[ \frac{T^2}{2} + \frac{\alpha \beta T^{\beta+2}}{\beta + 1)(\beta + 2)} \right] \tag{4.1.2.7} \]

and ordering cost; OC per order is

\[ OC = A \tag{4.1.2.8} \]

About interest charged and interest earned the following two cases arises:

Case 1: M \leq T \quad \text{(Fig 4.1.2(a))}

Case 2: M > T \quad \text{(Fig 4.1.2(b))}

Next we compute interest charged and interest earned in both the cases:

**Case 1:** M \leq T i.e. offered credit period is less than or equal to cycle time.
The retailer can sale units during $[0, M]$ at sale price; $P$ per unit which he can put at an interest rate $i_e$ per unit per annum in an interest bearing account. So total interest earned during $[0, M]$ is

$$IE_1 = PI_e \int_0^M R(t)dt = \frac{PI_e RM^2}{2} \quad (4.1.2.9)$$

During $[M, T]$, supplier will charge interest to the retailer on the remaining stock at the rate $i_c$ per unit per annum. Hence, total interest charges payable by the retailer during $[M, T]$ is

$$IC_1 = CI_c \int_M^T Q(t)dt$$

$$= CI_c \left[ \frac{(T - M)^2}{2} - \frac{\alpha MT}{\beta + 1} (T^\beta - M^\beta) + \frac{\alpha \beta}{(\beta + 1)(\beta + 2)} (T^{\beta+2} - M^{\beta+2}) \right] \quad (4.1.2.10)$$

The total cost ($K_1(T)$) per time unit is

$$K_1(T) = \frac{1}{T}[OC + IHC + CD + IC_1 - IE_1 - SV] \quad (4.1.2.11)$$

The necessary condition for $K_1(T)$ to be minimum is
\[
\frac{\partial K_1(T)}{\partial T} = \frac{-1}{T^2} \left[ hR \left( \frac{T^2}{2} + \frac{\alpha \beta T^{\beta+2}}{(\beta+1)(\beta+2)} \right) + \frac{C \alpha R T^{\beta+1}}{\beta + 1} (1 - \gamma) + A \right] \\
+ \frac{1}{T} \left[ R \left( \frac{T^2}{2} + \frac{\alpha \beta T^{\beta+2}}{(\beta+1)(\beta+2)} \right) - R \left( \frac{T M - \frac{M^2}{2}}{\beta + 1} \right) \right] \\
\times \left[ \left( \frac{T \beta^2 + T \beta + \left( -\beta - 1 \right) T^{\beta+1}}{T} \right) + \frac{\alpha \beta T^{\beta+2}}{(\beta+1)T} \right] \\
- \left[ \left( \frac{\alpha T T^{\beta} M + \left( -\beta - 1 \right) M^{\beta+1}}{(\beta+1)} \right) + \frac{\alpha T \beta M}{\beta + 1} \right] \\
= 0
\]
and solve it for $T$ by a mathematical software. For obtained $T$, $K_1(T)$ is minimum only if $\frac{\partial^2 K_1(T)}{\partial T^2} > 0$ for all $T$.

**Case 2:** $T > M$ i.e. offered credit period by the supplier to the retailer for settling the account is greater than cycle time.
Here, interest charges is zero i.e.
\[ IC_2 = 0 \]  \hspace{1cm} (4.1.2.12)
and interest earned; \( IE_2 \) is
\[ IE_2 = PI_e \left( \int R(t) dt + RT(M - T) \right) = PI_e R \left( M - \frac{T}{2} \right) \]  \hspace{1cm} (4.1.2.13)
Thus, total cost \( K_2(T) \) per time unit is
\[ K_2(T) = \frac{1}{T} \left[ OC + IHC + CD + IC_2 - IE_2 - SV \right] \]  \hspace{1cm} (4.1.2.14)
For optimal value \( T \), solve
\[
\frac{\partial K_2(T)}{\partial T} = \frac{1}{T^2} \left\{ hR \left( T^2 \frac{(\beta + 2)}{2} + \frac{\alpha \beta T^{\beta+2}}{\beta + 1} \right) + \frac{CaRT^{\beta+1}}{T} \left( 1 - \gamma \right) + A - PI_e R (M - \frac{T}{2}) \right\} + \frac{1}{T} \left\{ hR \left( T + \alpha T \right) \left( \frac{T^\beta \beta + T^\beta + (-\beta - 1)T^{\beta+1}}{\beta + 1} \right) + \frac{\alpha \beta T^{\beta+2}}{(\beta + 1)T} \right\}
\]
\[ + \frac{PI_e R}{2} \] = 0
\]
which minimizes $K_2(T)$ only if $\frac{\partial^2 K_2(T)}{\partial T^2} > 0$ for all $T$.

For $T = M$, we have

$$K_1(M) = K_2(M) = hR \left[ \frac{M}{2} + \frac{\alpha \beta T^{\beta+1}}{(\beta+1)(\beta+2)} \right] + \frac{C(1-\gamma)\alpha R M^\beta}{\beta + 1} + \frac{A}{T} - \frac{P_c R M}{2} \quad (4.1.2.15)$$

### 4.1.3 Computational Algorithm:

To find optimal cycle time and hence total cost, we follow decision policy as shown in the flow chart.
Inventory Models for Deteriorating Items...

Start

Give parametric values

Compute T from Case 1

Is $T > M$?

Yes $\rightarrow$ Case 1 is optimal Decision Policy

No $\rightarrow$ Compute T from Case 2

Is $T < M$?

Yes $\rightarrow$ Case 2 is optimal Decision Policy

No $\rightarrow$ Optimal decision is at $T=M$

Stop

Fig 4.1.3(a) Flow - chart for optimal decision policy.
4.1.4 Analytical Results:

4.1.4.1 The total cost is increasing function of deterioration rate ($\alpha$).

Proof:

$$\frac{\partial K_1(T)}{\partial \alpha} = \frac{1}{T} \left[ Cr + \left( \frac{T^{\beta+1} M + \left( \frac{-\beta - 1}{\beta + 1} \right) M^{\beta+1}}{\beta + 1} \right) \right] > 0 \text{ for all } \alpha$$

$$\frac{\partial K_2(T)}{\partial \alpha} = \frac{hR\alpha T^{\beta+2}}{(\beta + 1)(\beta + 2)} + \frac{CRT^{\beta+1}}{\beta + 1} - \frac{CyRT^{\beta+1}}{T} > 0 \text{ for all } \alpha.$$ 

4.1.4.2 The total cost is decreasing function of allowable delay period $M$.

Proof:

$$\frac{\partial K_1(T)}{\partial M} = -CcR \left( T - M + \frac{\alpha T \left( T^{\beta} + \frac{(-\beta - 1) M^{\beta+1}}{M} \right)}{\beta + 1} + \frac{\alpha \beta M^{\beta+2}}{(\beta + 1) M} - P \frac{RI}{RM} \right) < 0$$

for all $M$

$$\frac{\partial K_2(T)}{\partial M} = -\frac{P \frac{RI}{R}}{T} < 0$$

for all $M$

4.1.4.3 The total cost is decreasing function of salvage parameter $\gamma$.

Proof:
\[
\frac{\partial K_1(T)}{\partial \gamma} = \frac{\partial K_2(T)}{\partial \gamma} = -\frac{C_\alpha R T^{\beta + 1}}{\beta + 1} < 0 \quad \text{for } 0 < \gamma < 1
\]

In next section, we consider numerical example to validate analytical results.

4.1.5 Numerical Example:

Consider an inventory system with following parametric values in proper units:

\[
[A, C, h, P, \alpha, \beta, \gamma, R, I_c, I_e, M] = \]

\[
[250, 50, 5, 75, 0.02, 1.5, 0.1, 1000, 18\%, 14\%, 15/365]
\]

Then Case 1 is the optimal decision policy. The optimum cycle time \(T = 0.1853\) years < 0.0411 = \(M\) and minimum total cost of an inventory system is

\[
K_1(T) = \$ 2298.74 \text{ and } \frac{\partial^2 K_1(T)}{\partial T^2} = 1.5728 > 0.
\]

Next, we study variations of delay period; \(M\), deterioration rate; \(\alpha\), shape parameter; \(\beta\) and salvage value; \(\gamma\) on decision variable and objective function.

In tables \(K = K_1\).

<table>
<thead>
<tr>
<th>(M) (\alpha)</th>
<th>15 days</th>
<th>20 days</th>
<th>25 days</th>
<th>30 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>(T) 0.1853</td>
<td>0.1850</td>
<td>0.1345</td>
<td>0.1840</td>
</tr>
<tr>
<td>(K) 2298.74</td>
<td>2170.06</td>
<td>2040.81</td>
<td>1909.08</td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>(T) 0.1824</td>
<td>0.1820</td>
<td>0.1316</td>
<td>0.1810</td>
</tr>
<tr>
<td>(K) 2327.60</td>
<td>2198.79</td>
<td>2069.38</td>
<td>1937.48</td>
<td></td>
</tr>
<tr>
<td>0.06</td>
<td>(T) 0.1796</td>
<td>0.1770</td>
<td>0.1780</td>
<td>0.1783</td>
</tr>
<tr>
<td>(K) 2355.79</td>
<td>2227.14</td>
<td>2097.29</td>
<td>1965.51</td>
<td></td>
</tr>
<tr>
<td>0.08</td>
<td>(T) 0.1770</td>
<td>0.1767</td>
<td>0.1763</td>
<td>0.1758</td>
</tr>
<tr>
<td>(K) 2383.34</td>
<td>2254.26</td>
<td>2124.57</td>
<td>1992.35</td>
<td></td>
</tr>
</tbody>
</table>
Table 4.1.5.2: Effect of M and $\beta$ when $\alpha = 0.02$, $\gamma = 0.1$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>15 days</th>
<th>20 days</th>
<th>25 days</th>
<th>30 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>T 0.1854</td>
<td>0.1850</td>
<td>0.1847</td>
<td>0.1840</td>
</tr>
<tr>
<td></td>
<td>K 2298.74</td>
<td>2170.06</td>
<td>2075.17</td>
<td>1909.08</td>
</tr>
<tr>
<td>1.7</td>
<td>T 0.1861</td>
<td>0.1858</td>
<td>0.1854</td>
<td>0.1847</td>
</tr>
<tr>
<td></td>
<td>K 2288.84</td>
<td>2160.20</td>
<td>2065.34</td>
<td>1899.30</td>
</tr>
<tr>
<td>1.9</td>
<td>T 0.1867</td>
<td>0.1864</td>
<td>0.1360</td>
<td>0.1853</td>
</tr>
<tr>
<td></td>
<td>K 2282.32</td>
<td>2153.71</td>
<td>2058.87</td>
<td>1892.88</td>
</tr>
<tr>
<td>2.1</td>
<td>T 0.1871</td>
<td>0.1868</td>
<td>0.1365</td>
<td>0.1858</td>
</tr>
<tr>
<td></td>
<td>K 2278.00</td>
<td>2149.41</td>
<td>2057.59</td>
<td>1888.63</td>
</tr>
</tbody>
</table>

Table 4.1.5.3: Effect of M and $\gamma$ when $\alpha = 0.02$, $\beta = 1.5$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>15 days</th>
<th>20 days</th>
<th>25 days</th>
<th>30 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>T 0.1854</td>
<td>0.1850</td>
<td>0.1847</td>
<td>0.1840</td>
</tr>
<tr>
<td></td>
<td>K 2298.74</td>
<td>2170.06</td>
<td>2075.17</td>
<td>1909.08</td>
</tr>
<tr>
<td>0.3</td>
<td>T 0.1860</td>
<td>0.1857</td>
<td>0.1853</td>
<td>0.1846</td>
</tr>
<tr>
<td></td>
<td>K 2292.34</td>
<td>2163.68</td>
<td>2068.80</td>
<td>1902.75</td>
</tr>
<tr>
<td>0.5</td>
<td>T 0.1867</td>
<td>0.1863</td>
<td>0.1860</td>
<td>0.1852</td>
</tr>
<tr>
<td></td>
<td>K 2285.90</td>
<td>2157.26</td>
<td>2062.41</td>
<td>1896.39</td>
</tr>
<tr>
<td>0.7</td>
<td>T 0.1874</td>
<td>0.1870</td>
<td>0.1867</td>
<td>0.1860</td>
</tr>
<tr>
<td></td>
<td>K 2279.43</td>
<td>2150.81</td>
<td>2055.97</td>
<td>1890.00</td>
</tr>
</tbody>
</table>

Observations:

In table 4.1.5.1 when shape parameter ($\beta$) of deterioration and salvage parameter ($\gamma$) are fixed, total cost and cycle time decreases when credit period (M) given supplier to retailer for settling the account increases. At the same time as scale parameter ($\alpha$) increases total cost increases whereas cycle time decreases.

In table 4.1.5.2 when scale parameter ($\alpha$) of deterioration and salvage parameter ($\gamma$) are fixed, total cost and cycle time decreases when shape
parameter ($\beta$) increases. At the same time as credit period (M) given supplier to retailer for settling the account increases total cost decreases whereas cycle time increases.

In table 4.1.5.3 when shape parameter ($\beta$) and scale parameter ($\alpha$) of deterioration are fixed, total cost as well as cycle time decreases as credit period (M) increases. At the same time total cost decreases but cycle time increases as salvage parameter ($\gamma$) increases.

MODEL 4.2

Optimal Ordering Policy for the time dependent deterioration with associated Salvage value under scenario of progressive credit period

4.2.1 Assumptions and Notations:

The aforesaid model is developed using following assumptions:

- The inventory system deals with single item.
- The demand of $R$ units for an item is deterministic and known during the cycle time.
- Replenishment rate is infinite. Replenishment is instantaneous.
- Shortages are not allowed and lead – time is zero.
- The distribution of time for deterioration of the items is

$$\theta(t) = \alpha \beta t^{\beta-1}$$ (4.2.1.1)

where $\alpha$ ($0 \leq \alpha < 1$) denotes scale parameter i.e. rate of deterioration denotes the shape parameter ($\beta \geq 1$) and $t$ ($t > 0$) denotes the time to
deterioration. The salvage value, $\gamma C$ ($0 \leq \gamma < 1$), is associated to the deteriorated units where $C$ is the unit cost of an item.

- The deteriorated units can neither be repaired nor replaced during the cycle time.

- If the retailer pays by offered credit period $M$, then supplier does not charge any interest to the retailer. If the retailer pays after $M$ and before second credit period $N$ ($N > M$) then he can keep difference in unit sale price and unit cost in an interest bearing account at the interest rate of $1e$ per unit per annum. During $[M, N]$, the supplier charges the retailer an interest rate of $1c_1$ per unit per year. If the retailer pays after $N$, then supplier charges the retailer an interest rate of $1c_2$ per unit per annum with $1c_2 > 1c_1$.

The following notations are used in the formulation of the model:

- $R$: the demand rate per annum,
- $C$: the unit purchase cost,
- $\gamma C$: salvage value of deteriorated unit, $0 \leq \gamma < 1$.
- $P$: selling price per unit. ($P > C$)
- $h$: the inventory holding cost per unit per year excluding interest charges.
- $A$: ordering cost per order.
- $M$: the first offered credit period in settling account by the supplier to the retailer without any extra charges.
• N: the second allowable credit period to settle the account with an interest charge of IC₁ and M > N.

• IC₁: the interest charged per $ in stock per year by the supplier when the retailer pays during [M, N].

• IC₂: the interest charged per $ in stock per year the supplier when the retailer pays during [N, T]. Note that IC₂ > IC₁.

• Q: the procurement quantity (a decision variable).

• T: the replenishment cycle time (a decision variable).

• Q(t): the on-hand inventory level at any instant of time t, 0 ≤ t ≤ T.

• D(T): the number of units deteriorated during the cycle time T.

• K(T): the total inventory cost per time unit is the sum of: (a) ordering cost; OC, (b) inventory holding cost (excluding interest charges) IHC, (c) cost due to deterioration; DC, (d) interest charges; IC, for unsold items after the offered delay period M or N, minus (e) salvage value of deteriorated units; SV, and (f) interest earned from the sales revenue during the permissible delay period [0, M].

4.2.2 Mathematical Model:

The on-hand inventory depletes due to demand and deterioration of units. The instantaneous state of inventory at any instant of time t is governed by the differential equation.

\[ \frac{dQ(t)}{dt} + \theta(t)Q(t) = -R, \quad 0 \leq t \leq T \]  

(4.2.2.1)
with boundary condition $Q(0) = Q$ and $Q(T) = 0$ and $Q(t)$ is as given in (4.2.1.1).

Taking series expansion and ignoring second and higher powers of $\alpha$ (assuming $\alpha$ to be very small), the solution of differential equation (4.2.2.1), using boundary condition $Q(T) = 0$ is given by

$$Q(t) = R \left( T - t + \frac{aT^\beta}{\beta + 1} - (1 + \beta)t^\beta + \frac{\alpha \beta t^{\beta+1}}{\beta + 1} \right), \quad 0 \leq t \leq T. \tag{4.2.2.2}$$

Using $Q(0) = Q$, the procurement quantity is given by

$$Q = R \left( T + \frac{\alpha T^{\beta+1}}{\beta + 1} \right) \tag{4.2.2.3}$$

The number of units deteriorated; $D(T)$ during one cycle is

$$D(T) = Q - RT = \frac{\alpha RT^{\beta+1}}{\beta + 1} \tag{4.2.2.4}$$

The various costs of total inventory cost of the system per cycle are as follows:

(a) Ordering cost;

$$OC = A \tag{4.2.2.5}$$

(b) Inventory holding cost;

$$IHC = h \int_0^T Q(t) dt = hR \left[ \frac{T^2}{2} + \frac{\alpha \beta T^{\beta+2}}{(\beta + 1)(\beta + 2)} \right] \tag{4.2.2.6}$$

(c) Cost due to deterioration of units;

$$CD = \frac{CR\alpha T^{\beta+1}}{\beta + 1} \tag{4.2.2.7}$$

(d) Salvage value of the deteriorated units;

$$SV = \frac{\gamma CR\alpha T^{\beta+1}}{\beta + 1} \tag{4.2.2.8}$$
The calculations of interest charged and interest earned depend on the length of cycle time; $T$ and offered credit periods. There are three cases:

Case 1. $T \leq M$

Case 2. $M < T < N$

Case 3. $T \geq N$.

Next, we discuss computations of interest charged and interest earned in each case.

Case 1. $T \leq M$ i.e. cycle time ends before the first offered credit period. (Fig. 4.2.2(a))

Here, the retailer sells $Q$ units during $[0, T]$ and is paying $CQ$ to the supplier in full at time $M \geq T$. So interest charges are zero. i.e.

$$IC_1 = 0$$  \hspace{1cm} (4.2.2.9)

During $[0, T]$, the retailer sells unit at sale price; $P$ per unit and deposit the revenue into an interest earning account at the rate of $\ell e$ per $\$ \text{ per year}$. In the
period \([T, M]\), the retailer only deposits the total revenue into an account that
earns \(i_e\) per $ per year. Hence, interest earned during the cycle is

\[
I_E_1 = P_I e \left[ \int_0^T R_t dt + R(T - T) \right] = P_I e R \left( M - \frac{T}{2} \right) \tag{4.2.2.10}
\]

Hence, the total cost of an inventory system per time unit is

\[
K(T) = \frac{1}{T} \left( OC + IHC + CD + IC_1 - I E_1 - SV \right) \tag{4.2.2.11}
\]

The optimum value of \(T = T_1\) is the solution of non-linear equation

\[
\frac{\partial K_1(T)}{\partial T} = 0.
\]

The obtained \(T = T_1\) minimizes the total cost because

\[
\frac{\partial^2 K_1(T)}{\partial T^2} > 0 \text{ for all } T
\]

Case 2. \(M < T < N\). (Fig 4.2.2(b))

![Inventory level](image)

(Fig. 4.2.2(b)) \((M < T < N)\)

Here, interest earned, \(I E_2\), during \([0, M]\) is

\[
I E_2 = P_I e \int_0^M R_t dt = \frac{P_I e R M^2}{2}.
\]

The retailer has to pay for \(Q - \) units purchased at unit cost \(C\) $ to the supplier up to time \(M\),
the retailer sells RM - units and has revenue PRM plus interest earned IE₂ to pay
the supplier. Depending on the difference between the total purchase cost; CQ
and the revenue; PRM + IE₂, two sub - cases may arise:

Sub - case 2.1: Let PRM + IE₂ ≥ CQ. i.e. the retailer has sufficient amount in
his account to pay - off total purchase cost at M. Then interest charges,

\[ IC_{21} = 0 \] (4.2.2.12)

and interest earned,

\[ IE_{21} = IE₂ = \frac{PI₁RM^2}{2} \] (4.2.2.13)

Therefore, the total cost of an inventory system per time unit is

\[ K_{21}(T) = \frac{1}{T} \left( OC + IH + CD + IC_{21} - IE_{21} - SV \right) \] (4.2.2.14)

The optimal value of \( T = T_{21} \) is a solution of non - linear equation \( \frac{\partial K_{21}(T)}{\partial T} = 0 \)

and \( T = T_{21} \) minimizes the total cost \( K_{21} \) of an inventory system because

\[ \frac{\partial^2 K_{21}(T)}{\partial T^2} > 0 \] for all \( T \).

Sub - Case 2.2 : Let PRM + IE₂ < CQ. Here, the retailer does not have
sufficient money in his account to do payment at offered credit period; M the
supplier charges retailer on the unpaid balance \( U₁ = CQ - (PRM + IE₂) \) at the
interest rate \( ic₁ \) at time M, therefore, interest charges; \( IC_{2,2} \) per cycle is

\[ IC_{2,2} = \frac{U₁^3}{2P} \int_M Q(t) \, dt \]

\[ = \frac{U₁^3}{2P} \left( \frac{(T - M)^2}{2} + \frac{\alpha \beta}{(\beta + 1)(\beta + 2)}(T^{\beta+2} - M^{\beta+2}) - \frac{\alpha MT}{(\beta + 1)}(T^\beta - M^\beta) \right) \] (4.2.2.15)
and interest earned per cycle is

\[ IE_{22} = IE_2 = \frac{P}{2} RM^2 \]

(4.2.2.16)

Therefore, the total cost of an inventory system per time unit is

\[ K_{22} (T) = \frac{1}{T} \left( OC + IHC + CD + IC_{22} - IE_{22} - SY \right) \]

(4.2.3.17)

The optimum value of \( T = T_{22} \) can be obtained by solving the non-linear equation \( \frac{dK_{22}(T)}{dT} = 0 \) which minimizes total cost \( K_{22} \) because

\[ \frac{d^2K_{22}(T)}{dT^2} > 0 \] for all \( T \).

**Case 3.** \( T \geq N \) i.e. supplier offers his retailer two progressive credit periods to settle the account before retailer's inventory reaches zero. (Fig 4.2.2(c))

![Inventory level diagram](image)

(Fig. 4.2.2(c)) (\( T \geq N \))

Based on the total purchase cost, \( CQ \), total revenue \( PRM + IE_2 \) at \( M \) and total revenue \( PRN + IE_2 \) at \( N \), three sub-cases may arise.
Sub-case 3.1: Let \( PRM + IE_2 \geq CQ \). This sub-case is same as sub-case 2.1
(Note: Here decision variable and objective function are designated by 3.1)

Sub-case 3.2: Let \( PRM + IE_2 \leq CQ \) and
\[
PR(N - M) + \frac{PI_e R(N^2 - M^2)}{2} \geq CQ - (PRM + IE_2)
\]
This sub-case is same as sub-case 2.2. (Note: Here decision variable and objective functions are designated by 3.2.)

Sub-case 3.3: Let \( PRM + IE_2 \leq CQ \) and
\[
PR(N - M) + \frac{PI_e R(N^2 - M^2)}{2} < CQ - (PRM + IE_2)
\]
Here, retailer does not have sufficient money in his account to pay-off total purchase cost at time \( N \); he pays \( PRM + IE_2 \) at \( M \) and
\[
PR(N - M) + \frac{PI_e R(N^2 - M^2)}{2} \] at \( N \). Hence, retailer will have to pay interest charges on unpaid balance;
\[
U_1 = CQ - (PRM + IE_2) \] with interest rate \( Ic_1 \) during \([M, N]\) and on unpaid balance;
\[
U_2 = U_1 - \left( PR(N - M) + \frac{PI_e R(N^2 - M^2)}{2} \right) \) with interest rate \( Ic_2 \) during \([N, T]\).

Hence, total interest payable per cycle is
\[
IC_{33} = U_1 Ic_1 (N - M) + \frac{U_2^2}{PR} \int_T^N Q(t) \, dt
\]
\[
= U_1 Ic_1 (N - M) + \frac{U_2^2}{P} Ic_2 \left\{ \frac{(T - N)^2}{2} + \frac{\alpha \beta (T^{\beta + 2} - N^{\beta + 2})}{(\beta + 1)(\beta + 2)} + \frac{\alpha NT(N^\beta - T^\beta)}{\beta + 1} \right\}
\]
(4.2.2.18)
and interest earned, \( I_{E3,3} = I_{E2} = \frac{P_{14}RM^2}{2} \) \hspace{2cm} (4.2.2.19)

Therefore, the total cost of an inventory system per time unit is

\[
K_{33}(T) = \frac{1}{T}(OC + IHC + CD + IC_{33} - I_{E33} - SV)
\] \hspace{2cm} (4.2.2.20)

The necessary condition for \( K_{33}(T) \) to be minimum \( T = T_{33} \) is \( \frac{\partial K_{33}(T)}{\partial T} = 0 \) and

sufficiency condition is \( \frac{\partial^2 K_{33}(T)}{\partial T^2} > 0 \) for all \( T \).

In the next section, computation flow start is desired to search for the optimal solution.

**4.2.3 Computational flow chart:**
Fig. 4.2.2(d) Computational flow chart to search Best Optimal Policy
4.2.4 Numerical Example:

Consider the following parametric values in appropriate units:

\[ [A, C, h, P, R, \lambda] = [200, 50, 5, 75, 1000, 0.12] \]

The effect of various parameters on decision variable and total cost of inventory system is exhibited in the following tables:

Table 4.2.4.1: Effect of two credit periods on decision policy

\[ \alpha = 0.1, \ \beta = 1.5, \ \gamma = 0.2 \]

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
<th>28/365 (18%)</th>
<th>28/365 (20%)</th>
<th>28/365 (22%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/365 (15%)</td>
<td>T</td>
<td>0.1400</td>
<td>0.1506</td>
<td>0.1587</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>2010.90</td>
<td>1985.98</td>
<td>1982.11</td>
</tr>
<tr>
<td>20/365 (16%)</td>
<td>T</td>
<td>0.1396</td>
<td>0.1500</td>
<td>0.1580</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1891.30</td>
<td>1859.46</td>
<td>1851.55</td>
</tr>
<tr>
<td>25/365 (17%)</td>
<td>T</td>
<td>0.1395</td>
<td>0.1497</td>
<td>0.1575</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1783.11</td>
<td>1741.03</td>
<td>1726.82</td>
</tr>
</tbody>
</table>

Increase in first credit period; M decreases cycle time and total because retailer can earn more during offered credit period. Increase in second credit period; N increases cycle time significantly and decreases total cost of an inventory system.
Table 4.2.4.2: Effect of deterioration rate and first credit period on decision policy

\[ N = \frac{35}{365}, \text{lc}_2 = 22\%, \beta = 1.5, \gamma = 0.2 \]

<table>
<thead>
<tr>
<th>M</th>
<th>α</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/365 (15%)</td>
<td>T</td>
<td>0.1587</td>
<td>0.1560</td>
<td>0.1531</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1982.11</td>
<td>2084.8</td>
<td>2184.54</td>
</tr>
<tr>
<td>20/365 (16%)</td>
<td>T</td>
<td>0.1580</td>
<td>0.1552</td>
<td>0.1522</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1953.20</td>
<td>2051.80</td>
</tr>
<tr>
<td>25/365 (17%)</td>
<td>T</td>
<td>0.1575</td>
<td>0.1547</td>
<td>0.1516</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1726.82</td>
<td>1827.71</td>
<td>1925.52</td>
</tr>
</tbody>
</table>

It is observed that more deterioration of units forces retailer to put frequent orders. Hence cycle time decreases and total cost of inventory system increases significantly.

Table 4.2.4.3: Effect of first credit period and shape parameter on decision policy

\[ N = \frac{35}{365}, \text{lc}_2 = 22\%, \alpha = 0.1, \gamma = 0.2 \]

<table>
<thead>
<tr>
<th>M</th>
<th>β</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/365 (15%)</td>
<td>T</td>
<td>0.1587</td>
<td>0.1608</td>
<td>0.1613</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1982.11</td>
<td>1888.82</td>
<td>1878.05</td>
</tr>
<tr>
<td>20/365 (16%)</td>
<td>T</td>
<td>0.1580</td>
<td>0.1602</td>
<td>0.1606</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1759.10</td>
<td>1748.47</td>
</tr>
<tr>
<td>25/365 (17%)</td>
<td>T</td>
<td>0.1575</td>
<td>0.1597</td>
<td>0.1602</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1726.82</td>
<td>1634.97</td>
<td>1624.43</td>
</tr>
</tbody>
</table>

As shape parameter increases, cycle time increases and total cost of an inventory system decreases.
Table 4.2.4.4: Effect of first credit period and salvage value on decision policy

\( N = 35/365, \ i_c = 22\%, \ \alpha = 0.1, \ \beta = 1.5 \)

<table>
<thead>
<tr>
<th>( M )</th>
<th>( \gamma )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/365</td>
<td>T</td>
<td>0.1587</td>
<td>0.1593</td>
<td>0.1599</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1982.11</td>
<td>1956.74</td>
<td>1931.25</td>
</tr>
<tr>
<td>20/365</td>
<td>T</td>
<td>0.1580</td>
<td>0.1586</td>
<td>0.1592</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1826.37</td>
<td>1801.04</td>
</tr>
<tr>
<td>25/365</td>
<td>T</td>
<td>0.1575</td>
<td>0.1581</td>
<td>0.1587</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1726.82</td>
<td>1701.74</td>
<td>1676.51</td>
</tr>
</tbody>
</table>

Increase in salvage value of deteriorated units prolongs the cycle time and decreases total cost of inventory system significantly.

Table 4.2.4.5: Effect of second credit period and deterioration units on decision policy \( M = 20/365, \ i_c = 16\%, \ \beta = 1.5, \ \gamma = 0.2 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \alpha )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>28/365</td>
<td>T</td>
<td>0.1396</td>
<td>0.1377</td>
<td>0.1356</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1891.30</td>
<td>1976.05</td>
<td>2058.80</td>
</tr>
<tr>
<td>32/365</td>
<td>T</td>
<td>0.1500</td>
<td>0.1476</td>
<td>0.1450</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1859.46</td>
<td>1953.62</td>
<td>2055.25</td>
</tr>
<tr>
<td>35/365</td>
<td>T</td>
<td>0.1580</td>
<td>0.1552</td>
<td>0.1522</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1953.20</td>
<td>2051.80</td>
</tr>
</tbody>
</table>

The optimal cycle time and total cost of an inventory system are very sensitive to changes in offered progressive credit period and deterioration of units. Increase in deterioration decreases cycle time and increases cycle time and decreases total cost. Increase in second credit period increases the cycle time and decreases total cost.
Table 4.2.4.6: Effect of second credit period and shape parameter on decision policy \( M = 20/365, \, l_{c1} = 16\%, \, \alpha = 0.1, \, \beta = 0.2 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \beta )</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>28/365 (18%)</td>
<td>T</td>
<td>0.1396</td>
<td>0.1413</td>
<td>0.1416</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1891.30</td>
<td>1813.43</td>
<td>1805.54</td>
</tr>
<tr>
<td>32/365 (20%)</td>
<td>T</td>
<td>0.1500</td>
<td>0.1520</td>
<td>0.1523</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1859.46</td>
<td>1773.43</td>
<td>1764.05</td>
</tr>
<tr>
<td>35/365 (22%)</td>
<td>T</td>
<td>0.1580</td>
<td>0.1602</td>
<td>0.1606</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1759.10</td>
<td>1748.47</td>
</tr>
</tbody>
</table>

Here, cycle time increases and total cost decreases significantly with increase in shape parameter and second offered credit period.

Table 4.2.4.7: Effect of second credit period and salvage parameter on decision policy \( M = 20/365, \, l_{c1} = 16\%, \, \alpha = 0.1, \, \beta = 1.5 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \gamma )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>28/365 (18%)</td>
<td>T</td>
<td>0.1396</td>
<td>0.1401</td>
<td>0.1405</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1891.30</td>
<td>1870.38</td>
<td>1849.36</td>
</tr>
<tr>
<td>32/365 (20%)</td>
<td>T</td>
<td>0.1500</td>
<td>0.1505</td>
<td>0.1510</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1859.46</td>
<td>1836.17</td>
<td>1812.75</td>
</tr>
<tr>
<td>35/365 (22%)</td>
<td>T</td>
<td>0.1580</td>
<td>0.1586</td>
<td>0.1592</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1826.37</td>
<td>1801.04</td>
</tr>
</tbody>
</table>

Increase in salvage value of the deteriorated units increases cycle time and decreases total inventory cost.
Table 4.2.4.8: Effect of deterioration rate and shape parameter on decision policy

\[ M = 20/365, \ \text{lc}_1 = 16\%, \ N = 35/365, \ \text{lc}_1 = 22\%, \ \gamma = 0.2 \]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>T</td>
<td>0.1580</td>
<td>0.1602</td>
<td>0.1606</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1759.10</td>
<td>1748.47</td>
</tr>
<tr>
<td>0.2</td>
<td>T</td>
<td>0.1552</td>
<td>0.1597</td>
<td>0.1605</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1953.20</td>
<td>1771.16</td>
<td>1749.99</td>
</tr>
<tr>
<td>0.3</td>
<td>T</td>
<td>0.1552</td>
<td>0.1592</td>
<td>0.1604</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>2051.80</td>
<td>1783.12</td>
<td>1751.51</td>
</tr>
</tbody>
</table>

The model is very sensitive to deterioration of units in inventory and shape parameter. Increase in deterioration rate decreases cycle time and increases total cost. Increase in shape parameter increases cycle time and decreases total cost of an inventory system.

Table 4.2.4.9: Effect of deterioration rate and salvage parameter on decision policy

\[ M = 20/365, \ \text{lc}_1 = 16\%, \ N = 35/365, \ \text{lc}_2 = 22\%, \ \gamma = 0.2 \]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \gamma )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>T</td>
<td>0.1580</td>
<td>0.1536</td>
<td>0.1592</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1826.37</td>
<td>1801.04</td>
</tr>
<tr>
<td>0.2</td>
<td>T</td>
<td>0.1552</td>
<td>0.1534</td>
<td>0.1576</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1953.20</td>
<td>1904.00</td>
<td>1854.22</td>
</tr>
<tr>
<td>0.3</td>
<td>T</td>
<td>0.1552</td>
<td>0.1542</td>
<td>0.1561</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>2051.80</td>
<td>1979.54</td>
<td>1906.49</td>
</tr>
</tbody>
</table>
Increase in salvage parameter for deteriorated units increases cycle time and decreases total inventory cost.

Table 4.2.4.10: Effect of shape parameter and salvage parameter on decision policy $M = 20/365$, $l_{c1} = 16\%$, $N = 35/365$, $l_{c2} = 22\%$, $\alpha = 0.2$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>T</td>
<td>0.1580</td>
<td>0.1586</td>
<td>0.1592</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1851.55</td>
<td>1826.37</td>
<td>1801.04</td>
</tr>
<tr>
<td>2.5</td>
<td>T</td>
<td>0.1602</td>
<td>0.1603</td>
<td>0.1604</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1759.10</td>
<td>1756.16</td>
<td>1753.22</td>
</tr>
<tr>
<td>3.5</td>
<td>T</td>
<td>0.1606</td>
<td>0.1606</td>
<td>0.1606</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>1748.47</td>
<td>1748.10</td>
<td>1747.73</td>
</tr>
</tbody>
</table>

No significant change is observed in decision variable and objective function for fixed shape parameter and increasing salvage parameter.

4.3 Conclusions:

The economic order quantity for time dependent deteriorated units with associated salvage value when supplier offers credit period to the retailer to settle the account is analyzed in this study. The model is sensitive to deterioration rate $\alpha$ and credit period $M$. The retailer can keep an eye for low deterioration rate to reduce his total inventory cost. To reduce his total inventory cost, he can buy deteriorated units at a lower price and sell it off at the earliest.