CHAPTER 5

INVENTORY MODELS FOR STOCK–DEPENDENT

DEMAND UNDER PROGRESSIVE PAYMENT

SCHEME
5.0 Introduction:

In this chapter, we study two inventory models. Section 5.1 deals with formulation of optimal ordering policies for retailer when demand is stock-dependent and the supplier offers progressive credit periods to settle the account. The effects of permissible credit periods and stock dependent factor on optimum purchase quantity, demand and total cost of an inventory system are observed. In Section 5.2, joint pricing and replenishment policy for stock dependent demand under progressive payment scheme is developed.

5.1 Optimal Ordering Policy for Stock dependent demand under Progressive Payment Scheme:

In this section, a mathematical model is developed to formulate optimal ordering policies for retailer when demand takes following two forms:

**Form 1:** \( R(Q(t)) = a + b Q(t) \) i.e. it is partially constant 'a' and partially dependent on the stock, 'b', \( a \gg b \). 'b' is display factor.

**Form 2:** \( R(Q(t)) = \alpha Q(t)^\beta, \quad 0 \leq \beta < 1, \quad \alpha \gg \beta \), \( \alpha \) is fixed demand and \( \beta \) is display factor.

and the supplier offers progressive credit periods to settle the account. The cost minimization is considered to be an objective function. A flowchart is given to find the flow of optimal ordering policy. A numerical illustration is given to study the effect of various parameters on ordering policy and total cost of an inventory system.

Form 1:

5.1.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model:
• The demand rate, \( R(Q(t)) \), is deterministic and is a function of instantaneous stock-level \( Q(t) \); the demand function \( R(t) \) is given by

\[
R(Q(t)) = a + bQ(t), \quad 0 \leq t \leq T.
\]

where \( a, b > 0 \) and \( a > b \).

\( Q \) (equivalently, \( T \)) is decision variable.

5.1.2 Mathematical Formulation:

The on-hand inventory \( Q(t) \) depletes due to partially constant demand and partially stock dependent demand. Hence, the instantaneous state of inventory \( Q(t) \) at any instant of time \( t, 0 \leq t \leq T \) is governed by the differential equation

\[
\frac{dQ(t)}{dt} = -a - bQ(t), \quad 0 \leq t \leq T \tag{5.1.2.1}
\]

with the initial condition \( Q(0) = Q \) and boundary condition \( Q(T) = 0 \).

Consequently, the solution of (5.1 2 1) is given by

\[
Q(t) = \frac{a(e^{b(T-t)} - 1)}{b}, \quad 0 \leq t \leq T \tag{5.1.2.2}
\]

and the procurement quantity is

\[
Q = \frac{a(e^{bT} - 1)}{b} \tag{5.1.2.3}
\]

Next follows computation of various cost components of the total cost of the system.

• Ordering cost;

\[
OC = \frac{A}{T} \tag{5 1.2.4}
\]

• Inventory holding cost;

\[
IHC = \frac{h}{T} \int_0^T Q(t) \, dt = \frac{ha}{b^2T} [e^{bT} - bT - 1] \tag{5.1.2.5}
\]
Regarding interest charged and earned, based on the length of the cycle time $T$, three cases arise:

**Case 1:** $T \leq M$

**Case 2:** $M < T < N$

**Case 3:** $T \geq N$

We discuss each case in detail.

**Case 1:** $T \leq M$ (Fig. 5.1.2.1)

![Diagram](image)

**Figure 5.1.2.1**

Here, the retailer sells $Q$-units during $[0, T]$ and is paying for $CQ$-units in full to the supplier at time $M > T$. Therefore, interest charges are zero, i.e.

$$ IC_1 = 0 $$  \hspace{1cm} (5.1.2.6)

By selling $Q$-units during $[0, T]$ and depositing this revenue in an interest bearing account at the rate of $le$/$/year. In the period $[T, M]$, the retailer earns interest at the rate $le$/$/year$. Therefore, interest earned per time unit is

$$ IE_1 = \frac{Pe}{T} \left[ \int_{0}^{T} R(Q(t))t dt + Q(M - T) \right] $$

$$ = \frac{Pe}{bT} \left( \frac{1}{b} (e^{bT} - bT - 1) + (e^{bT} - 1)(M - T) \right) $$  \hspace{1cm} (5.1.2.7)

Using (5.1.2.4) – (5.1.2.7), total cost, $TC_1(T)$ per time unit of an inventory system

$$ TC_1(T) = OC + IHC + IC_1 - IE_1 $$  \hspace{1cm} (5.1.2.8)
The optimum value of $T = T_1$ is the solution of non-linear equation

$$
\frac{dTC_1(T)}{dT} = -A \frac{h_a(-e^{bT} + 1 + bT)}{T^2 b^2} - \frac{h_a(1 - e^{bT})}{bT} + \frac{IE_1}{T} + \frac{Ple a e^{bT}(M - T)}{T} = 0
$$

(5.1.2.9)

which minimizes $TC_1(T)$ provided

$$
\frac{d^2 TC_1(T)}{dT^2} = \frac{2A}{T^3} - \frac{2h_a(-e^{bT} + 1 + bT)}{T^3 b^2} + \frac{2h_a(1 - e^{bT})}{bT^2} + \frac{h_a e^{bT}}{T} - \frac{2IE_1}{T^3}
$$

$$
+ \frac{2Ple a e^{bT}(M - T)}{T^2} - \frac{Ple a e^{bT}(b(M - T) - 1)}{T}
$$

(5.1.2.10)

Case 2: $M < T < N$ (Fig 5.1.2.2)

The retailer sells units and deposits the revenue into an interest bearing account at an interest rate $r$ per unit/year during $[0, M]$. Therefore, the interest earned during $[0, M]$ is given by

$$
IE_2 = Ple \int_0^M R(Q(t))t \, dt = \frac{Ple a}{b^2} \{(bM + 1)e^{b(T - M)} - bT - 1\} \quad (5.1.2.11)
$$

Retailer has to pay for $Q$-units purchased at time $t = 0$ at the rate of $C$ $$/unit to the supplier during $[0, M]$. The retailer sells $R(Q(M))M$-units at sale.
price $P/\text{unit.} \text{ So he has generated revenue of } PR(Q(M)) M \text{ plus the interest earned, } IE_2, \text{ during } [0, M]. \text{ Two sub-cases may arise: }

\textbf{Sub-case 2.1: } \text{Let } PR(Q(M)) M + IE_2 \geq CQ \text{ i.e. the retailer has enough money to pay for all } Q\text{-units procured at time } t = 0. \text{ Then, interest charges, }

\[ IC_{2.1} = 0 \] \tag{5.1.2.12}

and the interest earned, \( IE_{2.1} \), per time unit is

\[ IE_{2.1} = \frac{IE_2}{T} \] \tag{5.1.2.13}

Using (5.1.2.4), (5.1.2.5), (5.1.2.12) and (5.1.2.13) total cost, \( TC_{2.1}(T) \) per time unit of an inventory system

\[ TC_{2.1}(T) = OC + IHC + IC_{2.1} - IE_{2.1} \] \tag{5.1.2.14}

The optimum value of \( T = T_{2.1} \) is the solution of non-linear equation

\[ dTC_{2.1}(T) = \frac{-A}{T^2} + \frac{ha(-e^{bT} + 1 + bT)}{T^2b^2} \left( \frac{ha(-e^{bT} + 1)}{bT} \right) - \frac{Ple}{bT^a(-1 + e^{b(T-M)}(bM + 1))} \\
+ \frac{Ple}{b^2T^2} \left( -a[(1 + bT) - e^{b(T-M)}(bM + 1)] \right) = 0 \] \tag{5.1.2.15}

\( T = T_{2.1} \) minimizes total cost; \( TC_{2.1}(T) \), provided

\[ \frac{d^2TC_{2.1}(T)}{dT^2} = \frac{2A}{T^3} - \frac{2ha(-e^{bT} + 1 + bT)}{T^3b^2} + \frac{2ha(1 - e^{bT})}{bT^2} + \frac{hae^{bT}}{T} \\
- \frac{Ple a e^{b(T-M)}(bM + 1)}{T} + \frac{2Ple}{bT^2 a(-1 + e^{b(T-M)}(bM + 1))} \\
- \frac{2Ple}{b^2T^3} a[e^{b(T-M)}(bM + 1) - (1 + bT)] \\
> 0, \text{ for all } T \] \tag{5.1.2.16}
Sub-case 2.2: \( PR(Q(M))M + IE_2 < CQ \)

Here, retailer will have to pay interest on the un-paid balance
\[ U_1 = CQ - \left[ PR(Q(M))M + IE_2 \right] \] at rate of \( Ic_1 \) at time \( M \) to the supplier. The interest to be paid, \( IC_{2,2} \), per time unit is:

\[ IC_{2,2} = \frac{U_1^2}{PQ} Ic_1 \int_{M}^{T} Q(t)dt = \frac{U_1^2 Ic_1}{Pb(e^{bT} - 1)} \left( e^{b(T-M)} - b(T-M) - 1 \right) \] (5.1.2.17)

and interest earned per time unit,

\[ IE_{2,2} = \frac{IE_2}{T} \] (5.1.2.18)

Using equations (5.1.2.4), (5.1.2.5), (5.1.2.17) and (5.1.2.18), total cost, \( TC_{2,2}(T) \) per time unit of an inventory system

\[ TC_{2,2}(T) = OC + IHC + IC_{2,2} - IE_{2,2} \] (5.1.2.19)

The optimum value of \( T = T_{2,1} \) is the solution of non-linear equation

\[
\frac{dTC_{2,2}(T)}{dT} = \frac{-A}{T^2} + \frac{ha(-e^{bT} + 1 + bT)}{T^2b^2} - \frac{ha(1-e^{bT})}{bT} - \frac{U_1^2 Ic_1 \%_1}{P(1-e^{bT})^2} - \frac{U_1^2 Ic_1(1-e^{b(T-M)})}{P(1-e^{bT})} - \frac{2U_1 Ic_1 \%_1}{Pb(1-e^{bT})} \\
\times \left[ Ca e^{bT} - Pab e^{b(T-M)} (bM + 1) \right] \%_2 \left( \frac{\%_2}{T} + \frac{IE_{2,2}}{T} \right) = 0 \] (5.1.2.20)

where

\[ \%_1 = (-e^{b(T-M)} - bM + 1 + bT) \]

\[ \%_2 = \frac{Pfe}{b} a(-1+e^{b(T-M)}(bM + 1)) \]

can be solved by suitable numerical method for \( T = T_{2,2} \) which minimizes \( TC_{2,2}(T) \)

provided
\[ \frac{d^2T_{22}(T)}{dT^2} = \frac{2A}{T^3} - \frac{2ha(-e^{bT} + 1 + bT)}{T^3b^2} + \frac{2ha(1 - e^{bT})}{bT^2} + \frac{hae^{bT}}{T} - \frac{2\%3^2Ic_1\%1}{Pb(e^{bT} - 1)} \]

\[ + \frac{4U_1lc_1\%1\%3e^{bT}}{P(1 - e^{bT})^2} - \frac{4U_1lc_1(1 - e^{b(T-M)})\%3}{P(1 - e^{bT})} \]

\[ + \frac{2U_1^2lc_1(b - be^{b(T-M)})e^{bT}}{P(1 - e^{bT})^2} + \frac{U_1^2lc_1\%1be^{bT}}{P(1 - e^{bT})^2} + \frac{U_1^2lc_1be^{b(T-M)}}{P(1 - e^{bT})} \]

\[ - \frac{Ple a e^{b(T-M)}(bM + 1)}{T} + \frac{2\%2}{T^2} - \frac{2IE_{2.2}}{T^3} \]

\[ - \frac{2U_1lc_1\%1[Cae^{bT} - Pabe^{b(T-M)}M - Pleae^{b(T-M)}(bM + 1)]}{Pb(1 - e^{bT})} \]

> 0, for all \( T \)

\[(5.1.2.21)\]

where

\[\%1 = (-e^{b(T-M)} - bm + 1 + bT)\]

\[\%2 = \frac{Ple}{b}a(-1 + e^{b(T-M)}(bm + 1))\]

\[\%3 = Cae^{bT} - Pabe^{b(T-M)}M - \%2\]

**Case 3:** \( T \geq N \) (Fig. 5.1.2.3)
Based on the total purchase cost, $CQ$, total money $PR(Q(M))M + IE_2$ in account at $M$ and total money in account at $N$ is $PR(Q(N))N + IE_2$, three sub-cases may arise.

**Sub-case 3.1:** Let $PR(Q(M))M + IE_2 \geq CQ$ Then this sub-case is same as sub-case 2.1. (Note: Decision variables and objective function are designated by subscript 3.1)

**Sub-case 3.2:** Let $PR(Q(M))M + IE_2 < CQ$ but

$$PR(Q(N-M))(N-M) + Pl e \int_{M}^{N} R(Q(t))dt \geq CQ - [PR(Q(M))M + IE_2]$$

This sub-case coincides with sub-case 3.2. (Note: Decision variables and objective function are designated by subscript 3.2)

**Sub-case 3.3:** Let $PR(Q(M))M + IE_2 < CQ$ but

$$PR(Q(N-M))(N-M) + Pl e \int_{M}^{N} R(Q(t))dt < CQ - [PR(Q(M))M + IE_2]$$

Here, the retailer does not have sufficient money in his account to pay off for total purchase cost at $N$. He will do payment of $[PR(Q(M))M + IE_2]$ at $M$ and

$$PR(Q(N-M))(N-M) + Pl e \int_{M}^{N} R(Q(t))dt$$

at $N$. So, he has to pay interest charges on un-paid balance $U_1 = CQ - [PR(Q(M))M + IE_2]$ with interest rate $Ic_1$ during $[M, N]$ and $U_2 = U_1 - \left[ PR(Q(N-M))(N-M) + Pl e \int_{M}^{N} R(Q(t))dt \right]$ with interest rate $Ic_2$ during $[N, T]$.

Therefore, total interest charges, $IC_{3.3}$, per time unit is given by
Development of Inventory Models Under a Progressive Interest Payable Scenario

\[ IC_{3,3} = \frac{U_1 Ic_1 (N-M)}{T} + \frac{U_2^2}{P Q} \int_0^T Q(t) dt \]  
(5.1.2.22)

and total interest earned per time unit is;

\[ IE_{3,3} = \frac{IE_2}{T} \]  
(5.1.2.23)

Using equations (5.1.2.4), (5.1.2.5), (5.1.2.22) and (5.1.2.23), total cost, \( TC_{3,3}(T) \) per time unit of an inventory system

\[ TC_{3,3}(T) = OC + IHC + IC_{3,3} - IE_{3,3} \]  
(5.1.2.24)

The optimum value of \( T = T_{3,3} \) is the solution of non-linear equation

\[
\frac{dT C_{3,3}(T)}{dT} = -\frac{A}{T^2} + \frac{ha(-e^{-bT}+1+bT)}{T^2b^2} - \frac{ha(1-e^{-bT})}{bT} - \frac{U_1 Ic_1 (N-M)}{T} + \frac{\%1 Ic_1 (N-M)}{T} \\
+ \frac{2U_2 Ic_2 \%3}{P (1-e^{-bT})} \left\{ \%1 - Pae^{b(T-N+M)}(N-M) \right\} \\
+ \frac{Pa [-e^{-bT} + e^{b(T-N+M)}(b(N-M)+1)]}{b} - \frac{U_2^2 Ic_2 (1-e^{-b(T-M)})}{P (1-e^{-bT})} \\
+ \frac{U_2^2 Ic_2 \%3e^{bT}}{P (1-e^{-bT})^2} - \frac{\%2}{T} + \frac{IE_{3,3}}{T} = 0
\]  
(5.1.2.25)

where

\[ \%1 = (-e^{-b(T-M)} - bM + 1 + bT) \]

\[ \%2 = \frac{Ple}{b} a(-1 + e^{b(T-M)}(bM + 1)) \]

\[ \%3 = Cae^{bT} - Pae^{b(T-M)}M - \%2 \]

The sufficiency condition for \( TC_{3,3}(T) \) to be minimum is
\[
\frac{d^2 \text{TC}_{3,3}(T)}{dT^2} = \frac{2A}{T^3} - \frac{2ha(-e^{bT} + 1 + bT)}{T^3b^2} + \frac{2ha(1-e^{-bT})}{bT^2} + \frac{hae^{bT}}{T} - \frac{2 \%3 \text{Ic}_1(N-M)}{T^2}
\]

\[
+ \frac{\{Cae^{bT} - Pae^{b(T-M)}M - Pleae^{b(T-M)}(bM + 1)\} \text{Ic}_1(N-M)}{T}
\]

\[
- \frac{2 \%3 \text{Ic}_1(N-M)}{T^2} + \frac{2U_1\text{Ic}_1(N-M)}{T^3} - \frac{2 \%4 \text{Ic}_2}{bP(e^{bT-1})}
\]

\[
- \frac{4U_2\text{Ic}_2(1-e^{b(T-N)})}{P(e^{bT-1})} + \frac{4U_2\text{Ic}_2 \%1 \%4 e^{bT}}{P(e^{bT-1})^2} - \frac{2U_2\text{Ic}_2 \%1 \%4}{bP(e^{bT-1})}
\]

\[
+ \frac{U_2^2 \text{Ic}_2}\%1 \%4 e^{b(T-N)} + \frac{2U_2^2 \text{Ic}_2 (-be^{b(T-N)} + b)e^{bT}}{P(e^{bT-1})^2}
\]

\[
\frac{2U_2^2 \text{Ic}_2 \%1 e^{bT}b}{P(e^{bT-1})^3} + \frac{U_2^2 \text{Ic}_2 \%1 e^{bT}b}{P(e^{bT-1})^2} - \frac{Pleae^{b(T-M)}(bM + 1)}{T}
\]

\[
+ \frac{2\%2}{T^2} - \frac{2\text{E}_{3,3}}{T^3}
\]

> 0, for all \(T\)  \hspace{1cm} (5.1.2.26)

where

\[
\%1 = (-e^{b(T-M)} - bM + 1 + bT)
\]

\[
\%2 = \frac{Pleae}{b} (-1 + e^{b(T-M)}(bM + 1))
\]

\[
\%3 = Cae^{bT} - Pabe^{b(T-M)}M - \%2
\]

\[
\%4 = \%3 - Pabe^{b(T-N+M)}(N-M)
\]

\[
\quad + \frac{Pleae}{b} [ - e^{bT} + be^{b(T-N+M)}(N-M) + e^{b(T-N+M)}]
\]

In the next section, computational flowchart is given to search for the optimal solution.
5.1.3 Flowchart:
5.1.4 Numerical Example and observations:

Consider the following parametric values in appropriate units.

\[ [A, C, P, h, a, Ie] = [200, 20, 30, 0.2, 1000, 12\%] \]

Table: 5.1.4.1

Effect of M and b on decision parameters with \( Ic_2 = 20\% \), \( N = 35/365 \)

<table>
<thead>
<tr>
<th>M</th>
<th>17/365 ( Ic_1 = 13% )</th>
<th>18/365 ( Ic_1 = 13.5% )</th>
<th>19/365 ( Ic_1 = 14% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b ), N</td>
<td>( Q )</td>
<td>( R )</td>
<td>( TC )</td>
</tr>
<tr>
<td>3.5</td>
<td>288.03</td>
<td>2008.10</td>
<td>980.02</td>
</tr>
<tr>
<td>4.0</td>
<td>309.74</td>
<td>2238.96</td>
<td>927.10</td>
</tr>
<tr>
<td>4.5</td>
<td>334.28</td>
<td>2504.26</td>
<td>905.55</td>
</tr>
</tbody>
</table>

Observations:

- Increase in first credit period decreases total cost and demand rate is insensitive for fixed consumption rate.
- For fixed credit period, increase in stock dependent consumption rate; \( b \) results increases in demand and procurement quantity significantly and decreases total cost of an inventory system.
### Table: 5.1.4.2

**Effect of N and b on decision parameters with Ic₁ = 13%, M = 17/365**

<table>
<thead>
<tr>
<th>N →</th>
<th>( b )</th>
<th>30/365</th>
<th>35/365</th>
<th>40/365</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Q )</td>
<td>251.31</td>
<td>288.03</td>
<td>328.99</td>
</tr>
<tr>
<td>3.5</td>
<td>( R )</td>
<td>1879.55</td>
<td>2008.11</td>
<td>2151.46</td>
</tr>
<tr>
<td></td>
<td>( TC )</td>
<td>1101.34</td>
<td>980.02</td>
<td>933.80</td>
</tr>
<tr>
<td>4.0</td>
<td>( Q )</td>
<td>267.11</td>
<td>309.73</td>
<td>358.05</td>
</tr>
<tr>
<td></td>
<td>( R )</td>
<td>2068.44</td>
<td>2238.92</td>
<td>2432.20</td>
</tr>
<tr>
<td></td>
<td>( TC )</td>
<td>1064.03</td>
<td>927.10</td>
<td>872.99</td>
</tr>
<tr>
<td>4.5</td>
<td>( Q )</td>
<td>284.33</td>
<td>334.28</td>
<td>391.37</td>
</tr>
<tr>
<td></td>
<td>( R )</td>
<td>2279.48</td>
<td>2504.26</td>
<td>2761.16</td>
</tr>
<tr>
<td></td>
<td>( TC )</td>
<td>1047.47</td>
<td>905.55</td>
<td>852.88</td>
</tr>
</tbody>
</table>

**Observations:**

- Increase in second allowable credit period, increases demand, optimal purchase units and decreases total cost of an inventory system.
- Increase in stock dependent consumption rate significantly increases demand, purchase quantity and decreases total cost of an inventory system.

**Form 2:**

### 5.1.5 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model.
• The demand rate, \( R(Q(t)) \), is deterministic and is a function of instantaneous stock level \( Q(t) \); the demand function \( R(Q(t)) \) is given by

\[
R(Q(t)) = \alpha Q(t)\beta, \quad 0 \leq t \leq T, \quad 0 \leq \beta < 1, \quad \alpha \gg \beta.
\]

\( Q \) (equivalently, \( T \)) is decision variable.

5.1.6 Mathematical Formulation:

The on-hand inventory \( Q(t) \) depletes due to partially constant demand and partially stock dependent demand. Hence, the instantaneous state of inventory \( Q(t) \) at any instant of time \( t \), \( 0 \leq t \leq T \) is governed by the differential equation

\[
\frac{dQ(t)}{dt} = -\alpha Q(t)\beta, \quad 0 \leq t \leq T \quad \text{(5.1.6.1)}
\]

with the initial condition \( Q(0) = Q \) and boundary condition \( Q(T) = 0 \).

Consequently, the solution of (5.1.2.1) is given by

\[
Q(t) = \alpha^{1-\beta} (1-\beta)^{1-\beta} (T-t)^{1-\beta}, \quad 0 \leq t \leq T \quad \text{(5.1.6.2)}
\]

and the procurement quantity is

\[
Q = \alpha^{1-\beta} (1-\beta)^{1-\beta} T^{1-\beta} \quad \text{(5.1.6.3)}
\]

Next follows computation of various cost components of the total cost of the system.

• Ordering cost;

\[
OC = \frac{A}{T} \quad \text{(5.1.6.4)}
\]

• Inventory holding cost;

\[
IHC = \frac{h}{T} \int_0^T Q(t) \, dt = \frac{1}{2-\beta} \alpha^{1-\beta} (1-\beta)^{1-\beta} T^{1-\beta} \quad \text{(5.1.6.5)}
\]
Regarding interest charged and earned, based on the length of the cycle time $T$, three cases arise:

**Case 1:** $T \leq M$

**Case 2:** $M < T < N$

**Case 3:** $T \geq N$

We discuss each case in detail.

**Case 1:** $T \leq M$ (See Fig. 5.1.2.1)

Here, the retailer sells $Q$–units during $[0, T]$ and is paying for $CQ$–units in full to the supplier at time $M > T$. Therefore, interest charges are zero. i.e.

$$IC_1 = 0 \quad (5.1.6.6)$$

The retailer sells $Q$–units during $[0, T]$ and deposits the revenue in an interest bearing account at the rate of $le/\$/$year. In the period $[T, M]$, the retailer deposits revenue into the account that earns interest at the rate $le/\$/$year. Therefore, interest earned per time unit is

$$IE_1 = \frac{Ple}{T} \left[ \int_0^T R(Q(t))t \, dt + Q(M - T) \right]$$

$$= P \frac{le}{T} \alpha^{1-\beta} (1-\beta)^{1-\beta} \beta T^{1-\beta} \left[ M - T \left( \frac{1-\beta}{2-\beta} \right) \right] \quad (5.1.6.7)$$

Using (5.1.2.4) – (5.1.2.7), total cost, $TC_1(T)$ per time unit of an inventory system

$$TC_1(T) = OC + IHC + IC_1 - IE_1 \quad (5.1.6.8)$$

The optimum value of $T = T_1$ is the solution of non-linear equation

$$\frac{dTC_1(T)}{dT} = -A + \frac{h%_1}{T^2} + \frac{IE_1}{(1-\beta)T(2-\beta)} + \frac{Ple \%_1 (1-\beta)}{(2-\beta)} = 0 \quad (5.1.6.9)$$

where $%_1 = \alpha^{1-\beta} (1-\beta)^{1-\beta} T^{1-\beta}$

which minimizes $TC_1(T)$ provided
\[
\frac{d^2TC_1(T)}{dT^2} = \frac{2A}{T^3} + \frac{h \beta \%1}{(1 - \beta)^2 T^2 (2 - \beta)} - \frac{\beta IE_1}{(1 - \beta)^2 T^2} + \frac{2 P Ie \%1}{T (2 - \beta)}
\]

> 0, \quad \text{for all } T \tag{5.16.10}

where \( \%1 = \alpha^{1-\beta} (1-\beta)^{1-\beta} T^{1-\beta} \)

**Case 2: \( M < T < N \) (See Fig. 5.1.2.2)**

The retailer sells units and deposits the revenue into an interest bearing account at an interest rate \( Ie/\text{unit/year} \) during \([0, M]\). Therefore, the interest earned during \([0, M]\) is given by

\[
IE_2 = P Ie \int_0^M R(Q(t)) t \, dt = P Ie \frac{1}{\alpha^{1-\beta} (1-\beta)^{1-\beta} M^{1-\beta}} \frac{2-\beta}{2-\beta} \tag{5.1.6.11}
\]

Retailer has to pay for \( Q \)-units purchased at time \( t = 0 \) at the rate of \( C \)/unit to the supplier during \([0, M]\). The retailer sells \( R(Q(M)) M \)-units at sale price \( P/\text{unit} \). So he has generated revenue of \( P R(Q(M)) M \) units. The interest earned, \( IE_2 \), during \([0, M]\). Two sub-cases may arise:

**Sub-case 2.1:** Let \( P R(Q(M)) M + IE_2 \geq CQ \), i.e., the retailer has enough money to pay for all \( Q \)-units procured at time \( t = 0 \). Then, interest charges,

\[
IC_{2,1} = 0 \tag{5.1.6.12}
\]

and the interest earned, \( IE_{2,1} \), per time unit is

\[
IE_{2,1} = \frac{IE_2}{T} \tag{5.1.6.13}
\]

Using (5.1.2.4), (5.1.2.5), (5.1.2.12) and (5.1.2.13) total cost, \( TC_{2,1}(T) \) per time unit of an inventory system

\[
TC_{2,1}(T) = OC + IHC + IC_{2,1} - IE_{2,1} \tag{5.1.6.14}
\]
The optimum value of \( T = T_{21} \) is the solution of non-linear equation

\[
\frac{dTC_{21}(T)}{dT} = -\frac{A}{T^2} + \frac{h\%_1}{(1-\beta)T(2-\beta)} - \frac{IE_{21}}{T} = 0
\]  \hspace{1cm} (5.1.6.15)

where \( \%_1 = \alpha^{1-\beta}/(1-\beta) \cdot T^{1-\beta} \).

\( T = T_{21} \) minimizes total cost; \( TC_{21}(T) \), provided

\[
\frac{d^2TC_{21}(T)}{dT^2} = \frac{2A}{T^3} + \frac{h\beta\%_1}{(1-\beta)^2T^2(2-\beta)} - \frac{2IE_{21}}{T^2} > 0, \hspace{0.5cm} \text{for all } T
\]  \hspace{1cm} (5 1.6 16)

where \( \%_1 = \alpha^{1-\beta}/(1-\beta) \cdot T^{1-\beta} \).

**Sub-case 2.2:** \( PR(Q(M)) M + IE_2 < CQ \)

Here, retailer will have to pay interest on the un-paid balance \( U_1 = CQ - [PR(Q(M)) M + IE_2] \) at rate of \( Ic_1 \) at time \( M \) to the supplier. The interest to be paid, \( IC_{22} \), per time unit is:

\[
IC_{22} = \frac{U_1^2}{PQ} \int_{M}^{T} Q(t)dt = \frac{U_1^2 Ic_1 (1-\beta)(T-M)^{2-\beta}}{P \alpha T^{1-\beta}(2-\beta)}
\]  \hspace{1cm} (5 1.6.17)

and interest earned per time unit:

\[
IE_{22} = \frac{IE_2}{T}
\]  \hspace{1cm} (5.1.6.18)

Using equations (5.1.2.4), (5.1.2.5), (5.1.2.17) and (5.1.2.18), total cost, \( TC_{22}(T) \) per time unit of an inventory system

\[
TC_{22}(T) = OC + IHC + IC_{22} - IE_{22}
\]  \hspace{1cm} (5.1.6.19)

The optimum value of \( T = T_{21} \) is the solution of non-linear equation
\[
\frac{dTC_{2,2}(T)}{dT} = -A + \frac{h\%1}{T^2} + \frac{2U_1 Ic_1 (1-\beta)(T-M)^{1-\beta}}{P \alpha T^{1-\beta} (2-\beta)}
\]

\[
\times \left[ \frac{C\%1}{(1-\beta)T} - \frac{P\%2}{(1-\beta)(T-M)} \right] U_1^2 Ic_1 (T-M)^{1-\beta} \frac{1}{(T-M)P \alpha T^{1-\beta}}
\]

\[
\frac{2-\beta}{U_1^2 Ic_1 (T-M)^{1-\beta}} \frac{1}{(2-\beta)P \alpha T^{1-\beta}} + \frac{IE_2}{(T-M)^2 (2-\beta)P \alpha T^{1-\beta}}
\]

(5.1.6 20)

where \(\%1 = \alpha^{1-\beta} (1-\beta)^{1-\beta} T^{1-\beta}\); \(\%2 = \alpha^{1-\beta} (1-\beta)^{1-\beta} (T-M)^{1-\beta}\)

can be solved by suitable numerical method for \(T = T_{22}\) which minimizes \(TC_{2,2}(T)\)

provided

\[
\frac{d^2TC_{2,2}(T)}{dT^2} = \frac{2A}{T^3} + \frac{h \beta \%1}{(1-\beta)^2 T^2 (2-\beta)} + \frac{2 \%3^2 Ic_1 (1-\beta)(T-M)^{1-\beta}}{P \alpha T^{1-\beta} (2-\beta)}
\]

\[
+ \frac{2-\beta}{P \alpha T^{1-\beta}} \left\{ \frac{1}{T-M} - \frac{1}{(2-\beta)T} \right\}
\]

\[
+ \frac{4U_1 Ic_1 (T-M)^{1-\beta}}{P \alpha T^{1-\beta} (2-\beta)} \left\{ \frac{2-\beta}{P \alpha T^{1-\beta} (2-\beta)} \right\}
\]

\[
\begin{align*}
&+ \frac{2-\beta}{P \alpha T^{1-\beta} (2-\beta)} \left\{ \frac{C\%1}{T^2} - \frac{P\%2}{(T-M)^2} \right\} \\
&+ \frac{2-\beta}{P \alpha T^{1-\beta} (T-M)^2 (1-\beta)} \left\{ \frac{1}{T-M} - \frac{2}{T^2} \right\} + \frac{2-\beta}{P \alpha T^{1-\beta} T^2 (1-\beta)}
\end{align*}
\]
\[ \frac{2IE_2}{(2 - \beta)T^3} > 0, \quad \text{for all } T \] (5 1 6 21)

where

\[ \%1 = \alpha^{1-\beta} (1 - \beta)^{-1/\beta} T^{-1/\beta}; \quad \%2 = \alpha^{1-\beta} (1 - \beta)^{-1/\beta} (T - M)^{-1/\beta}; \quad \%3 = \frac{C\%1}{(1 - \beta)T} - \frac{P\%2}{(1 - \beta)(T - M)} \]

**Case 3: \( T \geq N \)** (See Fig. 5.1.2.3)

Based on the total purchase cost, \( CQ \), total money \( PR(Q(M))M + IE_2 \) in account at \( M \) and total money in account at \( N \) is \( PR(Q(N))N + IE_2 \), three sub-cases may arise:

**Sub-case 3.1:** Let \( PR(Q(M))M + IE_2 \geq CQ \) Then this sub-case is same as sub-case 2.1. (Note: Decision variables and objective function are designated by subscript 3.1)

**Sub-case 3.2:** Let \( PR(Q(M))M + IE_2 < CQ \) but

\[ PR(Q(N-M))(N-M) + P I e \int_{M}^{N} R(Q(t))dt \geq CQ - \left[ PR(Q(M))M + IE_2 \right] \]

This sub-case coincides with sub-case 3.2. (Note: Decision variables and objective function are designated by subscript 3.2)

**Sub-case 3.3:** Let \( PR(Q(M))M + IE_2 < CQ \) but

\[ PR(Q(N-M))(N-M) + P I e \int_{M}^{N} R(Q(t))dt < CQ - \left[ PR(Q(M))M + IE_2 \right] \]

Here, the retailer does not have enough money in his account to pay off for total purchase cost at \( N \). He will do payment of \( \left[ PR(Q(M))M + IE_2 \right] \) at \( M \) and

\[ PR(Q(N-M))(N-M) + P I e \int_{M}^{N} R(Q(t))dt \] at \( N \). So, he has to pay interest
charges on un-paid balance \( U_1 = CQ - [P R(Q(M))M + IE_2] \) with interest rate \( Ic_1 \) during \([M, N]\) and \( U_2 = U_1 - \left[ P R(Q(N-M)) (N-M) + Pl e \int_{M}^{N} R(t) dt \right] \) with interest rate \( Ic_2 \) during \([N, T]\).

Therefore, total interest charges, \( IC_{3,3} \), per time unit is given by

\[
IC_{3,3} = \frac{U_1 Ic_1 (N-M)}{T} + \frac{U_2^2}{PQ} Ic_2 \int_{M}^{T} Q(t) dt
\]  

(5.1.6.22)

and total interest earned per time unit is;

\[
IE_{3,3} = \frac{IE_2}{T}
\]  

(5.1.6.23)

Using equations (5.1.2.4), (5.1.2.5), (5.1.2.22) and (5.1.2.23), total cost, \( TC_{3,3}(T) \) per time unit of an inventory system

\[
TC_{3,3}(T) = OC + IHC + IC_{3,3} - IE_{3,3}
\]  

(5.1.6.24)

The optimum value of \( T = T_{3,3} \) is the solution of non-linear equation

\[
\frac{dTC_{3,3}(T)}{dT} = \frac{-A}{T^2} + \frac{h}{(1-\beta)T(2-\beta)} + \frac{\%3 Ic_1 (N-M)}{T} - \frac{U_1 Ic_1 (N-M)}{T^2} + \frac{2-\beta}{P \alpha \ T^{1-\beta} (2-\beta)}
\]

\[
P \alpha \left\{ \frac{\beta}{1-\beta} \left[ \frac{\beta}{1-\beta} \right] + \frac{\beta}{1-\beta} \right\} \left( \frac{1}{1-\beta} \right) - \frac{1}{1-\beta} - \left( (T-N)^{1-\beta} - (T-M)^{1-\beta} \right)
\]

\[
\times P l a e \left[ \frac{1}{1-\beta} \left( \frac{1}{1-\beta} - \frac{U_2^2 Ic_2 (T-M)^{1-\beta}}{P \alpha \ T^{1-\beta}} \right) + \frac{1}{1-\beta} \right]
\]

\[
+ \frac{IE_2}{(2-\beta)T^2} = 0
\]  

(5.1.6.25)
where \( \%1 = \alpha^{1-\beta} (1-\beta)^{1-\beta} \frac{1}{T^{1-\beta}}; \) \( \%2 = \alpha^{1-\beta} (1-\beta)^{1-\beta} \frac{2-\beta}{(T-M)^{1-\beta}} \)

\( \%3 = \frac{C \%1}{(1-\beta)T} - \frac{P \%2}{(1-\beta)(T-M)}; \)

\( \%4 = \alpha^{1-\beta} (1-\beta)^{1-\beta} \frac{2-\beta}{(T-N+M)^{1-\beta}} \)

The sufficiency condition for \( TC_3(T) \) to be minimum is

\[
\frac{d^2 TC_3(T)}{dT^2} = \frac{2A}{T^3} + \frac{h \beta \%1}{(1-\beta)^2 T^2 (2-\beta)} + \left[ \frac{C \%1 \beta}{(1-\beta)^2 T^2} - \frac{P \%2 \beta}{(1-\beta)^2 T^2} \right] \frac{Ic_1(N-M)}{T} + \frac{2U_1 Ic_1(N-M)}{T^3} \frac{2-\beta}{P \alpha T^{1-\beta} (2-\beta)} + \frac{2Ic_2(1-\beta)(T-M)^{1-\beta} \%5^2}{P \alpha T^{1-\beta}} + \frac{4U_2 Ic_2(T-M)^{1-\beta} \%5}{T^3} \frac{1}{P \alpha T^{1-\beta}} + \frac{2Ic_2(1-\beta)(T-M)^{1-\beta}}{P \alpha T^{1-\beta} (2-\beta)} + \frac{P \%4 \beta}{(1-\beta)^2 (T-N+M)^2} - P \alpha \frac{1}{(1-\beta)^{1-\beta}} \frac{1}{(T-M)^{1-\beta}} \frac{1}{(T-N)(1-\beta)} + \frac{1}{(T-M)(1-\beta)} \]

\[
\times \left( \frac{N \beta(T-N)^{1-\beta}}{(1-\beta)^2} - \frac{M \beta(T-M)^{1-\beta}}{(1-\beta)^2} + \frac{2\beta-1}{(T-N)(1-\beta)} + \frac{1}{(T-M)(1-\beta)} \right) \}

\[
\frac{2\beta-1}{P \alpha T^{1-\beta}} \left\{ \frac{2-\beta}{1-\beta} \frac{1}{(T-M)^2} - \frac{2}{T(T-M)} \right\} \]
\[
\frac{1}{T^2(2 - \beta)(1 - \beta)} + \frac{1}{(2 - \beta)T^2} \quad - \quad \frac{2IE_2}{(2 - \beta)T^3}
\]

> 0, \quad \text{for all } T \quad (5.1.6.26)

where \( \%_1 = \alpha^{1-\beta} (1 - \beta) T^{1-\beta} \); \( \%_2 = \alpha^{1-\beta} (1 - \beta) (T - M)^{1-\beta} \)

\( \%_3 = \frac{C\%_1}{(1 - \beta)T} - \frac{P\%_2}{(1 - \beta)(T - M)} \); \( \%_4 = \frac{1}{T} \); \( \%_5 = \frac{1}{(1 - \beta)(T - N + M)^{1-\beta}} \)

\[
%5 = \left\{ \frac{C%_1}{(1 - \beta)T} - \frac{P%_2}{(1 - \beta)(T - M)} - \frac{P%_4}{(1 - \beta)(T - N + M)} - \frac{1}{T} \right\}
\]

\[
\times \left\{ \frac{2\beta - 1}{(1 - \beta)} - \frac{M(T - M)}{(1 - \beta)} + \frac{1}{(T - N)^{1-\beta}} - \frac{1}{(T - M)^{1-\beta}} \right\}
\]

In the next section, computational flowchart is given to search for the optimal solution.

5.1.7 Flowchart:

The flowchart and notations are same as defined in section 5.1.3

5.1.8 Numerical Example and observations:

Consider the following parametric values in appropriate units.

\([A, C, P, h, \alpha, \beta, IE] = [200, 20, 30, 0.2, 1000, 0.2, 12\%] \)
Table: 5.1.8.1

Effect of $M$ and $\beta$ on decision parameters with $Ic_2 = 20\%$, $N = 35/365$

<table>
<thead>
<tr>
<th>$\downarrow \beta$</th>
<th>$M \rightarrow$</th>
<th>15/365 $Ic_1 = 15%$</th>
<th>20/365 $Ic_1 = 16%$</th>
<th>25/365 $Ic_1 = 17%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>Q</td>
<td>77.81</td>
<td>78.19</td>
<td>78.55</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>154.56</td>
<td>154.64</td>
<td>154.71</td>
</tr>
<tr>
<td></td>
<td>TC</td>
<td>436.56</td>
<td>429.28</td>
<td>421.35</td>
</tr>
<tr>
<td>0.2</td>
<td>Q</td>
<td>99.82</td>
<td>100.37</td>
<td>100.86</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>251.10</td>
<td>251.37</td>
<td>251.62</td>
</tr>
<tr>
<td></td>
<td>TC</td>
<td>485.66</td>
<td>476.26</td>
<td>466.04</td>
</tr>
<tr>
<td>0.3</td>
<td>Q</td>
<td>133.29</td>
<td>134.00</td>
<td>134.60</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>433.95</td>
<td>434.64</td>
<td>435.23</td>
</tr>
<tr>
<td></td>
<td>TC</td>
<td>545.35</td>
<td>533.92</td>
<td>520.52</td>
</tr>
</tbody>
</table>

Observations:

- Increase in first credit period decreases total cost and demand rate is insensitive for fixed consumption rate.

- For fixed credit period, increase in stock dependent consumption rate; $\beta$ results increases in demand, procurement quantity and total cost of an inventory system significantly.
Table: 5.1.8.2

Effect of $N$ and $\beta$ on decision parameters with $Ic_1 = 16\%$, $M = 20/365$

<table>
<thead>
<tr>
<th>$N \rightarrow$</th>
<th>30/365 $Ic_2 = 18%$</th>
<th>35/365 $Ic_2 = 20%$</th>
<th>40/365 $Ic_2 = 22%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\downarrow \beta$</td>
<td>Q</td>
<td>R</td>
<td>TC</td>
</tr>
<tr>
<td>0.1</td>
<td>79.46</td>
<td>154.88</td>
<td>425.77</td>
</tr>
<tr>
<td>0.2</td>
<td>101.61</td>
<td>251.99</td>
<td>473.70</td>
</tr>
<tr>
<td>0.3</td>
<td>134.76</td>
<td>435.38</td>
<td>532.41</td>
</tr>
</tbody>
</table>

Observations:

- Increase in second allowable credit period decreases total cost and demand rate is insensitive for fixed consumption rate
- For fixed credit period, increase in stock dependent consumption rate; $\beta$ results increases in demand, procurement quantity and total cost of an inventory system significantly

5.2 Joint Pricing and Replenishment Policy for Stock-dependent demand Under Progressive Payment Scheme:

In this section, a mathematical model is developed to formulate optimal pricing and ordering policies for retailer when demand is dependent on selling price and stock, and the supplier offers two progressive credit periods to settle the account. The net profit is maximized with respect to be optimal selling price and order quantity. The effect of critical parameters viz. permissible credit...
periods and display factor on the objective function and decision variables are studied with the help of numerical example. A flowchart is given to explore the computational flow.

5.2.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model

- The demand rate, $R(P, Q(t))$, is deterministic and is a function of selling price and instantaneous stock-level $Q(t)$; the demand function $R(P, Q(t))$ is given by $R(P, Q(t)) = a - vP + bQ(t), 0 \leq t \leq T$.

where $a, b, v > 0$ and $a \gg b, a \gg v$. $a$ is fixed demand, $v$ is mark-up and $b$ is display factor or stock-dependent parameter.

- $GR = \text{the gross revenue}$.

- $NP(P, T) = \text{the net profit of a retailer per time unit}$.

$Q$ and $P$ are decision variables.

5.2.2 Mathematical Formulation:

The on-hand inventory $Q(t)$ depletes due to partially constant demand and partially stock-dependent demand. Hence, the instantaneous state of inventory $Q(t)$ at any instant of time $t$, $0 \leq t \leq T$ is governed by the differential equation

$$\frac{dQ(t)}{dt} = -R(P, Q(t)) = -a + vP - bQ(t), \quad 0 \leq t \leq T \quad (5.2.2.1)$$

with the initial condition $Q(0) = Q$ and boundary condition $Q(T) = 0$. The solution of differential equation (5.2.2.1) is given by

$$\frac{dQ(t)}{dt} = -R(P, Q(t)) = -a + vP - bQ(t), \quad 0 \leq t \leq T \quad (5.2.2.2)$$

and the procurement quantity is
Next follows computation of various cost components of the total cost of the system.

- Ordering cost;

\[ OC = \frac{A}{T} \]  

(5.2.2.4)

- Inventory holding cost;

\[ IHC = h \int_0^T Q(t) \, dt = \frac{h(a - vP)}{b^2 T} \left[ e^{bT} - bT - 1 \right] \]  

(5.2.2.5)

- Gross revenue;

\[ GR = (P - C) R(P, Q(t)) \]  

(5.2.2.6)

Regarding interest charged and interest earned, based on the length of the cycle time \( T \), three cases may arise:

**Case 1**: \( T \leq M \)

**Case 2**: \( M < T < N \)

**Case 3**: \( T \geq N \)

We discuss each case in detail.

**Case 1**: \( T \leq M \) (Fig. 5.2.2.1)
Here, the retailer sells \( Q \)-units during \([0, T]\) and is paying for \( CQ \)-units in full to the supplier at time \( M \geq T \). Therefore, interest charges are zero, i.e.

\[
IC_1 = 0 \tag{5.2.2.7}
\]

The retailer sells \( Q \)-units during \([0, T]\) and deposits the revenue in an interest bearing account at the rate of \( le/\$\)/year. In the period \([T, M]\), the retailer deposits revenue into the account that earns interest at the rate \( le/\$\)/year. Therefore, interest earned per year is

\[
IE_1 = \frac{P le}{T} \left[ \int_0^T R(P, Q(t)) dt + Q(M - T) \right]
\]

\[
= \frac{P le}{bT} \left( \frac{1}{b} (e^{bT} - bT - 1) + (e^{bT} - 1)(M - T) \right) \tag{5.2.2.8}
\]

The net profit; \( NP_1 \) is given by

\[
NP_1(P, T) = GR - OC - IHC - IC_1 + IE_1 \tag{5.2.2.9}
\]

\( P \) and \( T \) are continuous variables. Hence optimal values of \( P \) and \( T \) can be obtained by setting

\[
\frac{\partial NP_1(P, T)}{\partial P} = \frac{(e^{bT} - 1)}{bT} (a - 2vP - vC) - \frac{hv(bT + 1 - e^{bT})}{b^2T} + \frac{IE_1}{P} + \frac{P le v}{T} \left( \frac{(bT + 1 - e^{bT})}{b} - (e^{bT} - 1)(M - T) \right) = 0 \tag{5.2.2.10}
\]

\[
\frac{\partial NP_1(P, T)}{\partial T} = \frac{(P - C)(a - vP)}{T} \left[ e^{bT} - \frac{e^{bT} - 1}{bT} \right] + \frac{A}{T^2} \frac{h(a - vP)(-e^{bT} + 1 + bT)}{T^2b^2} + \frac{h(a - vP)(1 - e^{bT})}{bT} + \frac{IE_1}{T} + \frac{P le (a - vP)}{T}
\]

\[
\times \left[ \frac{(1 - e^{bT})}{b} - e^{bT}(M - T) - \frac{(e^{bT} - 1)}{b} \right] = 0 \tag{5.2.2.11}
\]

The obtained \( P = P_i \) and \( T = T_1 \), maximizes net profit; provided
where

\[
X = \frac{\partial^2 N P_1(P,T)}{\partial P^2} = 2v \frac{bT}{bT^2} \left[ -e^{bT-1} + \frac{Ie}{b} (bT+1-e^{bT}) - (e^{bT}-1)(M-T) \right]
\]

\[
Y = \frac{\partial^2 N P_1(P,T)}{\partial T^2} = \frac{(P-C)(a-vP)}{T^2} \left[ e^{bT}(bT-2)+\frac{2(e^{bT}-1)}{bT} \right] \frac{2A}{T^3} \frac{h(a-vP)e^{bT}}{T} + \frac{2h(a-vP)(-e^{bT}+1+bT)}{bT^3} \frac{2h(a-vP)(1-e^{bT})}{bT^2}
\]

\[
+ \frac{2Ple(a-vP)e^{bT}(b(M-T)-1)}{T} - \frac{Ple(a-vP)e^{bT}(M-T)}{T^2} \frac{2Ie_1}{T^2}
\]

\[
Z = \frac{\partial^2 N P_1(P,T)}{\partial T \partial P} = (a-2vP-vC) \left[ e^{bT} \frac{bT}{T} - (e^{bT}-1) \frac{bT}{bT^2} - \frac{hv}{bT} \left( (1-e^{bT}) + \frac{(bT+1-e^{bT})}{bT} \right) \right]
\]

\[
+ \frac{Ie}{h} \left[ (a-vP)e^{bT}(M-T) \right] - \frac{Ie_1}{PT} - \frac{Ple}{T} (v(1-e^{bT})(M-T))
\]

\[
- \frac{Ple}{T^2} \left( \frac{v(bT+1-e^{bT})}{b^2} - \frac{v(e^{bT}-1)(M-T)}{b} \right)
\]

Case 2: \( M < T < N \) (Fig. 5.2.2.2)

\[ XY - Z^2 < 0 \] (5.2.2.12)
The retailer sells units and deposits the revenue into an interest bearing account at an interest rate $i$ per unit/year during $[0, M]$. Therefore, the interest earned during $[0, M]$ is given by

$$IE_2 = P\int_0^M R(P, Q(t)) t \, dt = \frac{P}{b^2} \left[ e^{bT} - (bM + 1)e^{b(T - M)} \right]$$  \hspace{1cm} (5.2.13)

Buyer has to pay for $Q = R(P)T$ units purchased initially at the rate of $C$ \$/unit to the supplier during $[0, M]$. The retailer sells $R(P)M$ units at sale price $sP$/unit. So he has generated revenue of $P R(P, Q(t)) M$ plus the interest earned, $IE_2$, during $[0, M]$. Two sub-cases may arise:

**Sub-case 2.1:** Let $P R(P, Q(t)) M + IE_2 \geq CQ$, i.e. the retailer has enough money to pay for all $Q$–units procured at time $t = 0$. Then, interest charges,

$$IC_{2.1} = 0$$  \hspace{1cm} (5.2.2.14)

and the interest earned, $IE_{2.1}$, per time unit is

$$IE_{2.1} = \frac{IE_2}{T}$$  \hspace{1cm} (5.2.2.15)

Using (5.1.2.4) – (5.1.2.6), (5.1.2.14) and (5.1.2.15), the net profit; $NP_{2.1}(P, T)$ is given by

$$NP_{2.1}(P, T) = GR - OC - IHC - IC_{2.1} + IE_{2.1}$$  \hspace{1cm} (5.2.2.16)

The optimum values of $P = P_{2.1}$ and $T = T_{2.1}$ are solutions of

$$\frac{\partial NP_{2.1}(P, T)}{\partial P} = \frac{(e^{bT} - 1)}{bT} (a - 2vP - vC) - \frac{h}{b^2 T} (bT + 1 - e^{bT}) + \frac{IE_1}{P} + \frac{P}{T} \left[ \frac{(bT + 1 - e^{bT})}{b} - (e^{bT} - 1) (M - T) \right] = 0$$  \hspace{1cm} (5.2.2.17)
\[
\frac{\partial NP_{2,1}(P,T)}{\partial T} = \frac{(P-C)(a-vP)}{T} \left[ e^{bT} - \frac{e^{bT} - 1}{bT} \right] + \frac{A}{T^2} \frac{h(a-vP)(-e^{bT} + 1 + bT)}{T^2b^2} \\
+ \frac{h(a-vP)(1-e^{bT})}{bT} + \frac{P\ell (a-vP)}{T} \left[ \frac{1-e^{bT}}{b} - e^{bT}(M-T) - \frac{(e^{bT} - 1)}{b} \right] \\
+ \frac{IE_1}{T} = 0
\]

(5.2.2.18)

The obtained \( P = P_{2,1} \) and \( T = T_{2,1} \), maximizes net profit; provided

\[ XY - Z^2 < 0 \]

(5.2.2.19)

where \( X, Y \) and \( Z \) are same as defined earlier. \( NP_1(P,T) \) is replaced by \( NP_{2,1}(P,T) \)

**Sub-case 2.2:** \( PR(P,Q(M))M + IE_2 < CQ \)

Here, retailer will have to pay interest on the un-paid balance \( U_1 = CQ - [PR(P,Q(M))M + IE_2] \) at rate of \( Ic_1 \) at time \( M \) to the supplier. The interest to be paid, \( IC_{2,2} \), per time unit is:

\[
IC_{2,2} = \frac{U_1^2}{PQ} \int_0^T Q(t)dt = \frac{1}{Pb^2} \int_0^T Ic_1 (a-vP)^2 \left( e^{bT} - 1 \right) \left( e^{b(T-M)} - b(T-M) - 1 \right)
\]

(5.2.2.20)

and interest earned per time unit is:

\[
IE_{2,2} = \frac{IE_2}{T}
\]

(5.2.2.21)

Using (5.1.2.4) – (5.1.2.6), (5.1.2.20) and (5.1.2.21), the net profit; \( NP_{2,2}(P,T) \) is given by

\[
NP_{2,2}(P,T) = GR - OC - IHC - IC_{2,2} + IE_{2,2}
\]

(5.2.2.22)

The non-linear equation
\[
\frac{\partial N_{P_2}(P,T)}{\partial P} = \frac{(e^{bT} - 1)(a - 2vP - vC)}{bT} + \frac{h(-e^{bT} + 1 + bT)}{b^2T} - \frac{2U_1Ic_1(a - vP)\%1\%2}{pb^3} \\
- \frac{U_1^2Ic_1(a - vP)\%1}{p^2b^3} + \frac{U_1^2Ic_1(v + 1)\%1}{pb^3} \\
+ \frac{(a - 2vP)e(e^{bT} - e^{b(T - M)}(bM + 1))}{b^2T} = 0
\]

where

\%
1 = \left(\frac{e^{bT} - 1}{b}\right)(-a + vP)(e^{b(T - M)} + b(M - T) + 1)

\%
2 = \left\{ \frac{Cv(e^{bT} - 1)}{b} - e^{b(T - M)}(a - vP)M + Pe^{b(T - M)}vM \right\}

\[
\frac{\partial N_{P_2}(P,T)}{\partial T} = \frac{(P - C)(a - vP)}{T} \left[ e^{bT} - \frac{(e^{bT} - 1)}{bT} \right] + \frac{h(a - vP)}{bT} \\
- \frac{e^{bT}}{bT} \left[ 1 - e^{bT} - \frac{(-e^{bT} + 1 + bT)}{bT} \right] + \frac{A}{T^2} - \frac{2U_1Ic_1(a - vP)(1 - e^{bT})\%1\%2}{pb^3} \\
- \frac{U_1^2Ic_1(a - vP)e^{bT}\%1}{p^2b^2} + \frac{U_1^2Ic_1(a - vP)^2(e^{bT} - 1)(e^{b(T - M)} - 1)}{pb^2} \\
+ \frac{Ple(-a + vP)}{bT} \left\{ \left(\%3 + e^{b(T - M)}M \right) - \frac{(\%3 + e^{b(T - M)}bM)}{bT} \right\}
\]

where

\%
1 = (-a + vP)(-bM - e^{b(T - M)} + bT + 1)

\%
2 = C(a - vP)e^{bT} + Pbe^{b(T - M)}(-a + vP)M \\
- \frac{Ple(-a + vP)(-e^{bT} + e^{b(T - M)}bM + e^{b(T - M)})}{b}
\[ %3 = -e^{bT} + e^{b(T - M)} \]

can be solved by suitable numerical method for \( P = P_{22} \) and \( T = T_{22} \) which maximizes \( NP_{22}(P, T) \) provided

\[
EF - G^2 < 0
\] (5.2.2.25)

\[
E = \frac{\partial^2 NP_{22}(P, T)}{\partial P^2} = \frac{-2\nu(e^{bT} - 1)}{bT} + \frac{2\%2 I_{c1}(a - vP) \%1}{Pb^3} + \frac{4U_1 I_{c1}(a - vP) \%1 \%2}{P^2 b^3}
\]

\[
+ \frac{4U_1 I_{c1} \%1 \%2}{Pb^3} - \frac{4U_1(a - vP) \%3 \%2}{Pb^3} - \frac{2U_1 I_{c1}(a - vP) \%1}{Pb^3}
\]

\[
\times \left\{ 2e^{b(T - M)\nu M} \%4 \right\} - \frac{2U_1^2 I_{c1}(a - vP) \%1}{P^3 b^3} + \frac{2U_1^2(a - vP) \%3}{P^2 b^3}
\]

\[
+ \frac{2U_1^2 \nu \%3}{Pb^3} + \frac{2\nu(e^{bT} + e^{b(T - M)\nu M} + e^{b(T - M)})}{b^2 T}
\]

where

\[ \%1 = (e^{bT} - 1)(-a + vP)(a^{b(T - M)} + b(M - T) + 1) \]

\[ \%2 = \left\{ -\frac{C\nu(e^{bT} - 1)}{b} - e^{b(T - M)}(a - vP)M + Pe^{b(T - M)\nu M} \right\} \]

\[ \left\{ -\frac{Ie\left(-a + 2vP\right)(-e^{bT} + e^{b(T - M)\nu M} + e^{b(T - M)})}{b^2} \right\} \]

\[ \%3 = I_{c1}(e^{bT} - 1)\nu(-bM - e^{b(T - M)} + bT + 1) \]
\[
F = \frac{\partial^2 NP_{2,2}(P,T)}{\partial T^2} = \frac{(P-C)(a-vP)e^bT}{T} - \frac{2(P-C)(a-vP)e^bT}{T^2} - \frac{2A}{T^3}
\]

\[
+ \frac{2(P-C)(a-vP)(e^bT-1)}{bT^3} - \frac{h(a-vP)e^bT}{T} - \frac{2h(a-vP)(1-e^bT)}{b^2T^2}
\]

\[
+ \frac{2h(a-vP)(bT+1-e^bT)}{b^2T^3} + \frac{Ic_1}{Pb} \left\{ \frac{2\%2^2(e^bT-1)^%1}{b} \right\}
\]

\[
\frac{4U_1e^bT%1%2}{b} + \frac{4U_1(a-vP)^2(e^bT-1)(1-e^b(T-M))}{b} \frac{%2}{b}
\]

\[
+ \frac{2U_1(e^bT-1)^%1%2}{b^2} + U_1^2e^bT%1 + 2U_1^2(a-vP)^2 \left\{ e^bT(1-e^b(T-M)) - U_1^2(a-vP)^2(e^bT-1)e^b(T-M) \right\}
\]

\[
+ \frac{2%3-T%3}{bT^2} - \frac{2%3}{b^2T^2}
\]

where

\[
%1 = (a-vP)^2 (-bM - e^b(T-M)+bT+1)
\]

\[
%2 = C(a-vP)e^bT + Pbe^b(T-M)(-a+vP)M
\]

\[
\frac{Pla(-a+vP)(-e^bT+e^b(T-M)bM+e^b(T-M))}{b}
\]

\[
%3 = Pla(a-vP)(-e^bT+e^b(T-M)bM+e^b(T-M))
\]
\[
G = \frac{\partial^2 NP_{2,2}(P,T)}{\partial P \partial T} = \frac{(a-2vP+vQ)}{T} \left[ e^{bT} \left( \frac{(e^{bT}-1)}{bT} \right) - \frac{hv}{bT} \left( \frac{1-e^{bT}}{bT} \right) \right] \\
\frac{2U_1 L_1 e^{bT} \%1}{Pb^2} + \frac{2U_1 L_1 (a-vP) \%3}{Pb^3} + \frac{2U_1 L_1 (a-vP) \%3}{Pb^3} \\
\times \{ \%4 + C e^{bT} - e^{b(T-M)}(-a+vP)M - P e^{b(T-M)}vM \\
+ \frac{I e \%5(-a+2vP)}{b} \}\ + \frac{2U_1 L_1 (a-vP) \%3}{Pb^3} + \frac{U_2 L_1 (a-vP)e^{bT} \%1}{Pb^3} \\
\frac{U_2 L_1 (a-vP)^2 (e^{bT}-1)}{Pb^2} + \frac{U_2 L_1 (a-vP)(e^{bT}-1)}{Pb^2} + \frac{I e(-a+2vP) \%5}{b^2 T^2} \\
\times(bT-1) - \frac{U_2 L_1 e^{bT} \%1}{Pb^2} \left( \frac{2}{b} \left( e^{bT} - 1 \right) + U_1 e^{bT} \right) \\

\text{where} \\
\%1 = (a-vP) (-bM - e^{b(T-M)} + bT + 1) \\
\%2 = C(a-vP)e^{bT} + P e^{b(T-M)}(-a+vP)M \\
\frac{P e(-a+vP)(-e^{bT} + e^{b(T-M)})_bM + e^{b(T-M)}}{b} \\
\%3 = (e^{bT} - 1) \%1 \\
\%4 = \left\{ \frac{-Ce^{bT-1}}{b} - e^{b(T-M)(a-vP)M} + Pe^{b(T-M)}e^{b(T-M)}vM - \frac{I e(-a+2vP) \%5}{b^2} \right\} \\
\%5 = (-e^{bT} + e^{b(T-M)}bM + e^{b(T-M)} \right)
Case 3: \( T \geq N \) (Fig. 5.2.2.3)

Based on the total purchase cost, \( CQ \), total money \( PR(P,Q(M))M + IE_2 \) in account at \( M \) and total money in account at \( N \) is \( PR(P,Q(N))N + IE_2 \), three sub-cases may arise:

Sub-case 3.1: Let \( PR(P,Q(M))M + IE_2 \geq CQ \) Then this sub-case is same as sub-case 2.1. (Note: Decision variables and objective function are designated by subscript 3.1)

Sub-case 3.2: Let \( PR(P,Q(M))M + IE_2 < CQ \) but

\[
PR(P,Q(N-M))(N-M) + Pte \int_{M}^{N} R(P,Q(t))dt \geq CQ - (PR(P,Q(M))M + IE_2)
\]

This sub-case coincides with sub-case 3.2. (Note: Decision variables and objective function are designated by subscript 3.2)

Sub-case 3.3: Let \( PR(P,Q(M))M + IE_2 < CQ \) but

\[
PR(P,Q(N-M))(N-M) + Pte \int_{M}^{N} R(P,Q(t))dt < CQ - (PR(P,Q(M))M + IE_2)
\]
Here, the retailer does not have sufficient money in his account to pay off for total purchase cost at \( N \). He will do payment of \( PR(P, Q(M))M + IE_2 \) at \( M \) and \( PR(P, Q(N-M))(N-M) + Pte \int_{M}^{N} R(P, Q(t))dt \) at \( N \). So, he has to pay interest charges on un-paid balance \( U_1 = CQ - [PR(P, Q(M))M + IE_2] \) with interest rate \( Ic_1 \) during \([M, N] \) and \( U_2 = U_1 - \left[ PR(P, Q(N-M))(N-M) + Pte \int_{M}^{N} R(P, Q(t))dt \right] \) with interest rate \( Ic_2 \) during \([N, T]\).

Therefore, total interest charges, \( IC_3 \), per time unit is given by

\[
IC_{3.3} = \frac{U_1 Ic_1 (N-M)}{T} + \frac{U_2^2}{PQ} \int_{N}^{T} Q(t)dt \tag{5.2.2.26}
\]

and total interest earned per time unit is;

\[
IE_{3.3} = \frac{IE_2}{T} \tag{5.2.2.27}
\]

Using (5.1.2.4) – (5.1.2.6), (5.1.2.26) and (5.1.2.27), the net profit; \( NP_{3.3}(P, T) \) is given by

\[
NP_{3.3}(P, T) = GR - OC - IHC - IC_{3.3} + IE_{3.3} \tag{5.2.2.28}
\]

The optimum value of \( P = P_{3.3} \) and \( T = T_{3.3} \) is the solution of non-linear equation

\[
\frac{\partial NP_{3.3}(P, T)}{\partial P} = \frac{(e^{bT} - 1)(a - 2vP - vC)}{bT} - \frac{h(e^{bT} + 1 + bT)}{b^2T} - \frac{2Ic_1(N-M)\%2}{T} \]

\[
- \frac{2U_2^2}{bP} \left\{ \%2 - e^{b(T-N+M)}(N-M)(a-2vP) + \frac{Ie(a-2vP)\%4}{b^2} \right\} \]

\[
+ \frac{U_2^2}{bP^2} \left( \%2 - \frac{Ie(a-2vP)\%3}{b^2T} \right) = 0 \tag{5.2.2.29}
\]
\[ \frac{\partial N_{P,3}(P, T)}{\partial T} = \frac{(P-C)(a-vP)}{T} \left( e^{bT} \left( e^{bT} - 1 \right) \right) + h(a-vP) \left( 1 - e^{bT} \right) \left( -e^{bT} + 1 + bT \right) \]

\[ + \frac{A}{T^2} - \frac{2Ic_1(N-M)}{T} - \frac{U_1lc_1(N-M)}{T^2} - \frac{2U_2lc_2(bN + e^{b(T-N)} - bT - 1)}{bP(e^{bT} - 1)} \]

\[ \left\{ \%2 - Pe^{b(T-N+M)}(a-vP)(N-M) + \frac{Ple(a-vP)\%1}{b} \right\} \]

\[ + \frac{U_2lc_2(e^{b(T-N)} - 1)}{P(e^{bT} - 1)} + \frac{U_2lc_2(bN + e^{b(T-N)} - bT - 1)e^{bT}}{P(e^{bT} - 1)^2} \]

\[ - \frac{Ie(a-vP)\%3}{bT} \left[ 1 - \frac{1}{bT} \right] = 0 \]  

(5.2.2.30)

where

\[ \%1 = (-e^{bT} + e^{b(T-N+M)})bN - e^{b(T-N+M)}bM + e^{b(T-N+M)} \]

\[ \%2 = C(a-vP)e^{bT} + Pe^{b(T-M)}(-a+vP)M \]

\[ - \frac{Ple(-a+vP)(-e^{bT} + e^{b(T-M)})bM + e^{b(T-M)}}{b} \]

\[ \%3 = (-e^{bT} + e^{b(T-M)})bM + e^{b(T-M)} \]

The obtained \( P = P_3 \) and \( T = T_3 \) maximizes the net profit \( NP_{3,3} \) provided

\[ BK - J^2 < 0 \]  

(5.2.2.31)

\[ B = \frac{\partial^2 N_{P,3}(P, T)}{\partial P^2} = -2v(e^{bT} - 1) - \frac{Ic_1(N-M)}{T} \left( 2e^{b(T-M)} - \frac{2lev\%3}{b^2} \right) \]

\[ - \frac{2%1^2lc_2\%1}{bP(e^{bT} - 1)} + \frac{4U_2lc_2\%1\%4}{bP^2(e^{bT} - 1)} - \frac{2U_2lc_2\%1}{bP(e^{bT} - 1)} \]

\[ \left\{ 2e^{b(T-M)vM} - \frac{2lev\%3}{b^2} + 2e^{b(T-N+M)v(N-M)} - \frac{2lev\%5}{b^2} \right\} \]

\[ + \frac{2U_2^2lc_2\%1}{bP^3(e^{bT} - 1)} + \frac{2lev\%3}{b^2} \]
where

\[ \%1 = (bN + e^{b(T-N)} - bT - 1) \]

\[ \%2 = \left\{ \frac{Cv(e^{bT-1})}{b} - e^{b(T-M)}(a - vP)M + Pe^{b(T-M)vM} - \frac{I(e - a + 2vP) \%3}{b^2} \right\} \]

\[ \%3 = (-e^{bT} + e^{b(T-M)}bM + e^{b(T-M)}) \]

\[ \%4 = e^{bT - N + M}(a - vP)(N - M) + Pe^{b(T - N + M)v(N - M)} + \frac{I(e - a + 2vP)\%5}{b^2} \]

\[ \%5 = (-e^{bT} + e^{b(T - N + M)}bN - e^{b(T - N + M)}bM + e^{b(T - N + M)}) \]

\[ K = \frac{\partial^2 N P_{\%3}(P, T)}{\partial T^2} = \frac{2\%2 Ic_1(N - M)}{T^2} - \frac{2A}{T^3} - \frac{\%2 Ic_1(N - M)}{T} + \frac{4U_2 Ic_2 \%1 e^{bT}}{P(e^{bT-1})^2} \]

\[ - \frac{2U_2 Ic_2 \%1 e^{bT}}{P(e^{bT-1})} - \frac{2U_2 Ic_2 \%3 e^{bT - N} - 1}{P(e^{bT-1})} - \frac{2U_2 Ic_2 \%3 e^{bT}}{P(e^{bT-1})} \]

\[ \frac{U_2^2 Ic_2 e^{bT}(e^{b(T-N)-1})}{P(e^{bT-1})^2} + \frac{h(-a + vP)e^{bT}}{T} \]

\[ - \frac{2U_2^2 Ic_2 \%1 e^{2bT}}{P(e^{bT-1})^3} + \frac{U_2^2 Ic_2 e^{bT} \%1}{P(e^{bT-1})^2} - \frac{e^{bT}(P - C)(a - vP)}{T} \left[ \frac{2}{T} - 1 \right] \]

\[ + \frac{2(e^{bT-1})(P - C)(a - vP)}{bT^3} + \frac{2h(-a + vP)(1 - e^{bT})}{bT^2} \]

\[ - \frac{2h(-a + vP)(bT + 1 - e^{bT})}{b^2 T^3} - \frac{2U_1 Ic_1(N - M)}{T^3} \]

\[ + \frac{Pe(-a + vP)\%5}{T} \left[ 1 - \frac{2}{bT} + \frac{2}{b^2 T^2} \right] \]

where

\[ \%1 = (bN + e^{b(T-N)} - bT - 1) \]
\[
\%2 = C(a - vP)e^{bT} + Pbe^{b(T-M)}(-a + vP)M
\]

\[
Ple(-a + vP)(-e^{bT} + e^{b(T-M)})bM + e^{b(T-M)}
\]

\[
\%3 = \%2 + Pb(-a + vP)(N - M) - \frac{Ple(-a + vP)\%4}{b}
\]

\[
\%4 = (-e^{bT} + e^{b(T - N + M)})bN - e^{b(T - N + M)}bM + e^{b(T - N + M)}
\]

\[
\%5 = (-e^{bT} + e^{b(T - M)})bM + e^{b(T - M)}
\]

\[
J = \frac{\partial^2 N_{3,3}(P,T)}{\partial P \partial T} = \frac{(a - vP)(e^{bT-1})}{bT^2} + \frac{(a - vP) bT}{T} + \frac{U_2^2 Ic_2 \%1 v e^{bT}}{P(a - vP)(e^{bT-1})^2}
\]

\[
+ \frac{U_2^2 Ic_2 (e^{b(T - N - 1)v})}{P(a - vP)(e^{bT-1})^2} + \frac{U_2^2 Ic_2 \%1 v}{bP(e^{bT-1})} + \frac{2U_2 Ic_2 \%1 \%2}{bP(a - vP)(e^{bT-1})}
\]

\[
+ \frac{2U_2 Ic_2 \%1 \%5}{bP(e^{bT-1})} + \frac{2U_2 Ic_2 \%6 e^{bT}}{P(e^{bT-1})^2} + \frac{2U_2 Ic_2 \%1 \%2}{bP(a - vP)(e^{bT-1})}
\]

\[
+ (P - C)v(e^{bT-1}) - \frac{(P - C)e^{bT}}{bT^2} + hv(bT + 1 - e^{bT})
\]

\[
- \left\{ -\frac{Cv(e^{bT-1})}{b} - e^{b(T - M)}(a - vP)M + Pe^{b(T - M)}vM
\]

\[
\frac{le(-a + 2vP) \%3}{b^2} \right\} \frac{Ic_1(N - M)}{T^2} - \left\{ -Cv e^{bT}
\]

\[
+ be^{b(T - M)}(-a + vP)M + Pbe^{b(T - M)}vM + \frac{le(a - 2vP) \%3}{b}
\]

\[
\times \frac{Ic_1(N - M)}{T^2} + \frac{le(-a + 2vP) \%3}{bT} + \frac{le(-a + 2vP) \%3}{b^2 T^2}
\]

\[
+ \frac{U_2^2 Ic_2 (e^{b(T - N - 1)v})}{P^2(e^{bT-1})} + \frac{U_2^2 Ic_2 \%1 e^{bT}}{P(e^{bT-1})^2} + \frac{2U_2 Ic_2 \%1 \%2}{bP(a - vP)(e^{bT-1})}
\]

\[
+ \frac{U_2^2 Ic_2 v(e^{b(T - N - 1)v})}{P(a - vP)(e^{bT-1})} - \frac{2U_2 Ic_2 \%1 \%6}{P(e^{bT-1})} + \frac{2\%2 Ic_2 \%1 \%6}{bP(e^{bT-1})}
\]
where

\[ %1 = (bN + e^{b(T - N)} - bT - 1) \]

\[ %2 = C(a - vP)e^{bT} + Pbe^{b(T - M)}(-a + vP)M - \frac{Pe(-a + vP)(%3 + %4)}{b} + Pe^{b(T - N + M)}(-a + vP)(N - M) \]

\[ %3 = (-e^{bT} + e^{b(T - M)}bM + e^{b(T - M)}) \]

\[ %4 = (-e^{bT} + e^{b(T - N + M)}bN - e^{b(T - N + M)}bM + e^{b(T - N + M)}) \]

\[ %5 = \left\{ -Cve^{bT} + bMe^{b(T - M)}(-a + vP) + Pbe^{b(T - M)}vM + \frac{Ie(a - 2vP)(%3 + %4)}{b} + be^{b(T - N + M)}(a - vP)(N - M) + Pbe^{b(T - N + M)}v(N - M) \right\} \]

\[ %6 = \left\{ -C(e^{bT} - 1) - e^{b(T - M)}(a - vP)M + Pe^{b(T - M)}vM + e^{b(T - N + M)}(-a + vP)(N - M) + Pe^{b(T - N + M)}v(N - M) - \frac{Ie(-a + 2vP)(%3 + %4)}{b^2} \right\} \]

In the next section, computational flowchart is given to search for the optimal solution.
5.2.3 Flowchart:

The following notations are used in flowchart

\[ R_1 = R(P, Q(M)) \]

\[ R_2 = R(P, Q(N - M)) \]

\[ R_3 = R(P, Q(t)) \]

5.2.4 Numerical Example and observations:

Consider following parametric values in appropriate units.

\[ [A, C, h, Ie, a, v] = [200, 20, 0.2, 0.12, 1000, 22] \]
## Table 5.2.4.1

**Effect of allowable two progressive credit periods on decision variables**

*when \( b = 3.8 \)*

<table>
<thead>
<tr>
<th>M</th>
<th>15/365</th>
<th>17/365</th>
<th>19/365</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Ic_1 = 13% )</td>
<td>( Ic_1 = 13.5% )</td>
<td>( Ic_1 = 14% )</td>
</tr>
<tr>
<td>28/365</td>
<td>( T = 0.4540 )</td>
<td>( T = 0.4518 )</td>
<td>( T = 0.4479 )</td>
</tr>
<tr>
<td></td>
<td>( P = 37.22 )</td>
<td>( P = 37.19 )</td>
<td>( P = 37.15 )</td>
</tr>
<tr>
<td></td>
<td>( Q = 219.95 )</td>
<td>( Q = 218.52 )</td>
<td>( Q = 215.64 )</td>
</tr>
<tr>
<td></td>
<td>( R = 1016.97 )</td>
<td>( R = 1012.18 )</td>
<td>( R = 1002.11 )</td>
</tr>
<tr>
<td></td>
<td>( NP = 6329.02 )</td>
<td>( NP = 6314.07 )</td>
<td>( NP = 6277.81 )</td>
</tr>
<tr>
<td>30/365</td>
<td>( T = 0.4645 )</td>
<td>( T = 0.4631 )</td>
<td>( T = 0.4598 )</td>
</tr>
<tr>
<td></td>
<td>( P = 37.34 )</td>
<td>( P = 37.32 )</td>
<td>( P = 37.28 )</td>
</tr>
<tr>
<td></td>
<td>( Q = 227.48 )</td>
<td>( Q = 226.58 )</td>
<td>( Q = 224.27 )</td>
</tr>
<tr>
<td></td>
<td>( R = 1042.94 )</td>
<td>( R = 1039.96 )</td>
<td>( R = 1032.05 )</td>
</tr>
<tr>
<td></td>
<td>( NP = 6461.76 )</td>
<td>( NP = 6457.18 )</td>
<td>( NP = 6429.40 )</td>
</tr>
<tr>
<td>32/365</td>
<td>( T = 0.4776 )</td>
<td>( T = 0.4775 )</td>
<td>( T = 0.4749 )</td>
</tr>
<tr>
<td></td>
<td>( P = 37.49 )</td>
<td>( P = 37.48 )</td>
<td>( P = 37.45 )</td>
</tr>
<tr>
<td></td>
<td>( Q = 237.12 )</td>
<td>( Q = 237.02 )</td>
<td>( Q = 235.31 )</td>
</tr>
<tr>
<td></td>
<td>( R = 1075.91 )</td>
<td>( R = 1074.85 )</td>
<td>( R = 1070.28 )</td>
</tr>
<tr>
<td></td>
<td>( NP = 6622.92 )</td>
<td>( NP = 6620.32 )</td>
<td>( NP = 6612.31 )</td>
</tr>
</tbody>
</table>

## Table 5.2.4.2

**Effect of allowable two progressive credit periods on decision variables**

*when \( b = 4.0 \)*

<table>
<thead>
<tr>
<th>M</th>
<th>15/365</th>
<th>17/365</th>
<th>19/365</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Ic_1 = 13% )</td>
<td>( Ic_1 = 13.5% )</td>
<td>( Ic_1 = 14% )</td>
</tr>
<tr>
<td>28/365</td>
<td>( T = 0.4780 )</td>
<td>( T = 0.4755 )</td>
<td>( T = 0.4704 )</td>
</tr>
<tr>
<td></td>
<td>( P = 37.71 )</td>
<td>( P = 37.68 )</td>
<td>( P = 37.62 )</td>
</tr>
<tr>
<td></td>
<td>( Q = 245.63 )</td>
<td>( Q = 243.70 )</td>
<td>( Q = 239.75 )</td>
</tr>
<tr>
<td></td>
<td>( R = 1152.90 )</td>
<td>( R = 1145.85 )</td>
<td>( R = 1131.37 )</td>
</tr>
<tr>
<td></td>
<td>( NP = 6811.17 )</td>
<td>( NP = 6790.74 )</td>
<td>( NP = 6741.50 )</td>
</tr>
<tr>
<td>30/365</td>
<td>( T = 0.4913 )</td>
<td>( T = 0.4905 )</td>
<td>( T = 0.4861 )</td>
</tr>
<tr>
<td></td>
<td>( P = 37.87 )</td>
<td>( P = 37.85 )</td>
<td>( P = 37.81 )</td>
</tr>
<tr>
<td></td>
<td>( Q = 255.98 )</td>
<td>( Q = 255.70 )</td>
<td>( Q = 251.83 )</td>
</tr>
<tr>
<td></td>
<td>( R = 1190.77 )</td>
<td>( R = 1190.10 )</td>
<td>( R = 1175.48 )</td>
</tr>
<tr>
<td></td>
<td>( NP = 6983.70 )</td>
<td>( NP = 6975.64 )</td>
<td>( NP = 6935.64 )</td>
</tr>
<tr>
<td>32/365</td>
<td>( T = 0.5094 )</td>
<td>( T = 0.5087 )</td>
<td>( T = 0.5062 )</td>
</tr>
<tr>
<td></td>
<td>( P = 38.10 )</td>
<td>( P = 38.08 )</td>
<td>( P = 38.02 )</td>
</tr>
<tr>
<td></td>
<td>( Q = 269.89 )</td>
<td>( Q = 269.75 )</td>
<td>( Q = 268.84 )</td>
</tr>
<tr>
<td></td>
<td>( R = 1241.36 )</td>
<td>( R = 1241.25 )</td>
<td>( R = 1238.90 )</td>
</tr>
<tr>
<td></td>
<td>( NP = 7193.61 )</td>
<td>( NP = 7188.49 )</td>
<td>( NP = 7173.45 )</td>
</tr>
</tbody>
</table>
**Table 5.2.4.3**

**Effect of allowable two progressive credit periods on decision variables when \( b = 4.2 \)**

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M \rightarrow )</th>
<th>15/365</th>
<th>17/365</th>
<th>19/365</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Ic_1 = 13% )</td>
<td>( Ic_1 = 13.5% )</td>
<td>( Ic_1 = 14% )</td>
<td></td>
</tr>
<tr>
<td>28/365</td>
<td>( Ic_2 = 16% )</td>
<td>( T = 0.5063 )</td>
<td>( T = 0.5027 )</td>
<td>( T = 0.4966 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( P = 38.28 )</td>
<td>( P = 38.23 )</td>
<td>( P = 38.16 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Q = 242.90 )</td>
<td>( Q = 240.70 )</td>
<td>( Q = 236.50 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R = 1080.87 )</td>
<td>( R = 1073.62 )</td>
<td>( R = 1059.18 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( NP = 7375.52 )</td>
<td>( NP = 7344.92 )</td>
<td>( NP = 7277.60 )</td>
</tr>
<tr>
<td>30/365</td>
<td>( Ic_2 = 18% )</td>
<td>( T = 0.5252 )</td>
<td>( T = 0.5232 )</td>
<td>( T = 0.5175 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( P = 38.53 )</td>
<td>( P = 38.50 )</td>
<td>( P = 38.43 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Q = 254.88 )</td>
<td>( Q = 253.74 )</td>
<td>( Q = 249.93 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R = 1120.88 )</td>
<td>( R = 1117.22 )</td>
<td>( R = 1104.28 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( NP = 7603.22 )</td>
<td>( NP = 7589.92 )</td>
<td>( NP = 7534.66 )</td>
</tr>
<tr>
<td>32/365</td>
<td>( Ic_2 = 20% )</td>
<td>( T = 0.5497 )</td>
<td>( T = 0.5494 )</td>
<td>( T = 0.5456 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( P = 38.85 )</td>
<td>( P = 38.84 )</td>
<td>( P = 38.80 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Q = 270.55 )</td>
<td>( Q = 270.46 )</td>
<td>( Q = 267.79 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R = 1173.40 )</td>
<td>( R = 1173.13 )</td>
<td>( R = 1164.00 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( NP = 7885.91 )</td>
<td>( NP = 7874.48 )</td>
<td>( NP = 7854.35 )</td>
</tr>
</tbody>
</table>

**Observations:**

- Increase in first allowable credit period decreases cycle time, selling price and net profit.
- Increase in the second allowable credit period increase all the decision variables, viz. cycle time selling price and hence, net profit significantly.
- Similar pattern is followed when display parameter (\( b \)) increases.

**5.3 Conclusion:**

In this chapter, two models are developed under the conditions of two progressive credit periods offered by the supplier to the retailer. The demand is considered as a stock – dependent. In section 5.1, two forms of demand viz linear and exponential are considered. It is observed that stock – dependent parameter attracts customer to buy more and there by demand increases significantly. In section 5.2, the profit is maximized with respect to selling price of an item and cycle time (procurement quantity). Increase in display factor increases demand and net profit significantly. The model developed in 5.2 reduces to that in 5.1 if \( v = 0 \).