CHAPTER 2

OPTIMAL ORDERING POLICIES FOR

EXPONENTIALLY DETERIORATING ITEMS UNDER

SCENARIO OF PROGRESSIVE CREDIT PERIOD
2.0 Introduction:

In this chapter, we develop an EOQ model when units in inventory are subject to constant rate of deterioration and supplier offers two progressive credit periods to the retailer to settle the account.

2.1 Assumptions and Notations:

The assumptions are same as mentioned in A.1.

The following notations other than N.1 are used to develop the proposed mathematical model:

\[ D(T) = \text{the number of units deteriorated during the cycle time } T. \]

\[ Q \text{ and } T \text{ are decision variables.} \]

\[ K(T) = \text{the total inventory cost per time unit.} \]

The total inventory cost per time unit is sum of (a) ordering cost; \( OC \), (b) inventory holding cost (excluding interest charges); \( IHC \), (c) purchase cost; \( PC \), (d) interest charges; \( IC \), for un-sold items after the allowable delay period \( M \) or \( N \), minus (e) interest earned; \( IE \) from the sales revenue during the permissible delay period \( [0, M] \).

2.2 Mathematical Formulation:

The on-hand inventory \( Q(t) \), \( 0 \leq t \leq T \) depletes due to demand and deterioration of units at a constant rate \( \theta \). The instantaneous state of inventory \( Q(t) \), \( 0 \leq t \leq T \) at any time \( t \) is governed by the differential equation:

\[
\frac{dQ(t)}{dt} + \theta Q(t) = -R, \quad 0 \leq t \leq T
\]  

(2.2.1)

with the initial condition \( Q(0) = Q \) and boundary condition \( Q(T) = 0 \).

The solution of equation (2.2.1) is given by

\[
Q(t) = \frac{R}{\theta} \left( e^{\theta(T-t)} - 1 \right), \quad 0 \leq t \leq T
\]

(2.2.2)

and the procurement quantity is
\[ Q = \frac{R}{\theta} \left( e^{\theta T} - 1 \right) \]  
\[ (2.2.3) \]

The components of total inventory cost of the system per time unit are as follows:

- Ordering cost: \[ OC = \frac{A}{T} \]  
\[ (2.2.4) \]

- Inventory holding cost:
\[ IHC = h \int_{0}^{T} Q(t) dt = \frac{hR}{\theta^2 T} \left( e^{\theta T} - \theta T - 1 \right) \]  
\[ (2.2.5) \]

- Purchase cost:
\[ PC = \frac{CR}{\theta T} \left( e^{\theta T} - 1 \right) \]  
\[ (2.2.6) \]

Regarding interest charged and earned, based on the length of the cycle time \( T \), three cases arise:

**Case 1:** \( T \leq M \)

**Case 2:** \( M < T < N \)

**Case 3:** \( T \geq N \)

Now, we discuss each case in detail.

**Case 1:** \( T \leq M \) (Fig. 2.2.1)
Here, the retailer sells \( Q \)-units in cycle time \( T \) and are paying \( CQ \) to the supplier in full at time \( M \geq T \). So interest charges are zero i.e.,

\[
IC_1 = 0 \quad (2.2.7)
\]

During \([0, T]\), the retailer sells products at price \( P/L\) per unit and deposits the revenue into an interest earning account at the rate of \( le/s/year \). In the period \([T, M]\), the retailer only deposits the total revenue into an account that earns \( le/s/year \). Hence, interest earned per unit time is

\[
IE_1 = \frac{Ple}{T} \left( \int_0^T Rt \, dt + RT(M - T) \right) = PLeR \left( M - \frac{T}{2} \right) \quad (2.2.8)
\]

Using \((2.2.4) - (2.2.8)\), the total cost of an inventory system per time unit is

\[
K_1(T) = OC + IHC + PC + IC_1 - IE_1
\]

The optimum value of \( T = T_1 \) is solution of non-linear equation

\[
\frac{dK_1(T)}{dT} = \frac{PLeR}{2} + \frac{CRe \, e \, T_T}{T} + \frac{A}{\theta} - \frac{hR \, (1 - e \, T_T)}{\theta^2} = 0 \quad (2.2.10)
\]

The obtained \( T = T_1 \) minimizes the total cost because

\[
\frac{d^2K_1(T)}{dT^2} = \frac{CRe \, e \, T_T}{T} + \frac{hR \, (1 - e \, T_T)}{T^3} > 0 \quad \forall \ T \quad (2.2.11)
\]
Case 2: $M < T < N$ (Fig 2.2.2)

Here, interest earned, $IE_2$, during $[0, M]$ is

$$IE_2 = Pte \int_0^M Rt \, dt = \frac{PteRM^2}{2}.$$

Buyer has to pay for $Q$–units at time $t = 0$ at the rate of $C$ $$/unit to the supplier up to time $M$, the retailer sells $RM$– units and has $PRM$ plus interest earned $IE_2$ to pay the supplier. Depending on the difference between the total purchase cost; $CQ$, and the revenue; $PRM + IE_2$, two sub–cases may arise:

**Sub–case 2.1:** Let $PRM + IE_2 \geq CQ$.

Here, the retailer has sufficient amount in his account to pay off total purchase cost at $M$. Then Interest charges, $IC_{2.1} = 0$ \hspace{1cm} (2.2.12)

and interest earned per time unit is,

$$IE_{2.1} = \frac{Pte}{T} \int_0^T Rt \, dt = \frac{PteRM^2}{2T} \hspace{1cm} (2.2.13)$$

Therefore, the total cost of an inventory system per time unit is

$$K_{2.1}(T) = OC + IHC + PC + IC_{2.1} - IE_{2.1} \hspace{1cm} (2.2.14)$$

The optimum value of $T = T_{2.1}$ is solution of non–linear equation
\[ \frac{dK_{21}(T)}{dT} = \frac{CRe e^\theta T + e^\theta T}{T} - A \frac{CR(e^{\theta T} - 1)}{\theta} + \frac{hR(1 - e^\theta T)}{\theta^2} + \frac{PleRM^2}{2} \]

\[ = 0 \quad (2.2.15) \]

and \( T = T_{21} \) minimizes the total cost \( K_{21} \) of an inventory system because

\[ \frac{d^2 K_{21}(T)}{dT^2} = \frac{CRe e^\theta T + hRe e^\theta T}{T} - \frac{CRe e^\theta T}{T^2} - A \frac{CR(e^{\theta T} - 1)}{\theta} + \frac{hR(1 - e^\theta T)}{\theta^2} + \frac{PleRM^2}{2} \]

\[ > 0 \quad \forall \, T \quad (2.2.16) \]

**Sub-case 2.2:** Let \( PRM + IE_2 < CQ \)

In this case, the retailer does not have sufficient money in his account to do payment at given permissible credit period, \( M \), then supplier charges retailer on the un-paid balance, \( U_1 = CQ - [PRM + IE_2] \) at the interest rate \( ic_1 \) at time \( M \).

Therefore, interest charges; \( IC_{22} \) per time unit is

\[ IC_{22} = \frac{U_1^2 IC_1}{2PRT} \int_{M}^{T} Q(t) dt = \left\{ \frac{CR(e^{\theta T} - 1)}{\theta} - PRM - \frac{1}{2} PleRM^2 \right\}^{2} IC_1 \]

\[ \frac{PRT}{} \]

\[ = \left\{ \frac{-1}{\theta^2} + \left( e^{\theta(T-M) - M\theta} \right)\frac{R}{\theta^2} \right\} \quad (2.2.17) \]

and interest earned, \( IE_{22} \) per time unit is

\[ IE_{22} = \frac{Ple}{T} \int_{0}^{M} R(t) dt = \frac{PleRM^2}{2T} \quad (2.2.18) \]
Therefore, the total cost of an inventory system per time unit is

\[ K_{2,2}(T) = OC + IHC + PC + IC_{2,2} - IE_{2,2} \]  

(2.2.19)

The optimum value of \( T = T_{2,2} \) can be obtained by solving non-linear equation

\[
\frac{dK_{2,2}(T)}{dT} = \frac{PleRM^2 - 2A}{2T^2} + \frac{CRe \theta T}{\theta T^2} \left( T - 1 \right) + \frac{C + hR \left( e^{\theta T} + e^{-\theta T} \right)}{P\theta^2 T} \\
2U_1Ic_1 \left( -e^{\theta(T-M)} - M\theta + 1 + T\theta \right) \frac{CRe \theta T}{P\theta^2 T} \\
+ \frac{U_1^2 Ic_1 \left( e^{\theta(T-M)}(\theta T - 1) + 1 - M\theta \right)}{P\theta^2 T} = 0
\]

(2.2.20)

with suitable iterative method. The sufficiency condition is:

\[
\frac{d^2K_{2,2}(T)}{dT^2} = \frac{2A - PleRM^2}{T^3} - \frac{2CRe \theta T}{T^2} \left( C\theta + h \right) + \frac{2CR(e^{\theta T} - 1)}{\theta T^3} \\
+ \frac{2hR(e^{\theta T}(1-\theta T) - 1)}{\theta^2 T^3} - \frac{2CRe \theta T Ic_1 \%1(CRe \theta T + U_1\theta)}{P\theta^2 T} \\
+ \frac{2U_1Ic_1(4CRe \theta T - U_1)e^{\theta(T-M)}(\theta T - 1) + 1 - M\theta)}{P\theta^2 T^3} + \frac{U_1^2 Ic_1 e^{\theta(T-M)}}{PT} > 0 \quad \forall \ T
\]

(2.2.21)

where \( \%1 = (-e^{\theta(T-M)} - M\theta + 1 + \theta T) \),
**Case 3:** $T \geq N$ (Fig. 2.2.3)

![Figure 2.2.3](image)

We proceed as case - 2. Total purchase cost of $Q$-units is $CQ$, the amount of money in retailer's account at $N$ is $PRN + \frac{PleR\int_{N}^{2}}{2}$ The following three sub-cases will arise:

**Sub-case 3.1:** Let $PRM + IE_{2} \geq CQ$. This sub-case is same as sub-case 2.1.

Here, interest charges, $IC_{3,1} = 0$ \hspace{1cm} (2.2.22)

and interest earned, $IE_{3,1}$ per time unit,

$$IE_{3,1} = \frac{Ple}{T} \int_{N}^{M} \cdot \frac{PleRM^{2}}{2T}$$ \hspace{1cm} (2.2.23)

Therefore, the total cost of an inventory system per time unit is

$$K_{3,1}(T) = OC + \frac{IC_{3,1} = 0}{2}$$ \hspace{1cm} (2.2.24)

**Sub-case 3.2:** Let $PRM + IE_{2} < CQ$ and

$$PR (N-M) + \frac{PleR(N^{2} - M^{2})}{2} \geq CQ - (PRM + IE_{2})$$
Here, retailer does not have enough money in his account to settle the payment at time $M$ but he can do it before or at $N$. At $M$, retailer pays $(PRM + IE_2)$ and supplier charges for the un-paid balance $U_1 = CQ - (PRM + IE_2)$ with interest rate $Ic_1$. This situation is same as sub-case 2.2. The total cost $K_{3.2}(T)$, of an inventory system per time unit is

$$K_{3.2}(T) = OC + IHC + PC + IC_{3.1} - IE_{3.2}$$  \hspace{1cm} (2.2.25)

**Sub-case 3.3:** Let $PRM + IE_2 < CQ$ and

$$PRM (N-M) + \frac{PleR(N^2 - M^2)}{2} < CQ - (PRM + IE_2)$$

Here, retailer does not have sufficient money in his account to pay off total purchase cost at time $N$, he pays $(PRM + IE_2)$ at $M$ and $PRM (N-M) + \frac{PleR}{2} (N^2 - M^2)$ at $N$. Hence, the retailer will have to pay interest charges on the un-paid balance $U_1 = CQ - (PRM + IE_2)$ with interest rate $Ic_1$ during $[M, N]$ and un-paid balance,

$$U_2 = U_1 - \left(PRM (N-M) + \frac{PleR}{2} (N^2 - M^2)\right)$$

with interest rate $Ic_2$ during $[N, T]$.

Hence, total interest payable per time unit is

$$IC_{3.3} = \frac{U_1 Ic_1 (N-M)}{T} + \frac{U_2^2}{PRT} Ic_2 \int_N^T Q(t) dt$$  \hspace{1cm} (2.2.26)

and interest earned per time unit is,

$$IE_{3.3} = \frac{PleM}{T} \int_0^T Rtdt = \frac{PleRM^2}{2T}$$  \hspace{1cm} (2.2.27)

Therefore, the total cost of an inventory system per time unit is

$$K_{3.3}(T) = OC + IHC + PC + IC_{3.3} - IE_{3.3}$$  \hspace{1cm} (2.2.28)
The first order condition for $K_{33}(T)$ to be minimum is

$$
\frac{dK_{33}(T)}{dT} = P_{leR}M^2 - 2A + CRe \theta T (\theta T - 1) + CR \theta T^2 + \frac{hRe (e^{\theta T (\theta T - 1)} + 1)}{\theta T^2} \left\{ \begin{array}{l}
\frac{U_1 Ic_2 (N - M)}{T^2} + \frac{CRe \theta T^2 Ic_1 (N - M)}{T} + 2 \frac{\%_1 Ic_2 \%_2 CRe \theta T}{\theta T^2} \\
\%_1 Ic_2 \left( e^{\theta (T - N) (\theta T - 1) - N T - 1 - \theta T} \right) + \frac{\%_1^2 Ic_2 e^{\theta (T - N)}}{\theta T^2} = 0 \end{array} \right. $

(2.2.29)

where $\%_1 = \left\{ \begin{array}{l}
\frac{U_1 - PR(N - M)}{2}
\end{array} \right.$

and the sufficiency condition is

$$
\frac{d^2 K_{33}(T)}{dT^2} = \frac{2A - P_{leR}M^2}{T^3} + \frac{CRe \theta T (\theta T - 2)}{T^2} + \frac{2CR (e^{\theta T - 1})}{\theta T^3} + \frac{hRe \theta T}{T} + \frac{2hRe (e^{\theta T (1 - \theta T) - 1})}{\theta^2 T^3} + \frac{2U_1 Ic_1 (N - M)}{T^3} + \frac{CRe \theta T Ic_1 (N - M)(T - 1)}{T^2} + \frac{2 Ic_2 \left( e^{\theta (T - N) + N T - 1 - \theta T} \right) (CRe \theta T)^2 + \%_1^2}{\theta T^2} + \frac{\%_1 Ic_2 e^{\theta (T - N) - 1 (2CRTe \theta T - \%_1)}}{\theta T^2} + \frac{\%_1^2 Ic_2 e^{\theta (T - N)}}{\theta T^2} + \frac{\%_1 Ic_2 e^{\theta (T - N)(\theta T - 1) - N T - 1}}{\theta T^2} CRe \theta T (\theta T - 2)

> 0 \ \forall \ T

(2.2.30)

where $\%_1 = \left\{ \begin{array}{l}
\frac{U_1 - PR(N - M)}{2}
\end{array} \right.$

In next section, we present computational flowchart to search for optimal solution.
2.3 Flowchart:

2.4 Theoretical Results:

**Proposition 2.4.1** : For \( T \leq M \), \( K_1(T) \) is minimum

Proof : \[ \frac{d^2 K_1(T)}{dT^2} \] given by equation (2.2.11) is non-negative for all \( T \leq M \).

**Proposition 2.4.2** : For \( T \leq M \), \( K_1(T) \) is decreasing function of allowable credit period \( M \).
Proof: \[ \frac{d^2 K_1(T)}{dM} = -PleR < 0. \]

**Proposition 2.4.3**: For \( M < T < N \), if \( PRM + IE_2 > CQ \) then \( K_{21}(T) \) is minimum otherwise \( K_{22}(T) \) is minimum.

Proof: Clearly, from equation (2.2.16), \[ \frac{d^2 K_{21}(T)}{dT^2} > 0, \forall T. \]

Otherwise from equation (2.2.21), \[ \frac{d^2 K_{22}(T)}{dT^2} > 0, \forall T. \]

**Proposition 2.4.4**: For \( M < T < N \), \( K_{21}(T) \) (or \( K_{22}(T) \)) is decreasing function of credit period \( M \), for all \( T \).

Proof: \[ \frac{dK_{21}(T)}{dM} = - \frac{PleRM}{T} < 0, \forall T \text{ and } \]

\[ \frac{dK_{22}(T)}{dM} = - \frac{2U_1Ic_1}{\theta^2T} \left( e^{\theta(T-M)-M\theta+\theta T} - (R - RL\theta) \right) \]

\[ = \frac{PleRM}{T} \frac{U_1Ic_1}{P\theta^2T} \left( e^{\theta(T-M)-\theta} \right) \]

\[ < 0, \forall T. \]

**Proposition 2.4.5**: For \( T > N \), \( K_{32}(T) \) is minimum if \( PRM + IE_2 < CQ \) and

\[ PR(N-M) + \frac{PleR}{2} \left( N^2 - M^2 \right) \geq CQ - (PRM + IE_2) \]

\( K_{33}(T) \) is minimum if

\[ PR(N-M) + \frac{PleR}{2} \left( N^2 - M^2 \right) < CQ - (PRM + IE_2) \]

Proof: Obvious from equation (2.2.20) and (2.2.27).

**Proposition 2.4.6**: For \( T > N \), \( K_{32}(T) \) and \( K_{33}(T) \) are decreasing functions of \( M \) and \( K_{33}(T) \) is increasing function of \( N \).
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Proof:

\[
\frac{dK_{3.2}(T)}{dM} = \frac{2U_1 Ic_1 \left( e^{\theta (T - M) - M\theta + 1 + \theta T} \right) \left( -R - RleM \right)}{\theta^2 T} \\
- \frac{PleRM}{T} - \frac{U_2^2 Ic_1 \left( e^{\theta (T - N) - 1} \right)}{P\theta T}
\]

\[< 0, \forall T\]

\[
\frac{dK_{3.3}(T)}{dM} = \frac{(PR + PleRM)Ic_1(N - M)}{T} - \frac{U_1 Ic_1}{T} - \frac{PleRM}{T} < 0
\]

\[
\frac{dK_{3.3}(T)}{dN} = \frac{2 \%1 Ic_2 \left( e^{\theta (T - N) + N\theta - 1 - \theta T} \right) \left( -R - ReRN \right)}{\theta^2 T} \\
+ \frac{U_1 Ic_1}{T} + \frac{\%1^2 Ic_2 \left( -e^{\theta (T - N) + 1} \right)}{P\theta T}
\]

\[> 0
\]

\[
\%1 = \left( U_1 - PR(N - M) - \frac{PleR \left( N^2 - M^2 \right)}{2} \right)
\]

2.5 Conclusion:

In this chapter, an economic order quantity model is developed when units in inventory are subject to constant rate of deterioration and supplier offers two progressive credit periods to the retailer to settle the account. The decision policy depends on how much money retailer has in his account to pay for the units procured. It is observed that total cost of inventory system is decreasing function of first allowable credit period whereas total cost of an inventory system increases in the second allowable credit period. This happens because here retailer has to pay higher interest for the unpaid units.