CHAPTER 7

EOQ MODEL FOR TWO LEVEL CREDIT POLICIES

UNDER RANDOM SUPPLY
7.0 Introduction:

In this chapter, a mathematical model is developed to study the retailer's optimal replenishment decisions under two levels trade credit, i.e., retailer passes some credit offered by supplier to his customer to stimulate his own sale, when the production rate of the supplier is finite and units received by the retailer are uncertain. The optimal replenishment policy is established by minimizing the total expected cost of an inventory system. The numerical examples are given to validate the developed model.

7.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model:

- Replenishment rate \( P, P > R \) is finite
- The order size is \( Q \)-units per replenishment, however actual quantity received is \( Y \), which follows normal distribution with
  \[
  E(Y) = bQ \\
  V(Y) = \sigma_0^2 + \sigma_1^2 Q^2
  \]
  \[ (7.1.1) \]
  where \( b > 0 \) is the bias factor and \( \sigma_0^2 \) and \( \sigma_1^2 \) are known constants.
- \( Q(t/Y) \) = on-hand inventory of the system at time the of a cycle when actually \( Y \)-units are received.
- \( Ic \) = interest charges/\$ in stock/year by the supplier
- \( \rho = 1 - \frac{R}{P} \)
- \( T(Y) = \frac{Y}{R} \), cycle time when \( Y \)-units are received
- \( IHC(Y) \) = inventory holding cost when \( Y \)-units are received
- \( PC(Y) \) = purchase cost when \( Y \)-units are received
• $IE(Y) = \text{Interest earned when } Y\text{-units are received}$

• $IC(Y) = \text{Interest charged when } Y\text{-units are received}$

• $TC(Y) = \text{total cost of an inventory system when } Y\text{-units are received}$

• $TEC(Q) = \text{the total expected cost of the system when } Q\text{-units are received}$

• $Q^* = \text{the optimal purchase quantity}$

7.2 Mathematical Formulation:

The inventory cycle consists of two phases. During the phase – I, period $(0, t_1)$, units are produced at a constant rate $P$ and demand occurs at a constant rate of $R$-units, leaving a balance of $(P - R)$-units to enter into the inventory system. During phase – II, period $(t_1, T(Y))$, there is no production and demand occurring at a constant rate $R$ is met from the inventory accumulated during phase – I period. Let $Q(t/Y)$ denotes on-hand inventory of the system at time $t$ of a cycle when actually $Y$-units are received by the system. Then the differential equation that governs the instantaneous states of $Q(t/Y)$ at any instant of time $t$, $0 \leq t \leq T(Y)$ is given by

$$\frac{dQ(t/Y)}{dt} = \begin{cases} P - R, & 0 \leq t \leq t_1 \\ -R, & t_1 \leq t \leq T(Y) \end{cases} \quad (7.2.1)$$

with initial condition $Q(0/Y) = Y$ and boundary condition $Q(T(Y)/Y) = 0$. 

![Figure 7.2.1](image-url)
The solution of differential equation (7.2.1) is

\[ Q(t/Y) = \begin{cases} (P-R)t, & 0 \leq t \leq t_1 \\ R(T(Y)-t), & t_1 \leq t \leq T(Y) \end{cases} \] (7.2.2)

\( Q(t/Y) \), being a continuous function of \( t \), at \( t = t_1 \) we have \((P - R) t_1 = R (T(Y) - t_1) \). So \( t_1 = \frac{Y}{P} \). Hence, inventory, \( I_1(Y) \), during \([0, T(Y)]\) in the system is

\[ I_1(Y) = \int_0^{T(Y)} Q(t/Y) \, dt = \int_0^{t_1} (P-R)t \, dt + \int_{t_1}^{T(Y)} R(T(Y)-t) \, dt = \frac{y^2}{2R} \rho \]

Therefore, inventory holding cost excluding interest charges, \( IHC(Y) \), for the cycle is, \( IHC(Y) = \frac{hY^2}{2R} \rho \) (7.2.3)

and ordering cost; \( OC \), for the cycle is, \( OC = A \) (7.2.4)

The other two components of total cost of an inventory system, viz interest charged and interest earned are discussed in different situations as follows:

**Case 1: \( M \leq t_1 \leq T(Y) \) (Fig. 7.2.2)**

![Inventory Level vs Time](image)
Interest earned, $IE_1$, per cycle is

$$IE_1 = Clc \int_{N}^{M} R(t) dt = \frac{ClcQ}{2} (M^2 - N^2) \quad (7.2.5)$$

and interest charged, $IC_1$, per cycle is

$$IC_1 = Clc \int_{M}^{T(Y)} Q(t) dt = \frac{Clc(P - R)}{2} \left( \frac{Y^2}{P^2} - M^2 \right) \quad (7.2.6)$$

Using (7.2.3) – (7.2.6), the total cost, $TC_1(Y)$, of an inventory system when $Y$–units are received is

$$TC_1(Y) = OC + IHC + IC_1 - IE_1 \quad (7.2.7)$$

**Case 2:** $t_1 \leq M \leq T(Y)$ (Fig. 7.2.3)

![Figure 7.2.3](image)

Interest earned, $IE_2$, is same as $IE_1$ and interest charged, $IC_2$ over the cycle is

$$IC_2 = Clc \int_{M}^{T(Y)} R(T(Y) - t) dt = \frac{Clc}{2} (Y - RM)^2 \quad (7.2.8)$$

Arguing as in case–1, the total cost, $TC_2(Y)$, of an inventory system when $Y$–units are received is

$$TC_2(Y) = OC + IHC + IC_2 - IE_2 \quad (7.2.9)$$
**Case 3:** $N \leq T(Y) \leq M$ (Fig. 7.2.4)

Inventory Level

![Figure 7.2.4](image)

In this case, interest charged, $IC_3 = 0$ and interest earned, $IE_3$ is

$$IE_1 = Cle \left\{ \int_{N}^{T(Y)} Rdt + RT(Y)(M - T(Y)) \right\} = \frac{Cle}{2R} \left\{ -\frac{Y^2}{2} - \frac{R^2}{N^2} + 2YM + R \right\} \quad (7.2 \ 10)$$

Hence, the total cost, $TC_3(Y)$, of an inventory system when $Y$-units are received, is

$$TC_3(Y) = OC + IHC + IC_3 - IE_3 \quad (7.2 \ 11)$$

**Case 4:** $T(Y) \leq N \leq M$ (Fig. 7.2.5)

Inventory Level

![Figure 7.2.5](image)
Here, also interest charged, $IC_4 = 0$ Interest earned, $IE_4$ over the cycle is

$$IE_4 = C Ie R T(Y) (M - N) = C Ie Y(M - N)$$  \hspace{1cm} (7.2.12)

The total cost, $TC_4(Y)$, of an inventory system when $Y$-units are received is

$$TC_4(Y) = OC + IHC + IC_4 - IE_4$$ \hspace{1cm} (7.2.13)

Using (7.1.1), total expected cost when $Q$-units are ordered per time unit is given by

$$TEC_1(Q) = \frac{AR}{bQ} + \frac{h \rho}{2bQ} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) + \frac{C Ie (P - R) R}{2bQ} \left( \frac{\sigma_0^2 + (\sigma_1^2 + b^2)Q^2}{p^2} - M^2 \right)$$

$$- \frac{C Ie R^2 (M^2 - N^2)}{2bQ}, \hspace{1cm} M \leq t_1 \leq T$$ \hspace{1cm} (7.2.14)

$$TEC_2(Q) = \frac{AR}{bQ} + \frac{h \rho}{2bQ} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) + \frac{C Ie R}{2bQ}$$

$$\times \left( \sigma_0^2 + (\sigma_1^2 + b^2)Q^2 - 2bQMR + R^2 M^2 \right)$$

$$- \frac{C Ie R^2 (M^2 - N^2)}{2bQ}, \hspace{1cm} t_1 \leq M \leq T$$ \hspace{1cm} (7.2.15)

$$TEC_3(Q) = \frac{AR}{bQ} + \frac{h \rho}{2bQ} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) - \frac{C Ie}{2bQ}$$

$$\times \left( - \sigma_0^2 - (\sigma_1^2 + b^2)Q^2 - R^2 N^2 + 2bQMR \right), \hspace{1cm} N \leq T \leq M$$ \hspace{1cm} (7.2.16)

$$TEC_4(Q) = \frac{AR}{bQ} + \frac{h \rho}{2bQ} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) - C Ie R (M - N), \hspace{1cm} T \leq N \leq M \hspace{1cm} (7.2.17)$$

All $TEC_i(Q)$, $i = 1, 2, 3, 4$ are continuous and differentiable. Equations (7.2.14) – (7.2.17) give
\[
TEC'_1(Q) = \left( \frac{h(\sigma_1^2 + b^2)Q}{R} + \frac{Cle(P-R)(\sigma_1^2 + b^2)Q}{p^2} \right) \frac{R}{bQ} - \frac{1}{bQ^2} \\
\{ (AR + \frac{hP}{2b} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) + \frac{CleR}{2} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2 - M^2) \\
- \frac{CleR(M^2 - N^2)}{2} \} 
\]

\[
TEC'_2(Q) = \left( \frac{h(\sigma_1^2 + b^2)Q}{R} + Cl_e(\sigma_1^2 + b^2 - 2bMR) \right) \frac{R}{bQ} - \frac{1}{bQ^2} \\
\{ (AR + \frac{hP}{2b} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) + (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2 - 2bQMR + R^2 M^2) \\
\times \frac{CleR}{2} (M^2 - N^2) \} 
\]

\[
TEC'_3(Q) = \left( h(\sigma_1^2 + b^2)Q + Cl_e(\sigma_1^2 + b^2 - 2bMR) \right) \frac{1}{bQ} - \frac{1}{bQ^2} \\
\times \{ (AR + \frac{hP}{2b} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) \\
+ \frac{Cle}{2} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2 - 2bQMR + R^2 M^2) \} 
\]

\[
TEC'_4(Q) = \left( h(\sigma_1^2 + b^2)Q - ClebR(M - N) \right) \frac{1}{bQ} - \frac{1}{bQ^2} \\
\{ (AR + \frac{hP}{2} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) - ClebQ(M - N) \} 
\]

\[
TEC'_5(Q) = \frac{1}{P^2bQ^3} \left\{ 2ARP^2 + hP^2\sigma_0^2 - CleR^2 + CleRP\sigma_0^2 + CleR^2M^2P^2 \\
- CleRM^2P^2 + CleR^2N^2P^2 - CleR^2M^2P^2 \right\} 
\]
\[ TEC_2(Q) = \left( \frac{h(\sigma_1^2 + b^2) \rho}{R} + CIC(\sigma_1^2 + b^2) \right) \left( \frac{R}{bQ} - \frac{2R}{bQ^2} \frac{h(\sigma_1^2 + b^2)Q\rho}{R} + CIC(\sigma_1^2 + b^2) - 2bMR \right) \left\{ \left( A + \frac{h\rho}{2R} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) \right) \right\} \] 

\[ + \frac{CIC}{2} \left( \sigma_0^2 + (\sigma_1^2 + b^2)Q^2 - 2bQMR + R^2M^2 \right) - \frac{ClER^2(M^2 - N^2)}{2} \] 

\[ TEC_3(Q) = \frac{1}{bQ} \left( \frac{h(\sigma_1^2 + b^2) \rho + CIC(\sigma_1^2 + b^2)}{2} \right) - \frac{2}{bQ^2} h(\sigma_1^2 + b^2)Q\rho \] 

\[ + \frac{CIC(2(\sigma_1^2 + b^2)Q - 2bMR)}{2} - \frac{2R}{bQ^3} \left\{ \left( A + \frac{h\rho}{2R} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) \right) \right\} \] 

\[ + \frac{CIC}{2R} \left( \sigma_0^2 + (\sigma_1^2 + b^2)Q^2 - 2bQMR + R^2N^2 \right) \] 

\[ TEC_4(Q) = \frac{h(\sigma_1^2 + b^2) \rho}{bQ} - \frac{2R}{bQ^2} \left( \frac{h(\sigma_1^2 + b^2)Q\rho}{R} + CICb(M - N) \right) \] 

\[ - \frac{2R}{bQ^3} \left\{ \left( A + \frac{h\rho}{2R} (\sigma_0^2 + (\sigma_1^2 + b^2)Q^2) \right) + CICbQ(M - N) \right\} \] 

\[ (7.2.23) \]

\[ (7.2.24) \]

\[ (7.2.25) \]

7.3 Discussions:

Each of the \( TEC_i(Q) \), \( i = 1, 2, 3, 4 \) are convex. \( TEC_i(Q) \) is convex iff

\[ \alpha = 2ARP^2 + h\rho P^2 \sigma_0^2 - CICR^2 \sigma_0^2 + CICR^2 \sigma_0^2 + CICR^2M^2P^2 - CICRM^2P^3 \]

\[ + CICR^2P^2(N^2 - M^2) > 0 \]

7.3.1 Necessary and sufficient conditions: Equating \( TEC'_i(Q) = 0 \), \( i = 1, 2, 3, 4 \), we get,

For \( M \leq t_1 \leq T \)

\[ Q^*_i = \sqrt{\frac{2ARP^2 + h\rho P^2 \sigma_0^2 + CICR^2(P - R) - CICRM^2P^2(P - R) + CICR^2P^2(N^2 - M^2)}{(h\rho P^2 + CICR(P - R))(\sigma_1^2 + b^2)}} \]

\[ (7.3.1.1) \]
For $t_1 \leq M \leq T$

\[
Q_2^* = \sqrt{\frac{2AR + h\rho \sigma_0^2 + CIC \sigma_0^2 + CIC R^2 M^2 + C Ile R^2 (N^2 - M^2)}{(h\rho + ClcR)(\sigma_1^2 + b^2)}}
\]

(7.3 1.2)

For $N < T < M$

\[
Q_3^* = \sqrt{\frac{2AR + (h\rho + CIC) \sigma_0^2 + C Ile R^2 N^2}{(h\rho + Cle)(\sigma_1^2 + b^2)}}
\]

(7.3 1.3)

For $T < N < M$

\[
Q_4^* = \sqrt{\frac{2AR + h\rho \sigma_0^2}{h\rho(\sigma_1^2 + b^2)}}
\]

(7.3 1.4)

Also,

\[
TEC_1'\left(\frac{PM}{b}\right) = TEC_2'\left(\frac{PM}{b}\right)
\]

\[
= -\frac{1}{2bP^3M^2} \left\{ -h\rho P^2(M^2(\sigma_1^2 + 2b^2) + \frac{\sigma_0^2b^2}{P} - ClcR^2M^2(\sigma_1^2 + b^2) \\
+ ClcR^2P^2M^2(\sigma_1^2 + b^2) - ClcR\sigma_0^2b^2(P - R) - ClcRM^2P^2b^2(P - R) \\
- ClcR^2P^2b^2(M^2 - N^2) \right\}
\]

(7.3.1.5)

\[
TEC_2' (RM) = TEC_3' (RM)
\]

\[
= \frac{1}{2bR^2M^2} \left\{ h\rho[R^2M^2(\sigma_1^2 + b^2) - \sigma_0^2] - ClcR^3M^2(\sigma_1^2 + b^2) \\
- 2AR - ClcRM^2(\sigma_0^2 + R^2M^2) + ClcR^2(M^2 - N^2) \right\}
\]

(7.3.1.6)
\[ TEC'_3(RN) = TEC'_4(RN) \]
\[ = \frac{1}{2bR^2} \left\{ h\rho [R^2M^2(\sigma_1^2 + b^2) - \sigma_0^2] - CleR^2M^2(\sigma_1^2 + b^2) - 2AR - CleRM^2(\sigma_0^2 + R^2N^2) \right\} \] 

(7.3.1.7)

Define

\[ \Delta_1 = h\rho P^3[M^2(\sigma_1^2 + 2b^2) - \frac{\sigma_0^2b^2}{P}] + ClcRP^2M^2(\sigma_1^2 + b^2) - CleR^2PM^2(\sigma_1^2 + b^2) \]
\[ - 2Ab^2RP^2ClcRPb^2(\sigma_0^2 + MP^2) + CleR^2b^2(\sigma_0^2 - M^2P^2) \]
\[ + CleR^2P^2b^2(M^2 - N^2) \] 
(7.3.1.8)

\[ \Delta_2 = h\rho (R^2M^2(\sigma_1^2 + b^2) - \sigma_0^2) + ClcR^3M^2(\sigma_1^2 + b^2) - 2AR \]
\[ - CleR(\sigma_0^2 + R^2M^2) + CleR^2(M^2 - N^2) \] 
(7.3.1.9)

\[ \Delta_3 = h\rho (R^2N^2(\sigma_1^2 + b^2) - \sigma_0^2) + ClcP^2N^2(\sigma_1^2 + b^2) \]
\[ - 2AR - Cle(\sigma_0^2 + R^2N^2) \] 
(7.3.1.10)

In the next section, we exhibit the computational flowchart.

**7.4 Flowchart:**
7.5 Special Cases:

7.5.1 When \( P \to \infty \), \( M = N = 0 \), derived model reduces to that of Silver (1976)

7.5.2 When \( P \to \infty \), \( N = 0 \), \( \sigma^2_0 = \sigma^2_1 = 0 \) and \( b = 1 \), the developed model is same as that of Goyal (1985).
7.6 Numerical Examples:

Data is defined in proper units

Example 7.6.1: Consider $M = 0.1$, $N = 0.08$, $I_c = 15\%$, $I_e = 12\%$, $A = 30$, $R = 5000$,

$$P = 9000, C = 100, h = 5, \sigma_0^2 = 5, \sigma_1^2 = 0.1, b = 0.75.$$  

Then $\alpha < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$. Using algorithm, $Q = Q^*_2 = 614.28$ minimizes total expected cost $TEC_2(Q^*_2) = 3197214.86$.

Example 7.6.2: Consider $M = 0.1$, $N = 0.08$, $I_c = 15\%$, $I_e = 12\%$, $A = 100$,

$$R = 3000, P = 6000, C = 100, h = 5, \sigma_0^2 = 5, \sigma_1^2 = 0.1, b = 0.95.$$  

Then $\alpha < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$. Using algorithm, $Q = Q^*_3 = 298.05$ is optimum units to be procured and corresponding minimum cost is 960.49.

Example 7.6.3: Consider $M = 0.1$, $N = 0.08$, $I_c = 15\%$, $I_e = 12\%$, $A = 100$,

$$R = 2000, P = 7000, C = 100, h = 5, \sigma_0^2 = 5, \sigma_1^2 = 0.1, b = 0.90.$$  

Then $\alpha < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$. Hence optimum $Q = Q^*_2 = 209.68$ and corresponding minimum cost is 360972.76.

Example 7.6.4: Consider $M = 0.1$, $N = 0.08$, $I_c = 15\%$, $I_e = 12\%$, $A = 35$, $R = 1200$,

$$P = 2000, C = 40, h = 25, \sigma_0^2 = 5, \sigma_1^2 = 0.1, b = 0.95.$$  

Then $\alpha > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$. $Q = Q^*_4 = 91.29$ units and minimum cost is $TEC_4(Q^*_4) = 851.06$.

Example 7.6.5: Consider $M = 0.1$, $N = 0.08$, $I_c = 15\%$, $I_e = 12\%$, $A = 60$, $R = 4800$,

$$P = 5000, C = 50, h = 10, \sigma_0^2 = 5, \sigma_1^2 = 0.1, b = 0.95.$$
Then } \alpha > 0, \Delta_1 > 0, \Delta_2 > 0, \text{ and } \Delta_3 < 0 \text{. Hence, optimum procurement quantity is } Q = Q_3^* = 477.15 \text{ and minimum cost is } TEC_3(Q_3^*) = 342.55.

**Example 7.6.6:** Consider } M = 0.1, N = 0.08, Ic = 15\%, Ie = 12\%, A = 200,

\begin{align*}
R &= 2000, \quad P = 4000, \quad C = 100, \quad h = 5, \quad \sigma_0^2 = 5, \quad \sigma_1^2 = 0.1, \quad b = 0.90.
\end{align*}

Then } \alpha > 0, \Delta_1 < 0, \Delta_2 < 0, \text{ and } \Delta_3 < 0 \text{. Algorithm suggests that } Q = Q_2^* = 209.72 \text{ and corresponding minimum cost is } 361918.89.

**Example 7.6.7:** Consider } M = 0.1, N = 0.08, Ic = 15\%, Ie = 12\%, A = 200,

\begin{align*}
R &= 1500, \quad P = 2000, \quad C = 100, \quad h = 5, \quad \sigma_0^2 = 5, \quad \sigma_1^2 = 0.1, \quad b = 0.90.
\end{align*}

Then } \alpha > 0, \Delta_1 < 0, \Delta_2 < 0, \text{ and } \Delta_3 < 0 \text{. Using algorithm } Q = Q_1^* = 324.93 \text{ is optimal procurement quantity and optimal total expected cost of inventory system is } 1334.71.

**7.7 Conclusions:**

In this chapter, a mathematical model is formulated to study the retailer's optimal replenishment decisions when supplier offers retailer a credit period and retailer passes part of it to the customer to increase his sale. The replenishment rate is assumed to be finite and units received are uncertain. The model is exhibited with numerical examples to support the decision policy of the retailer.