CHAPTER 6

LOT–SIZE MODEL FOR PROGRESSIVE TRADE CREDIT SCHEME UNDER RANDOM SUPPLY
6.0 Introduction:

In this chapter, a model has been developed to formulate ordering policy for retailer when supplier offers progressive trade credit to settle the account and the quantity received by the retailer does not match with that of the ordered. The optimal replenishment policy is derived by minimizing total expected cost of an inventory system Analytic results are discussed to observe the effect of various parameters on the objective function.

6.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model

• The order size is $O$–units per replenishment, however, actual quantity received is $Y$, which follows normal distribution with

\[ E(Y) = bO \]  
\[ V(Y) = \sigma_0^2 + \sigma_1^2Q^2 \]

where $b > 0$ is the bias factor and $\sigma_0^2$ and $\sigma_1^2$ are known constants

• $Q(Y)$ = on-hand inventory of the system at time the of a cycle when actually $Y$–units are received.

• $T(Y) = \frac{Y}{R}$, cycle time when $Y$–units are received

• $IHC(Y)$ = inventory holding cost when $Y$–units are received

• $PC(Y)$ = purchase cost when $Y$–units are received.

• $IE(Y)$ = Interest earned when $Y$–units are received.

• $IC(Y)$ = Interest charged when $Y$–units are received.

• $TC(Y)$ = total cost of an inventory system when $Y$–units are received

• $TEC(Q)$ = the total expected cost of the system when $Q$–units are received

• $O$ = the optimal purchase quantity
6.2 Mathematical Formulation:

Let $Q(t/Y)$ be on-hand inventory of the system at time $t$ of a cycle when actually $Y$-units are received by the inventory system. Then the differential equation that governs the instantaneous state of $Q(t/Y)$, $0 \leq t \leq T(Y)$ is given by,

$$\frac{dQ(t/Y)}{dt} = -R, \quad 0 \leq t \leq T(Y)$$  \hspace{1cm} (6.2.1)

with initial condition $Q(0/Y) = Y$ and boundary condition $Q(T(Y)/Y) = 0$.

The solution of differential equation (6.2.1) is

$$Q(t/Y) = R(T(Y) - t), \quad 0 \leq t \leq T(Y)$$  \hspace{1cm} (6.2.2)

The inventory, $I_1(Y)$, during $[0, T(Y)]$ in the system is,

$$I_1(Y) = \int_0^{T(Y)} Q(t/Y) \, dt = \frac{Y^2}{2R}$$  \hspace{1cm} (6.2.3)

Therefore, inventory holding cost excluding interest charges, $IHC(Y)$, for the cycle is,

$$IHC(Y) = h \frac{Y^2}{2R}$$  \hspace{1cm} (6.2.4)

and ordering cost; $OC$, for the cycle is, $OC = \frac{A}{T}$  \hspace{1cm} (6.2.5)

Regarding interest charged and interest earned, based on the length of the cycle time $T$, three cases may arise.

Case 1: $T(Y) \leq M$

Case 2: $M < T(Y) < N$

Case 3: $T(Y) \geq N$

We discuss each case in detail as follows:
**Case 1: \( T(Y) \leq M \) (Fig. 6.2.1)**

Here, the retailer sells \( Y \)-units and hence paying for \( CY \)-units in full to the supplier at time \( M \geq T(Y) \). So, interest charges are zero. i.e

\[ IC_1(Y) = 0 \]  \hspace{1cm} (6.2.6)

The retailer sells \( Y \)-units during \([0, T(Y)]\) and deposits the revenue in an interest bearing account at the rate of \( l e / \text{$/year} \). In the period \([T(Y), M]\), the retailer deposits revenue into the account that earns interest at the rate \( l e / \text{$/year} \). Therefore, interest earned \( IE_1(Y) \), per year is,

\[ IE_1(Y) = P l e \left[ \int_{0}^{T(Y)} R_t dt + RT(Y)(M - T(Y)) \right] = P l e M Y - \frac{P l e Y^2}{2R} \]  \hspace{1cm} (6.2.7)

Using (6.2.4) – (6.2.7), total cost, \( TC_1(Y) \) of an inventory system when \( Y \)-units are received is

\[ TC_1(Y) = OC + IHC(Y) + IC_1(Y) - IE_1(Y) \]  \hspace{1cm} (6.2.8)

Using (6.1.1), total expected cost, \( TEC_1(Q) \), when \( Q \)-units are ordered per time unit is given by,
The optimum procurement quantity $Q_1$ is given by,

$$Q_1 = \sqrt{\frac{(h + P)e\sigma_0^2 + 2AR}{bQ^3}}$$  \hspace{1cm} (6.2.10)

The obtained $Q_1$ minimizes $TEC_1$ as

$$\frac{\partial^2 TEC_1(Q)}{\partial Q^2} = \frac{(h + P)e\sigma_0^2 + 2AR}{bQ^3} > 0, \text{ for all } Q$$  \hspace{1cm} (6.2.11)

**Case 2: $M < T < N$ (Fig. 6.2.2)**

The retailer sells $Y$–units and deposits the revenue into an interest bearing account at an interest rate $i$e/unit/year during $[0, M]$. Therefore, the interest earned during $[0, M]$ is

$$IE_{2.1}(Y) = Ple \int_0^M Rt \, dt = \frac{PleRM^2}{2}$$  \hspace{1cm} (6.2.12)

Buyer has to pay for $Y = RT(Y)$–units purchased at the rate of $C$ $$/\text{unit to the supplier during } [0, M]$. The retailer sells $RM$–units at sale price $P$/unit. So he
has generated revenue of $PRM$ plus the interest earned, $IE_{2,1}$, during $[0, M]$.

Two sub-cases may arise.

**Sub-case 2.1:** Let $PRM + IE_{2,1}(Y) \geq CY$ i.e. the retailer has enough money to pay for all $Y$–units requisitioned. Then, interest charges,

$$IC_{2,1}(Y) = 0$$

(6.2.13)

Using (6.2.4) – (6.2.7) and (6.2.12) – (6.2.13), total cost of an inventory system when $Y$–units are received is given by

$$TC_{2,1}(Y) = OC + IHC(Y) + IC_{2,1}(Y) - IE_{2,1}(Y)$$

(6.2.14)

Using (6.1.1), total expected cost, $TEC_{2,1}(Q)$, when $Q$–units are ordered per time unit is given by,

$$TEC_{2,1}(Q) = \frac{h \cdot Q(\sigma_1^2 + b^2)}{2b} + \frac{AR}{bQ} + \frac{h \sigma_0^2 - PR^2M^2}{2bQ}$$

(6.2.15)

The optimum procurement quantity $Q = Q_{2,1}$ is,

$$Q_{2,1} = \sqrt{\frac{2AR + h \sigma_0^2 - PR^2M^2}{bQ^2}}$$

(6.2.16)

which minimizes total expected cost $TEC_{2,1}(Q)$ because,

$$\frac{\partial^2 TEC_{2,1}(Q)}{\partial Q^2} = \frac{2AR + h \sigma_0^2 - PR^2M^2}{bQ^3} > 0, \text{ for all } Q$$

(6.2.17)

**Sub-case 2.2:** $PRM + IE_{2,1} < CY$

Here, retailer will have to pay interest on the un-paid balance $U_1 = CY - [PR(P)M + IE_{2,1}]$ at rate of $Ic_1$ at time $M$ to the supplier. The interest to be paid, $IC_{2,2}$, for a cycle is:

$$IC_{2,2}(Y) = \frac{U_1Ic_1}{PR} \int_{M}^{Y/R} Q(t/Y) dt$$
\[
\frac{U_1 Ic_1}{PM} \left( \frac{\frac{Y^2}{2R^2} - \frac{MY}{R} + \frac{M^2}{2}}{t} \right) \tag{6.2.18}
\]

Hence, total cost of an inventory system when \( Y \)-units are received is given by

\[
TC_{22}(Y) = OC + IHC(Y) + IC_{22}(Y) - IE_{21}(Y) \tag{6.2.19}
\]

Using (6.1.1), total expected cost, \( TEC_{22}(Q) \), when \( Q \)-units are ordered per time unit is given by,

\[
TEC_{22}(Q) = \frac{(C^2M^2Ic_1 + hP + 4Clc_1PM^2 + P^2Ic_1M^2)(\sigma_1^2 + b^2)Q}{2Pb} + \frac{AR}{bQ}
\]

\[
+ \frac{\left( hP + C^2Ic_1M^2 + P^2Ic_1M^2 + 4Clc_1PM^2 \right)\sigma_0^2}{2bQ} - \frac{PleR^2M^2}{2bQ} \tag{6.2.20}
\]

Differentiating (6.2.20) with respect to \( Q \) gives optimum procurement quantity,

\[
Q_{22} = \sqrt{\frac{hP\sigma_0^2 - P^2leR^2M^2 + (C^2 + P^2)Ic_1M^2\sigma_0^2 + 4Clc_1PM^2\sigma_0^2 + 2APR}{((C^2 + P^2)Ic_1M^2 + 4Clc_1PM^2 + hP)(\sigma_1^2 + b^2)}} \tag{6.2.21}
\]

which minimizes total expected cost \( TEC_{22}(Q) \) because,

\[
\frac{\partial^2 TEC_{22}(Q)}{\partial Q^2} = \frac{(hP + C^2Ic_1M^2 + P^2Ic_1M^2 + 4Clc_1PM^2)\sigma_0^2 - PleR^2M^2 + 2APR}{PbQ^3} > 0, \text{ for all } Q \tag{6.2.22}
\]

**Case 3:** \( T(Y) \geq N \) (Fig. 6.2.3)

![Figure 6.2.3](image-url)
Based on the total purchase cost, \( Y \)-units, total money in the account at \( N \), three sub-cases may arise,

**Sub-case 3.1:** Let \( PRM + IE_{21}(Y) \geq CY \) Then this sub-case is same as sub-case 2.1. (Note: Decision variables and objective function are designated by subscript 3.1)

**Sub-case 3.2:** Let \( PRM + IE_{21}(Y) < CY \) but

\[
P R (N - M) + \frac{Pl_e R(N - M)^2}{2} \geq CY - \left( PRM + IE_{21}(Y) \right)
\]

This sub-case is similar to sub-case 3.2. (Note: Decision variables and objective function are designated by subscript 3.2)

**Sub-case 3.3:** Let \( PRM + IE_{21}(Y) < CY \) but

\[
P R (N - M) + \frac{Pl_e R(N - M)^2}{2} < CY - \left( PRM + IE_{21}(Y) \right)
\]

Here, the retailer does not have enough money in his account to pay off for total purchase cost at \( N \). He will do payment of \( PRM + IE_{21}(Y) \) at \( M \) and

\[
P R (N - M) + \frac{Pl_e R(N - M)^2}{2} \] at \( N \). So, he has to pay interest charges on un-paid balance \( U_1 = CY - \left[ PRM + IE_{21}(Y) \right] \) with interest rate \( Ic_1 \) during \([M, N]\) and

\[
U_2 = U_1 - \left[ P R (N - M) + \frac{Pl_e R(N - M)^2}{2} \right] \] with interest rate \( Ic_2 \) during \([N, T(Y)]\).

Then, total interest charges, \( IC_{33} \), during the cycle is

\[
IC_{33}(Y) = U_1 Ic_1 (N - M) + \frac{U_2^2}{PR} Ic_2 \int_N^{T(Y)} Q(t) \, dt
\]

\[
= U_1 Ic_1 (N - M) + \frac{U_2^2}{2R^2} Ic_2 \left( \frac{Y^2}{2R^2} + \frac{NY}{R} + \frac{N^2}{2} \right) \tag{6.2.23}
\]
Using (6.2.4), (6.2.5), (6.2.12) and (6.2.23) and (6.1.1), total expected cost 
$T_{EC_{33}}(Q)$, of an inventory system per time unit when $Q$–units are ordered is given by,

\[
T_{EC_{33}}(Q) = \frac{\left( C^2 I_c R^2 N^2 + 4 C I_c P R^2 N^2 + P^2 I_c R^2 N^2 + h R^2 P \right) (\sigma_1^2 + b^2) Q}{2 R^2 Pb}
\]

\[
+ \frac{2 C I_c b PR^3 N - 2 C I_c b PR^3 M}{2 R^2 Pb} + \frac{1}{2 PbQ} \left\{ 2 P^2 I_c R^2 M^2 + 2 APR - 2 P^2 I_c R^2 MN + C^2 I_c N^2 \sigma_0^2 - P^2 I_c eR^2 M^2 N + h P \sigma_0^2 \right. \\
- P^2 I_c eR^2 M^2 + 4 C I_c N^2 P \sigma_0^2 + P^2 I_c N^2 \sigma_0^2 \right\}
\]

(6.2.24)

Then optimum procurement quantity $Q_{33}$ is given by,

\[
Q_{33} = \sqrt{\frac{\left\{ 2 I_c P^2 R^2 M + 2 APR + 2 I_c P^2 R^2 MN + C^2 I_c N^2 \sigma_0^2 - I_c P^2 I_eR^2 M^2 N \\
+ h P \sigma_0^2 - P^2 I_e R^2 M^2 + 4 I_c N^2 P \sigma_0^2 + I_c N^2 P \sigma_0^2 \right\}}{(hP + I_c C^2 N^2 + I_c P^2 N^2 + 4 I_c N^2 P)(\sigma_1^2 + b^2)}}
\]

(6.2.25)

and

\[
\frac{\partial^2 T_{EC_{33}}(Q)}{\partial Q^2} = \frac{1}{PbQ^3} \left\{ 2 P^2 I_c R^2 M^2 + 2 APR - 2 P^2 I_c R^2 MN + I_c C^2 N^2 \sigma_0^2 \\
- I_c P^2 I_e R^2 M^2 N + h P \sigma_0^2 - P^2 I_e R^2 M^2 + 4 I_c N^2 P \sigma_0^2 - I_c P^2 N^2 \sigma_0^2 \right\}
\]

> 0, for all $Q$

(6.2.26)

In the next section, computational flowchart is given to search for the optimal solution.
6.3 Flowchart:

Start

Compute $Q$, from Case 1

- Is $Q < \frac{RM}{b}$?
  - Yes: Optimal solution is From Case 1
  - No
    - Is $\frac{RM}{b} < Q < \frac{RN}{b}$?
      - Yes
        - Is $\frac{PRM+I_{E_{2.1}}}{b} \geq CbQ$?
          - Yes: Optimal solution is from Sub-case 2.1 or Sub-case 3.1
          - No: Compute $TEC(Q) = \min_i \{TEC_i(Q_i)\}$ where, $i = 1, 2.1, 2.2, 3.1, 3.2, 3.3$
      - No: Optimal solution is from Sub-case 3.2
    - No: Optimal solution is from sub-case 3.3

End
6.4 Theoretical Results:

**Proposition 6.4.1**  \( TEC_i(Q_i) \) is minimum for \( i = 1, 2.1, 2.2, 3.1, 3.2, \) and \( 3.3 \)

Proof: It follows from the equations (6.2.27), (6.2.28), (6.2.29)

**Proposition 6.4.2** For \( T(Y) > N, TEC_{3.3}(Q) \) is decreasing function of \( M \) and increasing function of \( N \).

Proof: \[
\frac{\partial TEC_{3.3}(Q)}{\partial M} = - \left\{ Clc_1 R + \left[ 2Ic_1 PR^2 N + \frac{2Ic_1 P R^2 MN + 2PR^2 M - 4Ic_1 PR^2 M}{2bQ} \right] \right\} < 0, \text{ for all } Q
\]
and
\[
\frac{\partial TEC_{3.3}(Q)}{\partial N} = Clc_1 R + \frac{\left( Ic_2 C^2 N + 4Ic_2 CPN + Ic_2 P^2 N \right)}{Pb} \times \left( \sigma_1^2 + b^2 \right)Q + \frac{1}{2bQ} \left[ -2Ic_1 PR^2 M + 2Ic_2 C^2 N \sigma_0^2 \right. \\
\left. -Ic_1 P R^2 M^2 + 8Ic_2 CP \sigma_0^2 + 2Ic_2 NP \sigma_0^2 \right] > 0, \text{ for all } Q
\]

6.5 Conclusion:

In this chapter, the mathematical model is developed when units received does not match with what is ordered but assumes normal distribution with specific mean and variance. It is observed that total expected cost of an inventory system is decreasing function of credit period and also, extended credit period.