CHAPTER 2

PRESENT VALUE FORMULATION OF LOT SIZE INVENTORY MODELS
2.0 Introduction:

In this chapter, three lot size inventory models are formulated under the present value approach. They are

Section 2.1: Present value formulation of Harris and Wilson's classical EOQ model.

Section 2.2: Present value formulation of lot size inventory model with constant rate of deterioration of items.

Section 2.3: Present value formulation of lot size inventory model with variable rate of deterioration of items.

2.1 PRESENT VALUE FORMULATION OF HARRIS AND WILSON'S CLASSICAL EOQ MODEL

This section deals with present value formulation of Harris and Wilson's classical EOQ model. The said model is studied under both the approaches, i.e. average cost approach (AVC) and present value (PV) approach and differences in the optimum cost and procurement quantities are discussed. A numerical example is given to illustrate the theoretical results.

2.1.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to develop the proposed mathematical model.

1. Shortages are not allowed.

2. Time horizon is infinite.

3. The replenishment size (order quantity) is $Q$ which is a decision variable where

   $Q_{AVC}$ : Procurement quantity under AVC approach

   $Q_{PV}$ : Procurement quantity under PV approach
$Q^*_AVC$: Optimum procurement quantity under AVC approach

$Q^*_PV$: Optimum procurement quantity under PV approach

4. The length of inventory cycle is $T$, which is a decision variable where

- $T_{AVC}$: Cycle time under AVC approach
- $T_{PV}$: Cycle time under PV approach
- $T^*_AVC$: Optimum Cycle time under AVC approach
- $T^*_PV$: Optimum Cycle time under PV approach

2.1.2 Mathematical Formulation:

2.1.2.1 Mathematical Formulation under AVC Approach:

Following Naddor (1966), $K(Q_{AVC})$ is the cost of an inventory system per time unit given by

$$K(Q_{AVC}) = \text{Ordering Cost (OC)} + \text{Purchase Cost (PC)} + \text{Inventory Holding Cost (IHC)}$$

$$K(Q_{AVC}) = \frac{AR}{Q_{AVC}} + CR + \frac{CiQ_{AVC}}{2}$$

(2.1.2.1.1)

Then, optimum procurement quantity and optimum cycle time are given by

$$Q^*_AVC = \sqrt{\frac{2AR}{Ci}}$$

and

$$T^*_AVC = \sqrt{\frac{2A}{iCR}}$$

(2.1.2.1.2)

The cost of inventory system for one cycle is given by

$$K(T^*_AVC) = A + CRT^*_AVC + \frac{iCRT^*_AVC}{2}$$

(2.1.2.1.3)
2.1.2.2 Mathematical Formulation under PV Approach:

At the beginning of each inventory cycle, there will be cash-out-flows of the ordering cost $A$ and the purchase cost $CRT$. The out-of-pocket inventory holding costs per time unit are $CiR(T-t)$.

Thus, present value of all cash-out-flows for one cycle is

$$PV(T) = \left[-A - CRT - Ci \int R(T-t)e^{-rt} dt \right]_0^T$$

$$= -A - CRT - Ci \frac{R}{r^2} \left[e^{-rT} - e^{-rT + rT - 1} \right]$$

Consider $T = T_{PV}$, we get

$$PV(T_{PV}) = -A - CRT_{PV} - Ci \frac{R}{r^2} \left[e^{-rT_{PV}} + rT_{PV} - 1 \right]$$

Therefore, the present value of all future cash-out-flows

$$PV_\infty (T_{PV}) = \sum_{n=0}^{\infty} PV(T_{PV}) \cdot e^{-nrT_{PV}}$$

$$= \frac{PV(T_{PV})}{1 - e^{-rT_{PV}}}$$

$$= -A - CRT_{PV} - Ci \frac{R}{r^2} \left[e^{-rT_{PV}} + rT_{PV} - 1 \right]$$

$$\therefore \ PV_\infty (T_{PV}) = \frac{-A - CRT_{PV} - Ci \frac{R}{r^2} \left[e^{-rT_{PV}} + rT_{PV} - 1 \right]}{1 - e^{-rT_{PV}}}$$

$$= - \left[ \frac{Ar + CRT_{PV} (i + r)}{r (1 - e^{-rT_{PV}})} \right] + \frac{iCR}{r^2}$$

(2.1.2.2.2)
Letting \( \frac{d PV(T_{PV})}{d T_{PV}} = 0 \), we obtain the following optimality condition

\[
1 - e^{-r T_{PV}} = \frac{r [CR T_{PV} (i + r) + r A]}{(i + r) CR}
\]

\[
e^{r T_{PV}} - 1 = r T_{PV} + \frac{r^2 A}{(i + r) CR}
\]

\[
e^{r T_{PV}} = 1 + r T_{PV} + \frac{r^2 A}{(i + r) CR}
\] (2.1.2.2.3)

The left hand side of the equation (2.1.2.2.3) is approximated as

\[e^{r T_{PV}} \approx 1 + r T_{PV} + \frac{r^2 T^2 P_{PV}}{2}\]

Then, optimum procurement quantity and optimum cycle time are given by

\[Q_{PV}^* = \sqrt{\frac{2 AR}{C (i + r)}} \quad \text{and} \quad T_{PV}^* = \sqrt{\frac{2 A}{C (i + r) R}}\]

(2.1.2.2.4)

Further, \( \frac{d^2 PV(T_{PV})}{d T_{PV}^2} = -2 r^2 (i + r) CR e^{-r T_{PV}} \)

which is < 0 for all values of \( T_{PV} = T_{PV}^* \)

Hence, \( PV_{\infty}(T_{PV}) \) obtained at equation (2.1.2.2.2) is minimum at \( T_{PV} = T_{PV}^* \).

2.1.3 Special Cases:

(1) If we consider \( r = 0 \), i.e. there is no discounting, the model 2.1.2.2 reduces to that of Naddor (1966).
2.1.4 Numerical Example and Observations:

In this section, we consider numerical analysis to support the comparison between classical EOQ model under AVC approach and under PV approach. Consider an inventory system with the following parametric values in appropriate units.

\[ [C, i, A, R] = [20, 10\%, 250, 1000] \]

\[ T_{AVC}^* = 0.5, \quad Q_{AVC}^* = 500, \quad K(T_{AVC}^*) = 10750 \]

Table 2.1.4.1

<table>
<thead>
<tr>
<th>( r )</th>
<th>( T_{PV}^* )</th>
<th>( Q_{PV}^* )</th>
<th>( PV(T_{PV}^*) )</th>
<th>( Q_{AVC}^* - Q_{PV}^* )</th>
<th>( K(T_{AVC}^<em>) - PV(T_{PV}^</em>) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.422577</td>
<td>422</td>
<td>8879</td>
<td>78</td>
<td>1871</td>
</tr>
<tr>
<td>0.05</td>
<td>0.408248</td>
<td>408</td>
<td>8580</td>
<td>92</td>
<td>2170</td>
</tr>
<tr>
<td>0.06</td>
<td>0.395284</td>
<td>395</td>
<td>8310</td>
<td>105</td>
<td>2439</td>
</tr>
</tbody>
</table>

It is observed from the Table 2.1.4.1 that as discounting rate increases optimum order quantity and PV decreases. As compared to classical EOQ model under AVC approach, significant difference is observed in optimum procurement quantities and considerable cost saving is attained due to the discounting which is not explicitly considered in the classical EOQ model under AVC approach.

2.2 Present Value Formulation of Lot Size Inventory Model with Constant Rate of Deterioration of Items

In this section, a lot size inventory model is developed for deteriorating items with constant rate of deterioration under present value approach. This model is an extension
of the model given in section 2.1. The effects of deterioration and discounting factor are studied in determining optimal order quantity and optimal inventory cost. A numerical example is given to support the theoretical findings.

2.2.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to develop the proposed mathematical model.

1. Shortages are not allowed.
2. Time horizon is infinite.
3. \( \theta \) is the constant rate of deterioration of items \((0 \leq \theta < 1)\).
4. The replenishment size (order quantity) is \( Q \) which is a decision variable where
   \begin{align*}
   Q_{AVC} & : \text{Procurement quantity under AVC approach} \\
   Q_{PV} & : \text{Procurement quantity under PV approach} \\
   Q_{AVC}^* & : \text{Optimum procurement quantity under AVC approach} \\
   Q_{PV}^* & : \text{Optimum procurement quantity under PV approach}
   \end{align*}
5. The length of inventory cycle is \( T \), which is a decision variable where
   \begin{align*}
   T_{AVC} & : \text{Cycle time under AVC approach} \\
   T_{PV} & : \text{Cycle time under PV approach} \\
   T_{AVC}^* & : \text{Optimum Cycle time under AVC approach} \\
   T_{PV}^* & : \text{Optimum Cycle time under PV approach}
   \end{align*}
2.2.2 Mathematical Formulation:

2.2.2.1 Mathematical Formulation under AVC Approach:

Let $Q(t)$ be the on-hand inventory at any instant of time $t$ ($0 \leq t \leq T$). It is assumed that depletion due to demand and due to deterioration will occur simultaneously. The instantaneous state of $Q(t)$ for any instant of time follows the differential equation

$$\frac{dQ(t)}{dt} + \theta Q(t) = -R, \quad 0 \leq t \leq T \tag{2.2.2.1.1}$$

with initial condition $Q(0) = Q$ and boundary condition $Q(T) = 0$

Following Jaggi and Aggarwal (1994), consider $Q = Q_{AVC}$ and $T = T_{AVC}$

The solution of equation (2.2.2.1.1) using boundary condition $Q(T_{AVC}) = 0$ is given by

$$Q(t) = \frac{R}{\theta} \left[ \exp \left( \theta \left( T_{AVC} - t \right) \right) - 1 \right], \quad 0 \leq t \leq T_{AVC} \tag{2.2.2.1.2}$$

using $Q(0) = Q$; we get,

$$Q_{AVC} = \frac{R}{\theta} \left[ \exp \left( \theta T_{AVC} \right) - 1 \right] \tag{2.2.2.1.3}$$

The total inventory during time interval $[0, T_{AVC}]$ is given by

$$I(T_{AVC}) = \int_{0}^{T_{AVC}} Q(t) \, dt$$

$$= \frac{R}{\theta^2} \left[ \exp \left( \theta T_{AVC} \right) - \theta T_{AVC} - 1 \right] \tag{2.2.2.1.4}$$

The number of units that deteriorated during $[0, T_{AVC}]$ is given by

$$D(T_{AVC}) = Q - RT_{AVC}$$

$$= \frac{R}{\theta} \left[ \exp \left( \theta T_{AVC} \right) - \theta T_{AVC} - 1 \right] \tag{2.2.2.1.5}$$
Thus, total cost $K(T_{AVC})$ per time unit of the inventory system is

$$K(T_{AVC}) = \text{Ordering Cost} + \text{Inventory Holding Cost} + \text{Deterioration Cost}$$

$$K(T_{AVC}) = \frac{A}{T_{AVC}} + \frac{C R (\theta + i)}{2 T_{AVC}} \left[ \exp \left( \frac{\theta T_{AVC}}{T_{AVC}} \right) - \frac{\theta T_{AVC}}{T_{AVC}} - 1 \right] \quad (2.2.2.1.6)$$

For $\theta << 1$, using Taylor series approximation, $\exp (\theta T_{AVC})$ in equation (2.2.2.1.6) can be replaced by

$$\exp (\theta T_{AVC}) = 1 + \theta T_{AVC} + \frac{\theta^2 T_{AVC}^2}{2}$$

Therefore, we have equation (2.2.2.1.6) as

$$K(T_{AVC}) \approx \frac{A}{T_{AVC}} + \frac{C R (\theta + i)}{2 T_{AVC}} \left( 1 + \frac{\theta T_{AVC}}{2} \right) + \frac{C i R T_{AVC}}{2} \quad (2.2.2.1.7)$$

Set $\frac{dK(T_{AVC})}{dT_{AVC}} = 0$, the approximate optimal cycle time and optimum procurement quantity are given by

$$T_{AVC}^* \approx \sqrt[3]{\frac{2A}{C R (\theta + i)}} \quad \text{and} \quad Q_{AVC}^* = R T_{AVC}^* \approx \sqrt[3]{\frac{2 A R}{C (\theta + i)}} \quad (2.2.2.1.8)$$

$T_{AVC}^*$ minimizes $K(T_{AVC})$ given in equation (2.2.2.1.7) because

$$\frac{d^2 K(T_{AVC})}{dT_{AVC}^2} = \frac{2 A}{T_{AVC}^3}$$

which is $> 0$ for all $T_{AVC} = T_{AVC}^*$.

::: Optimum cost function for one cycle under AVC approach is given by

$$K(T_{AVC}^*) = A + CR T_{AVC}^* \left( 1 + \frac{\theta T_{AVC}^*}{2} \right) + \frac{C i R T_{AVC}^*}{2} \quad (2.2.2.1.9)$$
2.2.2.2 Mathematical Formulation under PV Approach:

At the beginning of each cycle there will be cash-out-flow of ordering cost \( A \) and procurement cost \( CQ_{PV} \). Further, since the inventory carrying cost is proportional to the inventory, the out-of-pocket inventory carrying cost per unit time at \( t \) is \( CQ(t) \). The present value of the out-of-pocket inventory carrying cost is obtained by discounting \( CQ(t) \) at an opportunity cost (i.e. discounting rate) \( r \) which is given by \( CQ(t) e^{-rt} \).

Thus, the present value of cash flows for the first cycle \( T_{PV} \) is

\[
PV(T_{PV}) = - [\text{Ordering Cost} + \text{Procurement Cost} + \text{Inventory Holding Cost}]
\]

\[
PV(T_{PV}) = - A - CQ - iC \sum_{t}^{T_{PV}} Q(t)e^{-rt} dt
\]

\[
= - A - \frac{CR}{\theta} \left( e^{\theta T_{PV}} - 1 \right) - \frac{CIR}{\theta} \left[ \frac{e^{-rT_{PV}}}{r(\theta+r)} + \frac{e^{\theta T_{PV}}}{\theta+r} - \frac{1}{r} \right] \tag{2.2.2.1}
\]

The present value of all future cash-out-flow is given by

\[
P V_{\infty}(T) = \sum_{n=0}^{\infty} PV(T) e^{-nrT}
\]

\[
= \frac{PV(T)}{1 - e^{-rT}}
\]

Since \( r < 1 \), \( rT < 1 \). Using series expansion of exponential series (ignoring higher powers of \( rT \)), we get

\[
\frac{1}{1 - e^{-rT}} = \frac{1}{rT} \left( 1 - \frac{rT}{2} \right)^{-1} = \frac{1}{rT} + \frac{1}{2} + \frac{rT}{4}
\]

\[
\therefore PV_{\infty}(T) = \left( \frac{1}{rT} + \frac{1}{2} + \frac{rT}{4} \right) PV(T) \tag{2.2.2.2}
\]

Consider \( T = T_{PV} \); since \( \theta < 1 \), \( \theta T < 1 \). Using series expansion of exponential series
of the terms in equation (2.2.2.1) ignoring second and higher powers of \( \theta T \), we get

\[
P V_{\infty}(T_{PV}) = -\frac{A}{rT_{PV}} - \frac{CR}{r} - \frac{C\theta T_{PV}}{2r} - \frac{CiRT_{PV}}{2} - \frac{A}{2} - \frac{CRT_{PV}}{2}\tag{2.2.2.3}
\]

Theoretically, the optimum value of \( T_{PV} = T^*_PV \) can be obtained by differentiating equation (2.2.2.2) with respect to \( T_{PV} \) and equating it with zero. Hence, we get

\[
\frac{d PV_{\infty}(T_{PV})}{dT_{PV}} = 0 \Rightarrow T_{PV} = T^*_PV = \frac{\sqrt{2A}}{CR(\theta + 1 + r)}
\]

and

\[
Q^*_PV = R T^*_PV = \frac{\sqrt{2AR}}{C(\theta + 1 + r)}\tag{2.2.2.4}
\]

Equation (2.2.2.4) minimizes \( PV_{\infty}(T^*_PV) \) because

\[
\frac{d^2 PV_{\infty}(T_{PV})}{dT_{PV}^2} = -\frac{2A}{rT_{PV}^2}\tag{2.2.2.5}
\]

which is < 0 for all \( T_{PV} = T^*_PV \)

2.2.3 Special Cases:

(1) If we consider \( r = 0 \), i.e. there is no discounting, the model 2.2.2.2 reduces to that of Ghare and Schrader (1963).

(2) If we consider \( r = 0 \), i.e. there is no discounting and \( \theta = 0 \), i.e. no deterioration, the model 2.2.2.2 reduces to that of Naddor (1966).

2.2.4 Numerical Example and Observations:

Consider an inventory system with following parametric values in appropriate units.

\[ [R, A, C, i, h] = [1000, 250, 20, 15\%, 3] \]
Table 2.3.4.1

<table>
<thead>
<tr>
<th>$\theta \to r \downarrow$</th>
<th>Decision Variables $\downarrow$</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without PV (no discount factor)</td>
<td>$T^*_{AVC}$</td>
<td>0.4564</td>
<td>0.4385</td>
<td>0.4226</td>
</tr>
<tr>
<td></td>
<td>$Q^*_{AVC}$</td>
<td>456.44</td>
<td>438.53</td>
<td>422.58</td>
</tr>
<tr>
<td></td>
<td>$K(T^*_{AVC})$</td>
<td>9628.71</td>
<td>9270.58</td>
<td>8951.54</td>
</tr>
<tr>
<td>0.04</td>
<td>$T^*_{PV}$</td>
<td>0.3953</td>
<td>0.3835</td>
<td>0.3727</td>
</tr>
<tr>
<td></td>
<td>$Q^*_{PV}$</td>
<td>395.28</td>
<td>383.48</td>
<td>372.68</td>
</tr>
<tr>
<td></td>
<td>$PV(T^*_{PV})$</td>
<td>8343.10</td>
<td>8110.95</td>
<td>7898.27</td>
</tr>
<tr>
<td></td>
<td>$Q^<em>_{AVC} - Q^</em>_{PV}$</td>
<td>61.15</td>
<td>55.05</td>
<td>49.90</td>
</tr>
<tr>
<td></td>
<td>$K(T^<em>_{AVC}) - PV(T^</em>_{PV})$</td>
<td>1285.61</td>
<td>1159.63</td>
<td>1053.28</td>
</tr>
<tr>
<td>0.05</td>
<td>$T^*_{PV}$</td>
<td>0.3835</td>
<td>0.3727</td>
<td>0.3627</td>
</tr>
<tr>
<td></td>
<td>$Q^*_{PV}$</td>
<td>456.44</td>
<td>372.68</td>
<td>362.74</td>
</tr>
<tr>
<td></td>
<td>$PV(T^*_{PV})$</td>
<td>8095.62</td>
<td>7884.00</td>
<td>7689.07</td>
</tr>
<tr>
<td></td>
<td>$Q^<em>_{AVC} - Q^</em>_{PV}$</td>
<td>72.95</td>
<td>65.81</td>
<td>59.84</td>
</tr>
<tr>
<td></td>
<td>$K(T^<em>_{AVC}) - PV(T^</em>_{PV})$</td>
<td>1353.09</td>
<td>1386.58</td>
<td>1262.48</td>
</tr>
<tr>
<td>0.06</td>
<td>$T^*_{PV}$</td>
<td>0.3727</td>
<td>0.3627</td>
<td>0.3536</td>
</tr>
<tr>
<td></td>
<td>$Q^*_{PV}$</td>
<td>372.68</td>
<td>362.74</td>
<td>353.55</td>
</tr>
<tr>
<td></td>
<td>$PV(T^*_{PV})$</td>
<td>7869.55</td>
<td>7675.52</td>
<td>7496.07</td>
</tr>
<tr>
<td></td>
<td>$Q^<em>_{AVC} - Q^</em>_{PV}$</td>
<td>83.76</td>
<td>75.79</td>
<td>69.02</td>
</tr>
<tr>
<td></td>
<td>$K(T^<em>_{AVC}) - PV(T^</em>_{PV})$</td>
<td>0.4564</td>
<td>1595.07</td>
<td>1455.48</td>
</tr>
</tbody>
</table>
From the Table 2.3.4.1, it is observed that under AVC approach, as deterioration rate increases, a decrease is observed in optimum cycle time as well as in optimum procurement quantity. Similar results are also observed under the PV approach. For fixed value of discounting rate, as deterioration rate increases, optimum cycle time, optimum procurement quantity and optimum cost decreases. However, for fixed rate of deterioration, as discount factor $r$ increases, optimum cycle time, optimum procurement quantity and optimum cost of inventory system decreases considerably which results into a significant saving in both order quantity as well as cost of the inventory system.

2.3 PRESENT VALUE FORMULATION OF LOT SIZE INVENTORY MODEL WITH VARIABLE RATE OF DETERIORATION OF ITEMS

This section deals with mathematical development of an economic lot size inventory model for time dependent deterioration rate of items under present value approach. This model is an extension of the model given in section 2.2. The time dependent deterioration rate of items is assumed to be a continuous random variable and it follows two parameter weibull distribution. At the end, a numerical example is given and sensitivity analysis of various parameters on the optimal solutions is carried out.

2.3.1 Assumptions and Notations:

The following additional assumptions and notations other than those given in A.1 and N.1 are used to develop the proposed mathematical model.

1. Shortages are not allowed.
2. Time horizon is infinite.
3. $T$ and $Q$ are decision variables.
2.3.2 Mathematical Formulation:

Let $Q(t)$ be the on-hand inventory at any instant of time $t$ ($0 \leq t \leq T$). It is assumed that depletion due to demand and due to deterioration will occur simultaneously. The instantaneous state of $Q(t)$ for any instant of time follows the differential equation

$$\frac{d Q(t)}{dt} + \theta(t)Q(t) = -R, \quad 0 \leq t \leq T \quad (2.3.2.1)$$

with initial condition $Q(0) = Q$ and boundary condition $Q(T) = 0$, where $\theta(t)$ is as given by equation (1.3.1) earlier.

Taking series expansion and ignoring second and higher powers of $\alpha$ (assuming $\alpha$ to be very small), the solution of equation (2.3.2.1) using boundary condition $Q(T) = 0$ is

$$Q(t) = R \left[ T - t + \frac{\alpha T}{\beta + 1} \left( T \beta - (1 + \beta) t^\beta \right) + \frac{\alpha \beta t^{\beta + 1}}{\beta + 1} \right] \quad (2.3.2.2)$$

using $Q(0) = Q$; we get

$$Q = R \left[ T + \frac{\alpha T}{\beta + 1} \right] \quad (2.3.2.3)$$

The total demand during one cycle is $RT$. Hence, the number of units that deteriorated during one cycle $D(T)$ is given by

$$D(T) = Q - RT = \frac{R \alpha T}{\beta + 1} \quad (2.3.2.4)$$

Under the PV approach, at the beginning of each cycle there will be cash-out-flow of both the ordering cost $A$ and the purchase cost $CQ$. Since the inventory carrying cost is proportional to the value of the inventory, the out-flow inventory cost per unit time at any instant $t$ is $h Q(t)$ and the present value of the out-of-pocket carrying cost
is \( h Q(t) e^{-rt} \) (following Chung (1989)). Therefore, the present value of cash-out-flows

\[ PV(T) \]

for the first cycle is given by

\[
PV(T) = -\left( A + CQ + h \int_0^T Q(t) e^{-rt} dt \right)
\]

\[
= -A - CR \left[ T + \frac{\alpha T^{\beta + 1}}{\beta + 1} \right] - hR \left[ \frac{T^2}{2} + \frac{\alpha \beta T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{rT^3}{6} \right] \quad (2.3.2.5)
\]

Following equation(2.2.2.2), the present value of all future cash-out-flows is given by

\[
P V_{\infty}(T) = \left( \frac{1}{rT} + \frac{1}{2} + \frac{r T^2}{4} \right) PV(T)
\]

\[
\therefore PV_{\infty}(T) = -\frac{A}{rT} - \frac{CR}{r} \left( 1 + \frac{\alpha T^{\beta}}{\beta + 1} \right) - \frac{hR}{r} \left( \frac{T}{2} + \frac{\alpha \beta T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{rT^3}{6} \right) - \frac{A}{2}
\]

\[
- \frac{CR}{2} \left( T + \frac{\alpha T^{\beta + 1}}{\beta + 1} \right) - \frac{hR}{2} \left( \frac{T^2}{2} - \frac{\alpha \beta T^{\beta + 2}}{(\beta + 1)(\beta + 2)} - \frac{rT^3}{6} \right) - \frac{Ar T}{4} - \frac{CR r T^2}{4} - \frac{hR r T^3}{8}
\]

(2.3.2.6)

The optimum value of \( T \) can be obtained from the solution of the following equation

\[
\frac{d PV_{\infty}(T)}{dT} = -\frac{A}{rT^2} - \frac{CR \alpha \beta T^{\beta - 1}}{r(\beta + 1)} - \frac{hR}{r} \left( \frac{1 + \alpha \beta T^{\beta}}{\beta + 2} - \frac{rT}{3} \right) - \frac{CR}{2} \left( 1 + \alpha T^{\beta} \right)
\]

\[
- \frac{hR}{2} \left( T + \frac{\alpha \beta T^{\beta + 1}}{\beta + 1} - \frac{rT^2}{2} \right) - \frac{Ar}{4} - \frac{CR r T^2}{2} - \frac{3hR r T^2}{8} = 0 \quad (2.3.2.7)
\]

which is difficult to solve for \( T \) and therefore we solve it numerically.

Clearly the obtained value of \( T \) satisfies

\[
\frac{d^2 PV_{\infty}(T)}{dT^2} = -\frac{2A}{r T^3} - \frac{CR \alpha \beta (\beta - 1) T^{\beta - 2}}{r (\beta + 1)} - \frac{hR}{r} \left( \frac{\alpha \beta T^{\beta - 1}}{\beta + 2} - \frac{r}{3} \right) - \frac{CR \alpha \beta T^{\beta - 1}}{2}
\]

\[
- \frac{hR}{2} \left( 1 + \alpha \beta T^{\beta} \right) - \frac{CR r}{2} - \frac{3}{4} hR r T \quad (2.3.2.8)
\]

which is \(< 0\) for all values of \( T \).
Hence, equation (2.3.2.6) is minimized for the obtained value of $T$. Theoretically, it is difficult to find the optimum value of $T$ because of non-linearity of equation (2.3.2.7). Optimum solution is obtained using series expansion of exponent and ignoring second and higher powers of both $aT$ and $rT$.

2.3.3 Special Cases:

(1) If we consider $r = 0$, i.e. there is no discounting, this model reduces to that of Covert and Philip (1973).

(2) If we consider $r = 0$, i.e. there is no discounting and $\alpha = \theta$ and $\beta = 1$, i.e. constant deterioration, this model reduces to that of Ghare and Schrader (1963).

(3) If we consider $r = 0$, i.e. there is no discounting and $\alpha = \theta$ and $\beta = 1$, i.e. no deterioration, this model reduces to that of Naddor (1966).

2.3.4 Numerical Example and Observations:

Consider an inventory system with following parametric values in appropriate units.

$[ R, A, C, i, h ] = [2000, 200, 20, 15\%, 3]$  

$T = $ Cycle time; $Q = $ Procurement quantity and $P = PV_\infty (T) = $ Present value

<p>| Variations in $\alpha$ and $\beta$ for given value of $r = 0.03$ |
|-----------------|---|---|---|
| $\beta \rightarrow$ | $\alpha \downarrow$ | 1.5 | 2.0 | 2.5 |
| $T$ | 0.228 | 0.231 | 0.233 |
| $Q$ | 456.39 | 462.16 | 466.07 |
| $P$ | 1391258 | 1390541 | 1390254 |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.08</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>0.02</td>
<td>(T)</td>
<td>0.228</td>
<td>0.222</td>
<td>0.217</td>
<td>0.210</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Q)</td>
<td>456.39</td>
<td>444.37</td>
<td>434.35</td>
<td>420.32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(P)</td>
<td>1391258</td>
<td>1044610</td>
<td>836600</td>
<td>524532</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>(T)</td>
<td>0.225</td>
<td>0.219</td>
<td>0.214</td>
<td>0.205</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Q)</td>
<td>450.58</td>
<td>458.54</td>
<td>428.51</td>
<td>410.46</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(P)</td>
<td>1391842</td>
<td>1045032</td>
<td>836927</td>
<td>524707</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>(T)</td>
<td>0.221</td>
<td>0.216</td>
<td>0.211</td>
<td>0.200</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Q)</td>
<td>442.73</td>
<td>432.69</td>
<td>422.65</td>
<td>400.57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(P)</td>
<td>1392414</td>
<td>1045446</td>
<td>837247</td>
<td>524884</td>
</tr>
</tbody>
</table>

**Table 2.3.4.2**

Variations in \(\alpha\) and \(r\) for given value of \(\beta = 1.5\)
Table 2.3.4.3
Variations in $\beta$ and $r$ for given value of $\alpha = 0.02$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$r \rightarrow$</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.08</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>$T$</td>
<td>0.228</td>
<td>0.222</td>
<td>0.217</td>
<td>0.210</td>
<td>0.204</td>
</tr>
<tr>
<td></td>
<td>$Q$</td>
<td>456.39</td>
<td>444.37</td>
<td>434.35</td>
<td>420.32</td>
<td>408.04</td>
</tr>
<tr>
<td></td>
<td>$P$</td>
<td>1391258</td>
<td>1044610</td>
<td>836600</td>
<td>524532</td>
<td>420212</td>
</tr>
<tr>
<td>2.0</td>
<td>$T$</td>
<td>0.231</td>
<td>0.225</td>
<td>0.219</td>
<td>0.213</td>
<td>0.209</td>
</tr>
<tr>
<td></td>
<td>$Q$</td>
<td>462.16</td>
<td>450.15</td>
<td>438.14</td>
<td>426.13</td>
<td>418.05</td>
</tr>
<tr>
<td></td>
<td>$P$</td>
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<td>1044088</td>
<td>836195</td>
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<td>420233</td>
</tr>
<tr>
<td>2.5</td>
<td>$T$</td>
<td>0.233</td>
<td>0.227</td>
<td>0.221</td>
<td>0.217</td>
<td>0.213</td>
</tr>
<tr>
<td></td>
<td>$Q$</td>
<td>466.07</td>
<td>454.06</td>
<td>432.06</td>
<td>428.05</td>
<td>426.05</td>
</tr>
<tr>
<td></td>
<td>$P$</td>
<td>1390254</td>
<td>1043882</td>
<td>836037</td>
<td>524210</td>
<td>420258</td>
</tr>
</tbody>
</table>

From the tables given above, it is observed that among scale parameter $\alpha$, shape parameter $\beta$ and discounting factor $r$; discounting factor plays a very significant role in deciding $T$, $Q$ and $PV$. From Table 2.3.4.1, it is clear that as $\beta$ increases for the given values of $\alpha$ and $r$, then $T$ and $Q$ increases but $PV$ decreases. Similarly from Table 2.3.4.2, it can be observed that as $\alpha$ increases for the given values of $r$ and $\beta$, then $T$ and $Q$ decreases but $PV$ increases. Also it can be seen from Table 2.3.4.3, as $r$ increases for the given value of $\alpha$ and $\beta$, then $T$ and $Q$ decreases and $PV$ also decreases very rapidly i.e. discounting factor $r$ plays more significant role in determination of $PV$ than the scale parameter $\alpha$ and shape parameter $\beta$. 
2.4 Conclusion:

In this chapter, three lot size inventory models have been proposed and formulated under the present value approach. The logical basis of considering inventory as an investment over infinite planning horizon seems to be a practical situation. Another natural aspect is that of opportunity cost of an item and time dependent deterioration of units in an inventory system. It appears that the present value cost function is less sensitive to deterioration rate and most sensitive to discounting rate. The present value approach seems to be more advantageous and realistic in determining optimum order quantities in present market situations.