CHAPTER II

NULL FIELDS

1. Newtonian Law of Gravitation

It is well known that the advances in pure mathematics influence the progress of theoretical physics to a great extent. On the other hand instances are known where theoretical physicists, during the course of investigating solutions of physical problems, have themselves evolved new mathematics. As a matter of fact in the cases of Fourier, Gauss, Newton and others, the motives of a mathematician and the curiosity of a theoretical physicist are so much inter-woven that it is difficult to judge whether physics inspired them to be mathematicians or mathematics inspired them to be physicists.

The Newtonian theory of gravitation is based on the inverse square law

\[ F = \frac{Gmm'}{\lambda^2} . \]

The only geometrical entity which occurs in the enunciation of this law is the distance \( \lambda \), which has its own limitations in the sense that it takes no cognisance of the neighbourhood pervading between the end points of the distance.
Now the propagation of any physical phenomenon requires transmission of physical effects through infinitesimal neighbourhoods, for which the statement of the physical law in a differential form is pre-requisite. Since such is not the case with the Newtonian law of gravitation, gravitational action at a distance remains instantaneous instead of assuming the form of propagation through intervening space.

2. Maxwellian Null Electromagnetic Fields

Though the fundamental law of electric or magnetic forces namely the Coulomb's law, takes the same form as Newton's law of gravitation, Faraday's explanation of these forces through the idea of the lines of force pervading the intervening space leads to a field theory-formulation of electrodynamics. The structure of the lines of force could divert the attention from charges and magnetic poles to the space around them. The surrounding space could be recognised as the seat of energy by imagining the lines of force to be in a state of stress. Consequently an electro-magnetic field theory was formulated, which predicted the propagation of electromagnetic waves. Equations of the electromagnetic field theory developed by Maxwell on the basis of Faraday's hypothesis are
\[
\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0, \quad (2.1)
\]

\[
\frac{\partial F_{\mu\nu}}{\partial x^\nu} = J^\mu, \quad (2.2)
\]

where the field tensor of the electro-magnetic field \( F_{\mu\nu} \) is given in terms of the potential \( A^\mu \) by

\[
\frac{\partial A^\mu}{\partial x^\nu} - \frac{\partial A^\nu}{\partial x^\mu} = F_{\mu\nu}, \quad (2.3)
\]

\( J^\mu \) is the charge current vector, and the raising and lowering of suffixes is done through the Minkowskian line element

\[
d\mathbf{s}^2 = -dx^2 - dy^2 - dz^2 + dt^2. \quad (2.4)
\]

The energy-momentum contents of the field are described by a tensor \( T^\mu_\nu \) given by

\[
T^\mu_\nu = -F_{\mu\lambda}F^{\nu\lambda} + \frac{1}{4} \eta^\mu_\lambda F_{\lambda\rho}F^{\nu\rho}. \quad (2.5)
\]

In a field theory, the form of any tensor is not as important as the invariant formed by it, since the value of this invariant remains uninfluenced by co-ordinate transformations. Two such invariants of the Maxwellian field theory are \( F_{\mu\nu}F^{\mu\nu} \) and \( F_{\mu\nu}F^{\mu\nu} \), where

\[
F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\gamma\delta} F^{\gamma\delta}, \quad (2.6)
\]
and $\varepsilon_{\alpha\beta\gamma\delta}$ is the usual permutation symbol. These invariants when couched in terms of the electric 3-vector $\mathbf{E}$ and the magnetic 3-vector $\mathbf{H}$ turn out to be as follows:

$$-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = E^2 - H^2,$$

$$\frac{1}{4} F_{\mu\gamma} F^{\mu\gamma} = E_\xi H^\xi.$$  

(2.7)

(2.8)

Obviously these invariants vanish if $E = H$ and $E \perp H$. Such electromagnetic fields for which both the invariants vanish are known as null electromagnetic fields.

Structure of null electro-magnetic fields has been studied in detail by Lichnerowicz (1955) and Synge (1953) by considering the eigen values of the energy tensor $T_{\mu\nu}$. The eigen value $\lambda$ and any principal vector $\xi_{\nu}$ of $T_{\mu\nu}$ satisfy the relation

$$T_{\rho\nu} \xi_{\nu} = \lambda \xi_{\rho}.$$  

(2.9)

The equation for $\lambda$ is

$$\det \left| T_{\mu\nu} - \lambda \delta_{\mu\nu} \right| = 0.$$  

(2.10)

The four eigen values of the electromagnetic energy tensor are found to be real and equal, being
\[ x = \pm K , \quad x = \pm K , \]

where

\[ K = \frac{1}{4} \left[ (F_{\mu \nu} F^{\mu \nu})^2 + (F_{\mu \nu}^* F^{\mu \nu})^2 \right] . \]  

(2.11)

In case, the field under consideration be a null electromagnetic field, all the eigen values vanish. (Synge, 1958). Out of the totality of the principal vectors \( \xi^\alpha \), there exists one vector, say \( \omega^\alpha \), which is null, and orthogonal to all the principal vectors \( \xi^\beta \). Hence

\[ \omega^\alpha \xi^\alpha = 0 . \]  

(2.12)

The kinematic interpretation of this null vector is that it represents the propagation of the Poynting vector with the fundamental velocity. Also from equation (2.9) we see that for a null electromagnetic field

\[ \nabla^\alpha \Gamma^\beta_\mu = 0 . \]  

(2.13)

The solution of the above equation is

\[ \Gamma^\beta_\mu = \sigma \omega^\alpha \mu^\beta . \]  

(2.14)

Finally the electromagnetic field tensor \( F_{\mu \nu} \) and \( \Gamma^\alpha_\mu \) and the principal null vector are related as follows:

\[ \omega^\alpha F_{\nu}^\alpha = 0 , \]  

(2.15)

\[ \omega^\alpha F_{\mu}^\alpha = 0 . \]  

(2.16)
Equation (2.16) can also be expressed as

\[ \omega_{\mu} F_{\mu q} + \omega_{\nu} F_{\nu \lambda} + \omega_{\lambda} F_{\lambda \rho} = 0 . \]  

(2.16a)

Maxwell's equations are invariant under Lorentz's transformation and hence can be expressed in the notations of four vectors and tensors, with the background geometry as Euclidean geometry. Now Einstein's general theory of relativity postulates that the presence of matter or of energy in any form would not permit a Euclidean 4-fold as the background but would necessitate the use of a Riemannian 4-fold as this background. The curvature in the Euclidean background caused by the presence of a distribution of electromagnetic energy is connected with the electromagnetic field through the Einstein-Maxwell equations

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu , \]  

(2.17)

\[ R^\mu - \frac{1}{2} g^\mu R = -8\pi \left[ -F_{\mu \nu} F^{\nu \lambda} + \frac{1}{4} g_{\mu} F_{\rho} F_{\rho \beta} \right] , \]  

(2.18)

\[ F_{\mu \nu, \sigma} + F_{\nu \sigma, \mu} + F_{\sigma \mu, \nu} = 0 , \]  

(2.19)

\[ F_{\mu \nu} = J^\mu , \]  

(2.20)
where a semi-colon indicates covariant differentiation.

In the case of a null electromagnetic field, we have seen that equations (2.18) will be equivalent to

$$ R^\mu_\nu = -8\pi\sigma\omega_\nu\omega^\nu, $$

and

$$ \omega_\mu\omega^\mu = 0, \quad \omega^\nu(C_\omega^\mu), \quad \varphi = 0. \tag{2.22} $$

In the succeeding chapters, we are investigating a series of solutions of the above equations.

3. Null Gravitational Fields

It is not possible to formulate a field theory for gravitation by a procedure similar to that adopted in the case of electromagnetic field theory. The reason is very simple. In the case of electromagnetic field theory, circulation of electric 3-vector gives rise to changes in the magnetic 3-vector and vice versa. Such is not the case with the gravitational field, because the circulation of gravitational force vanishes. Due to this difficulty, field theory of gravitation was formulated much later than the electromagnetic field theory, even though the gravitational inverse square law preceded the Coulombian inverse square law. One could get a field theory if the infinitesimal
neighbourhood, through which the gravitational effects are supposed to propagate, could be recognised in a refined manner. The affine connections of the Riemannian geometry link a mathematical quantity at a point with its representative within an infinitesimal neighbourhood around the point, and provide a refined mathematical tool for the formulation of a field theory. To use Prof. Narlikar's words "Field physics of general relativity and Riemannian geometry may be looked upon as mathematical twins". (Narlikar, 1953).

Field equations of gravitation for empty space-time as given by Einstein are

$$ R_{\mu \nu} = 0, \quad (2.23) $$

where $R_{\mu \nu}$ is the contracted Riemann tensor. It is quite natural to anticipate propagation of gravitational effects in the form of waves in the scheme of this field theory. Einstein in 1918, by adopting a certain scheme of approximation and imposing certain coordinate conditions reduced the differential equation (2.21) to the well known wave equation

$$ \square G_{\mu \nu} = 0, \quad (2.24) $$

where $\square$ denotes D'Alambertian operator and thus was
led to the conclusion that the mathematical scheme of general relativity does permit propagation of gravitational effects by waves.

Pirani (1957)* gave an axiomatic basis to the theory of gravitational radiation. He postulated that (1) gravitational radiation is characterized by the Riemann tensor (2) radiation must be propagated along the null cone. Using Lichnerowicz's (1955) conclusion that the characteristic surfaces of the Einstein's equations are the null 3-surfaces, he deduced from the above two postulates that a gravitational wave front manifests itself as a discontinuity in the Riemann tensor across a null 3-surface. By using suitable tetrads he calculated the discontinuities in the physical components of $R_{ijkl}$ and classified them in order to characterize the wave fronts of gravitational radiation.

Lichnerowicz in 1958 studied the general problem of the discontinuity $[R_{ijkl}]$ in the Riemann tensor across a null 3-surface and found that they satisfied the equations

$$l_m [ R^h_{i j k} ] + l_j [ R^h_{i k m} ] + l_k [ R^h_{i m j} ] = 0 \quad (2.24)$$

$$l_h [ R^h_{i j k} ] = 0 \quad (2.25)$$

* See also Witten (1962)
where \( l_\alpha \) is a null vector normal to the null 3-surface. From (2.24) and (2.25) Lichnerowicz postulated that if the Riemann tensor \( R^{h ijk} \) everywhere satisfied the equations

\[
\ell_m R^{h \ ij \kappa} + \ell_j R^{h \ ik \mu} + \ell_k R^{h \ im \j} = 0, \tag{2.26}
\]

\[
\ell_n R^{h \ ij \kappa} = 0, \tag{2.27}
\]

the space-time field will be said to be pervaded by "Total Radiation". From (2.26) and (2.27) it follows that

\[
R_{ij} = \sigma l_i l_j. \tag{2.28}
\]

Thus when \( \sigma = 0 \), we get gravitational radiation in empty space time while for a null electromagnetic field we have already seen, in equation (2.21), that \( R_{ij} \) is given by the same form as that of the equation (2.28).

The similarity of Lichnerowicz's equations (2.26) and (2.27) with the conditions (2.1$) and (2.16a) satisfied by \( F_{ij} \) in the case of null electromagnetic field tensor leads one to name all the gravitational fields, irrespective of \( R_{ij} \) vanishing or not, for which \( R_{hijk} \) satisfy relations of the type (2.26), and (2.27) as null gravitational fields.

In this thesis we are working out solutions of the equations
and thus trying to understand in a simple way both null electromagnetic and gravitational fields.

4. Spherically Symmetric Solutions

From the first principles Tolman (1934) has given $\mathbf{T}_\mu^\lambda$ for a directed flow of electromagnetic radiation in proper coordinates. Vaidya (1951) transformed these particular expressions from proper coordinates to general coordinates at a point and derived the expression,

$$\mathbf{T}_\mu^\lambda = \sigma \, \mathbf{v}_\mu \mathbf{v}^\lambda, \quad (2.30)$$

$$\mathbf{v}_\mu \mathbf{v}^\mu = 0, \quad \mathbf{v}^\mu \left( \mathbf{v}^\lambda \right)_\mu = 0, \quad (2.31)$$

for the energy tensor of a directed flow of radiation. This form (2.30) is the same as the form (2.14) derived by Lichnerowicz (1955) for null electromagnetic fields from a more general consideration. Using the tensor (2.30), Vaidya (1951) has derived the spherically symmetric solution of the field equations

$$-8\pi \mathbf{T}_\mu^\lambda = \mathbf{R}_\mu^\lambda - \frac{1}{2} \mathbf{g}_{\mu}^\lambda \mathbf{R}. \quad (2.32)$$
The solution is

\[ ds^2 = \frac{1}{1 - \frac{2m}{\kappa r}} \, dr^2 - \kappa^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) + \frac{m^2}{f^2} \left( 1 - \frac{2m}{\kappa r} \right) dt^2, \tag{2.33} \]

where \( m(\kappa, t) \) is a conserved function and satisfies the equation

\[ m'(1 - \frac{2m}{\kappa r}) = f(m), \quad m' = \frac{\partial m}{\partial \kappa}, \quad \dot{m} = \frac{\partial m}{\partial t}. \tag{2.34} \]

In case the conserved function \( m \) itself is taken to be the new time coordinate, the line element (2.33) is transformed to the form

\[ ds^2 = - \kappa^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) + \left[ 1 + \frac{\dot{x}(t)}{\kappa} \right] dt^2 + 2 \, dr \, dt \tag{2.35} \]

(Vaidya, 1953). The wave fronts will now be represented by the equation

\[ t = \text{constant}. \tag{2.36} \]

Investigations regarding spherically symmetric solutions for null electromagnetic fields have been carried out by several workers (Roychoudhary, 1953; Israel, 1958; Synge, 1957; Vaidya and Shah, 1960).

In the present thesis we are interested in studying rotationally symmetric solutions of null electromagnetic fields characterized by an energy momentum tensor of the
form (2.30). Since we are now considering a non-spherical geometry, our investigations will essentially differ from the investigations of the abovementioned authors in a very important way viz. our solutions will contain investigations of null gravitational fields also which are obtained by putting $\sigma = 0$ in the equation (2.30).

We shall begin the next chapter by considering a line element essentially of the form (2.35), which contains a small deviation representing slight departure from spherical symmetry. This investigation will point to a method of obtaining rigorous non-spherical solutions for null fields, which will be discussed in the succeeding chapters.
References: