CHAPTER - I

INTRODUCTION

In the scheme of the general theory of relativity, spherically symmetric solutions have been worked out with a view to describe the gravitational field of a radiating star such as the Sun. Now it is known that the Sun rotates around its axis with a period of twenty five days approximately. This would mean that the gravitational field of the Sun cannot be strictly spherically symmetric. In the present thesis our programme is to investigate the axially symmetric solutions describing propagation of waves emitted by a radiating star. We begin by introducing a slight departure from spherical symmetry in the known solution of the radiating star (Vaidya, 1953) viz.

\[ ds^2 = -\lambda^2 \left[ d\theta^2 + \sin^2 \theta \, dt^2 \right] + \left[ 1 + \frac{\alpha \, \psi}{\lambda} \right] dt^2 + 2 \, dx \, dt \quad (1) \]

We have taken a line-element of the form

\[ ds^2 = -\lambda^2 \left[ 1 + h_{\mu\nu} + \ldots \right] \left[ \, d\theta^2 + \sin^2 \theta \, dt^2 \right] \]

\[ + \left[ 1 + \frac{\alpha \, \psi}{\lambda} + h_{\mu\nu} + \ldots \right] dt^2 + 2 \left[ 1 + h_{\mu\nu} + \ldots \right] dx \, dt \quad (2) \]

and have obtained solutions of the field equations

\[ R_{\mu\nu} = \sigma \, \nu_\mu \nu_\nu \], \quad \nu_\mu \nu_\mu = 0 \], \quad \nu_\mu (\nu_\mu) = 0 \]. \quad (3) \]
The investigations are given in chapter III. These investigations have suggested that at every stage $h_{14}$ remains arbitrary. Further $\mathcal{G}_{14} = 1$ gives $\Theta^2 = 0$, thus making the direction of the flow vector and the direction of $x$-coordinate coincident.

In chapter IV, we have therefore taken up the choice of a non-holonomic coordinate system which considerably simplifies the resulting field equations and enables us to obtain their rigorous solutions for the problem at hand. Taking suggestions from the approximate solution of chapter III, we have chosen coordinates in such a manner that,

1. surfaces $x^4 = \text{constant}$ become null surfaces,
2. a null vector $\Psi^\lambda$ has components $(\Psi^1, 0, 0, 0, c)$,
3. further the coordinate surfaces $(x^2, x^3)$ could become surfaces of revolution. (Vaidya and Pandya, 1960).

On these assumptions, we have arrived at a canonical coordinate system in which the fundamental tensor $q_{ijl}$ takes the form

$$ q_{ijl} = \begin{pmatrix} o & o & o & \alpha \\ o & -B & o & -\lambda \\ o & o & -B & -\mu \\ \alpha & -\lambda & -\mu & \gamma \end{pmatrix} $$ (4)

in these coordinates the field equations (3) give $R_{44} = 0$, 
rest of the components of $R_{ij}$ vanishing. Further if we take

$$R_{\mu}^\nu = -\frac{1}{8\pi} \left[ F_{\mu\lambda} F_{\nu}^{\lambda\ast} - \frac{i}{4} q^9 F_{\rho\lambda} \epsilon^{\rho\mu\nu} \right], \quad (5)$$

we find that, in the present coordinates one can take $F_{24}$ as the only non-zero component of $F_{\mu\nu}$. In case, the null electromagnetic field is switched off by equating $R_{44}$ to zero, all components of $R_{ij}$ vanish, and this situation leads to the requisite field equations for the propagation of gravitational waves. Thus in the present scheme, it is very convenient to study null electromagnetic fields as well as gravitational radiation side by side.

In chapter V, we have considered the solutions of the field equations (3) and we find that, the first field equation $R_{11} = 0$ leads to

$$B = \left[ x^1 e^\alpha + \beta \right]^2, \quad (6)$$

where $\alpha$ and $\beta$ are functions of $x^2, x^3, x^4$. Two cases are immediately suggested.

Case I : $B = \beta (x^2, x^3, x^4). \quad (6.1)$

Since plane fronted waves and waves with cylindrical symmetry come under this case as particular cases, we have called solutions of this case as solutions giving open fronted waves.
Case II: \( B = (x^1)^2 e^q \). \( (6.2) \)

Since solutions with spherical symmetry could be included in this case, we have called solutions of this case as solutions with closed fronted waves. The details of the cases I and II are given in chapters V and VI respectively.

The solution corresponding to the case I, is

\[
dx^2 = - \beta \left[ (dx^2)^2 + (dx^3)^2 \right] + \left[ 1 + \psi(x^2, x^3, x^4) \right] (dx^4)^2 + 2dx^1 dx^4 - 2\lambda dx^2 dx^4 - 2\mu dx^3 dx^4, \quad (7)
\]

where \( \psi \) is left undetermined and \( \beta, \lambda \) and \( \mu \) are functions of \( (x^2, x^3, x^4) \) satisfying the equations

\[
\frac{\partial^2 \log \beta}{\partial (x^2)^2} + \frac{\partial^2 \log \beta}{\partial (x^3)^2} = \sigma,
\]

\[
\frac{\partial \lambda}{\partial x^2} + \frac{\partial \mu}{\partial x^3} = \sigma, \quad \frac{\partial \lambda}{\partial x^3} - \frac{\partial \mu}{\partial x^2} = \sigma. \quad (8)
\]

The only non-vanishing component of \( R_{ij} \) is

\[
R_{44} = - \frac{1}{2\beta} \left[ \frac{\partial^2 \psi}{\partial (x^2)^2} + \frac{\partial^2 \psi}{\partial (x^3)^2} \right]. \quad (9)
\]

If \( \psi \) satisfies Laplace's equation \( R_{44} \) will vanish and the line element represents the geometry of the space-time structure of gravitational waves (Pandya and Vaidya, 1962).
If we take

$$\frac{\partial}{\partial x^3}\left[ B \left( \frac{\partial \psi}{\partial (\alpha')^2} + \frac{\partial \psi}{\partial \alpha(x)^2} \right) \right] = 0,$$

(10)

the line element will represent a null electromagnetic field.

The solution corresponding to closed wave fronts with rotational symmetry is as follows:

$$ds^2 = -(x^1)^2 e^\xi \left[ (dx^2)^2 + (dx^3)^2 \right] + \left[ \frac{\nabla^2 \phi}{x^2} + \frac{\phi_x}{x^2} \right] + \phi + \frac{m(x)}{x^2} - x^2 e^\xi \frac{\partial \phi}{\partial x^2},$$

$$- x^1 \eta \frac{\partial \phi}{\partial x^3} - (x^1)^2 e^\xi \left( \xi^2 + \eta^2 \right) - x^1 \left( \frac{\partial \phi}{\partial x^2} + \frac{\partial \eta}{\partial x^3} \right) \right] (dx^4)^2$$

$$+ 2 \phi_x dx^2 dx^4 + 2(x^1)^2 e^\xi \phi_x dx^2 dx^3 - 2(x^1)^2 e^\xi \phi_x dx^3 dx^4,$$

(11)

where $\phi$ is undetermined function of $x^2, x^3, x^4$ and $\xi$

and $\eta$ satisfy the equations

$$\frac{\partial \xi}{\partial x^2} = \frac{\partial \eta}{\partial x^3}, \quad \frac{\partial \xi}{\partial x^3} = \frac{\partial \eta}{\partial x^2},$$

and $\nabla^2$ stands for the operator $\frac{\partial^2}{\partial (\alpha(x))^2} + \frac{\partial^2}{\partial (\phi(x))^2}$. The only non-vanishing component of $F_{0i}$ is

$$R_{0i} = -\frac{\nabla^2 (\frac{\phi}{x})}{4(x^1)^2 - 1} \frac{\partial m}{\partial (\alpha')^2} + \frac{\partial m}{\partial x^4} + \frac{\partial m}{\partial x^3} \left[ \frac{\xi \partial \phi}{\partial x^2} + \frac{\xi \partial \phi}{\partial x^3} \right].$$

We have considered particular simple cases of the above
solution for physical interpretation, by taking $\xi = 0$, 
$\eta = 0$, and $\alpha = \alpha (x^2)$. Using the usual spherical polar 
coordinates

$$x^4 = \rho, \quad x^2 = \log \tan \frac{\theta}{2}, \quad x^3 = \phi, \quad x^4 = t,$$

we get the line element reduced to the form

$$ds^2 = -c^2 e^\mu \left[ d\sigma^2 + \sin^2 \theta \, d\phi^2 \right] + 2 \, dx \, dt +$$

$$\left[ e^{-\mu} - \left( e^{-\mu/2} \right)^2 \frac{\partial^2 \mu}{\partial \sigma^2} + \frac{\partial \mu}{\partial \sigma} \cos \phi \right] \, dt^2,$$

where $e^\mu = e^\mu \cos^2 \sigma$ and $\mu = \mu(\theta)$. For the different 
choices of the function $\mu(\theta)$, we get various types of the 
surfaces of revolution as the wave fronts of either gravi-
tational waves if $R_{44} = 0$ or corresponding to null electro-
magnetic wave field if $R_{44} \neq 0$. In this particular case, 
since the metric tensor is independent of $x^3 = \phi$, the 
only non-zero component of $F_{\mu\nu}$, namely $F_{24}$, is 
independent of $\phi$, hence Maxwell's equations are 
obviously satisfied.

Thus in these two chapters we have given a new 
scheme of working out regorous solutions of the field 
equations of relativity which describe null gravitational 
fields or null electromagnetic fields.
In the last chapter of the thesis we use Petrov's method to ascertain the types of waves represented by our solutions. For the gravitational waves we represent the physical components of $R h_{ij}^R$ in the six dimensional formalism. For the null electromagnetic field we similarly treat the physical components of Weyl's tensor. We find that the gravitational waves as well as the null electromagnetic fields of chapter V belong to null type $-N$. The gravitational waves or the null electromagnetic fields with rotational symmetry developed in chapter VI also belong to type $-N$ if $m = 0$, otherwise they are of type $\Pi$ if $m \neq 0$. 
References:

