oscillating models of massive spheres in a cosmological background

1. Introduction

Following the Bonnor's conjecture 'The gravitational contraction of large masses can be halted by adding a charge density to the material', we endeavoured in chapter-II to obtain an internal solution of charged fluid spheres embedded in a Nordstrom's exterior field. But we could not succeed in our attempt to get a model where the gravitational contraction would be halted by the electrical repulsion. However from this internal solution of charged fluid spheres we obtained as a particular case the generalized Nordstrom's solution in an expanding universe which we have discussed in chapter-III. In this generalization of Nordstrom's solution, the mass and charge of the particle are related by \( e = -m \). This solution led us to investigate a further generalization of Nordstrom's metric with \( e^2 \neq m^2 \) which we have discussed in chapter-IV.

The purpose of the present chapter is to discuss the gravitational contraction of charged fluid spheres in the background of an expanding universe and to show that if one considers the gravitational contraction of charged fluid spheres under the influence of large scale repulsion of the cosmological back-ground, one gets a model without any
space-time singularity. Further we will see that not only in the case of charged fluid spheres but also in the case of uncharged fluid spheres in the back-ground of the non-static universe, we get an oscillatory behaviour without any singularity anywhere in the field, so that it is the large scale cosmic repulsion which is responsible for halting the gravitational contraction to a singularity.

However, to treat a general case we shall consider charged fluid spheres in the back-ground of expanding universe. For the field inside the contracting fluid sphere we use the line-element.

\[ ds^2 = (F+G)^{-2} dt^2 - (F+G)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) \quad (V-1.1) \]

which is given by the equation (II-3.10) in chapter-II.

Where \( F = F(r) \)
\( G = G(t) \)
are undetermined functions of their arguments.

The fluid pressure, matter density, electric intensity and charge density as given by (II-3.11) to (II-3.14) in chapter-II are

\[ 8\pi p_i = -\frac{k}{(F+G)^2} - 2\ddot{G}(F+G) - 3\dot{G}^2 \quad (V-1.2) \]
\[ 8\pi \rho_i = -2(F+G)^{-3} \left[ F^n(1-kr^2) + (2-3kr^2) \frac{F'}{r} \right] + 3k(F+G)^{-2} + 3G^2 \]  

(V-1.3)

\[ (F'_{14})_i \frac{1}{(F+G)^2} \]  

(V-1.4)

\[ 4\pi \sigma_i = 8(F+G)^{-3} \left[ F^n(1-kr^2) + (2-3kr^2) \frac{F'}{r} \right] \]  

(V-1.5)

where we have used the suffix \( i \) to emphasise that the quantity concerned is for the internal field. Since we are using co-moving coordinates, the boundary of this sphere will be given by \( r = a \), \( a \) being the constant radius in comoving coordinates.

For the regions outside this sphere i.e. for \( r > a \), we have the line-element describing the gravitational field of a charged particle in the expanding universe. In chapter-IV we have worked out two such line-elements. One describes the field of a charged particle in the Einstein-deSitter universe (cf. equation (IV-4.10) of chapter-IV) and the other describes the field in the case when the cosmic back-ground has a non-zero curvature (cf. equation (IV-5.4) of chapter-IV). We shall here consider the back-ground universe to be Einstein-deSitter universe. Therefore if the total mass and charge of the fluid distribution within the sphere embedded in the Einstein-deSitter universe are \( m \) and \( \xi \) respectively, then the appropriate external solution for our problem is
\[
\begin{align*}
\text{d}s^2 &= \frac{(1 - \frac{\mu^2 - e^2}{4x^2})}{(1 + \frac{\mu^2 - e^2}{4x^2})} \text{d}\tau^2 - e^g \left(1 + \frac{\mu^2 - e^2}{4x^2}\right) (\text{d}x^2 + x^2 \text{d}\Omega^2) \\
&= (V-1.6)
\end{align*}
\]

where \( g = g(\tau), \quad \mu = me^{-g/2}, \quad e = c^{-g/2} \)

There is no reason for the coordinates \((r, \theta, \phi, t)\) of \((V-1.1)\) which are comoving with respect to the fluid distribution to be the same as the coordinates \((x, \theta, \phi, \tau)\) which are comoving with respect to the cosmic background.

In the exterior field described by \((V-1.6)\), the pressure \( p_e \), matter density \( \rho_e \), electric intensity \( F_{14}^e \) and the charge density \( \sigma_e \) as given by \((IV-4.11)\) to \((IV-4.14)\) in chapter-IV are

\[
\begin{align*}
\delta \pi p_e &= -e^{-g/2} g_{rr} - \frac{3}{4} e^2 \\
&= (V-1.7) \\
\delta \pi \rho_e &= \frac{3}{4} e^2 \\
&= (V-1.8) \\
\left(F_{14}\right)_e &= \frac{c e}{x^2} \\
&= (V-1.9) \\
\sigma_e &= 0 \\
&= (V-1.10)
\end{align*}
\]

Here and in what follows subscript \( \tau \) denotes differentiation with respect to \( \tau \) and
The problem which we have to work-out in this chapter is to fit the interior field (V-1.1) with the exterior field (V-1.6) over the fluid boundary \( r = a \) and to see the effect of the cosmic back-ground on the gravitational contraction.

As we have discussed in chapter-II, the boundary conditions necessary to ensure the uniqueness of the interior field described by (V-1.1) embedded in an exterior field described by (V-1.6) are as follows:

(i) The coordinates used on the two sides of the boundary must be continuous across the fluid boundary \( r = a \).

Having established a continuous coordinate system across the boundary we have the following additional conditions.

(ii) \( g_{ik} \) are continuous across the fluid boundary \( r = a \).

(iii) The electromagnetic field tensor \( F_{14} \) is continuous across the fluid boundary \( r = a \).

(iv) The fluid pressure \( p \) is continuous across the fluid boundary \( r = a \).
The second condition - the continuity of $g_{ik}$ follows from the first condition - the continuity of the coordinates used to describe both the fields. But the interior field described by (V-1.1) and the exterior field described by (V-1.6) have been obtained with reference to the comoving coordinates $(r,t)$ and $(x,\gamma)$ respectively and the two sets of the coordinates are in general not the same. To establish the continuity of $g_{ik}$, the main thing which we have to do is to transform $(x,\gamma)$ of (V-1.6) into $(r,t)$ of (V-1.1) or conversely. Now this is not an easy task. But we have already obtained the coordinate transformation of $(r,t)$ to $(R,T)$ of the line-element

$$ds^2 = e^{\nu} dT^2 - e^\lambda dR^2 - R^2 d\eta^2$$  \hspace{1cm} (V-1.11)

and established the continuity of $g_{ik}$, $F_{ik}$ and pressure over the boundary $r = a$ in chapter-II. Therefore we shall transform both the coordinates $(r,t)$ and $(x,\gamma)$ to the system $(R,T)$ on the boundary $r = a$ and get the $g_{ik}$, $F_{ik}$ and pressure transformed accordingly and then apply the boundary conditions mentioned above.

2. Coordinate transformations and boundary conditions

We now transform the line-element (V-1.1) into the form (V-1.11) and quote the relevant results from chapter-II. On $r = a$, 

\[ R = as, \quad s = F(a) + G(t) \]  
\[ R' = s + ac, \quad c = F'(a) \]  
\[ \dot{R} = as \]

\[ (e^{-\lambda})_i = \frac{(s+ac)^2 (1-ka^2)}{s^2} - a^2 s^2 s^2 \]

\[ (e^{-\nu})_i = -\frac{T'(1-ka^2)}{s^2} + \frac{T'^2 s^2}{s^2} \]

\[ 0 = -(s+ac)(1-ka^2) T' + a ss^2 T \]

\[ (\tilde{F}^l)_i = - (\pm) \frac{c}{s^2} \sqrt{1-ka^2} (e^2)^i \]

\[ 8\pi p_i = - \frac{k}{s^2} - 2ss - 3s^2 \]

Similarly taking the coordinate transformation from \((x, \tau)\) to \((R, T)\) and transforming \(g_{ik}, F_{ik}\) and the pressure of the exterior solution, we get

\[ R = xe^{g/2} (1+ \frac{\alpha^2 - e^2}{x^2}) \]

\[ R_x = e^{g/2} (1- \frac{\alpha^2 - e^2}{x^2}) \]

\[ R_T = \frac{1}{2}xe_T e^{g/2} (1- \frac{\alpha^2 - e^2}{x^2}) \]
\[ \frac{1}{4} x \frac{2}{e_x^2} \frac{2}{e_x^2} (1+ \frac{\alpha}{x} + \frac{\alpha^2 - e^2}{4x^2}) \]

i.e.

\[ (e^{-\lambda})_e = (1- \frac{2m}{R} + \frac{\alpha^2}{R^2}) - \frac{1}{4} g^2 R^2 \]  

\[ (e^{-\lambda})_e = - \frac{T_x}{g} e^{\frac{1}{2}} (1+ \frac{\alpha}{x} + \frac{\alpha^2 - e^2}{4x^2}) \]

\[ 0 = - \frac{2}{x} (1- \frac{\alpha^2 - e^2}{4x^2}) \frac{T_x}{g} + e^2 g \frac{1}{4} \frac{T_x}{g} \frac{1}{4} \frac{T_x}{g} \]

\[ (\bar{F}^1) = - \frac{2}{R^2} (T_x - \frac{1}{g} x^2 T_x) \]

\[ 8 \pi p_e = - (1- \frac{2m}{R} + \frac{\alpha^2}{R^2}) \frac{1}{4} \frac{g_r}{g} - \frac{3}{4} g \frac{2}{g} \]

Here and in what follows the subscript \( x \) denotes differentiation with respect to \( x \).

From \((V-2.13)\) and \((V-2.14)\) we get
Using these values in (V-2.15) we get

\[
\left( \frac{R_{14}}{t} \right)_{e} = - \left( \frac{1}{R_{2}} \right) \frac{-\nu + \lambda}{(e^{2})_{e}}
\]  

(V-2.19)

We now apply the boundary conditions (i) to (iv) for the two transformed sets given by (V-2.1) to (V-2.8) and (V-2.9) to (V-2.19) on the boundary \( R = \alpha s \).

Continuity of \( \epsilon_{11} \) (i.e. \( \lambda \)) gives

\[
\frac{1}{4} g_{rr}^{2} = s^{2} - \frac{1}{a^{3} s^{3}} \left\{ 2m + 2(1-ka^{2}) a^{2} c \right\} + \frac{k}{s^{2}}
\]  

(V-2.20)

Continuity of \( g_{44} \) (i.e. \( e^{\nu} \)) and hence the continuity of \( F_{14} \) leads to

\[
\epsilon = ca^{2} \sqrt{1-ka^{2}}
\]  

(V-2.21)

Continuity of pressure gives

\[
- \frac{k}{s^{2}} - 2s \dot{s} - 3s^{2} = -(1- \frac{2m}{as} + \frac{\epsilon^{2}}{a^{2}s^{2}})^{-\frac{1}{2}} g_{rr} = \frac{3}{4} \frac{g_{r}}{s}
\]  

(V-2.22)
Since our model is embedded in the background of Einstein-deSitter's universe, we choose

\[ g_{rr} = -\frac{3}{4} g_{r}^{2} \quad (V-2.23) \]

Substituting this in (V-2.22), we get

\[ \frac{k}{s^{2}} + 2 s \ddot{s} + 3 \dot{s}^{2} = \frac{3}{4} g_{r}^{2} \left[ 1 - (1 - \frac{2m}{as} + \frac{\epsilon^{2}}{a^{2}s^{2}})^{-\frac{1}{2}} \right] \quad (V-2.24) \]

Eliminating \( g_{r}^{2} \) from (V-2.20) and (V-2.24) we get

\[ 2 s \ddot{s} + \frac{3}{a^{3}s^{3}} \left\{ 2m + 2 (1-ka^{2}) a^{2}c \right\} - \frac{2k}{s^{2}} = -3 \left( 1 - \frac{2m}{as} + \frac{\epsilon^{2}}{a^{2}s^{2}} \right)^{-\frac{1}{2}} \left[ s^{2} - \frac{1}{a^{3}s^{3}} \left\{ 2m + 2(1-ka^{2}) a^{2}c \right\} + \frac{k}{s^{2}} \right] \quad (V-2.25) \]

We can write this equation as

\[ \frac{s}{s} \left[ 2 s \ddot{s} + \frac{3}{a^{3}s^{4}} \left\{ 2m + 2 (1-ka^{2}) a^{2}c \right\} \dot{s} - \frac{2k}{s^{3}} \right] = -3 \left( 1 - \frac{2m}{as} + \frac{\epsilon^{2}}{a^{2}s^{2}} \right)^{-\frac{1}{2}} \left[ s^{2} - \frac{1}{a^{3}s^{3}} \left\{ 2m + 2(1-ka^{2}) a^{2}c \right\} + \frac{k}{s^{2}} \right] \quad (V-2.26) \]

On putting

\[ \dot{s}^{2} - \frac{1}{a^{3}s^{3}} \left\{ 2m + 2(1-ka^{2}) a^{2}c \right\} + \frac{k}{s^{2}} = \chi \quad (V-2.27) \]
and differentiating it with respect to \( t \) we get

\[
2\ddot{s} + \frac{3}{a^3 s^4} \left\{ 2m + 2(1-ka^2) a^2 c \right\} \dot{s} - \frac{2k}{s^3} \dot{s} = \dot{\chi}
\]

Thus equation (V-2.29) takes the form

\[
\frac{s}{\dot{s}} \dot{\chi} = -3 \left(1- \frac{2m}{as} + \frac{c^2}{a^2 s^2}\right)^{-\frac{3}{2}} \chi
\]

The solution of this equation is

\[
\chi = \frac{B(m^2-c^2)^{3/2}}{a^3 s^3} \left[ 1 - \frac{m}{as} + \sqrt{1 - \frac{2m}{as} + \frac{c^2}{a^2 s^2}} \right]^{-3}
\]

where \( B \) is a constant of integration.

The substitution of \( \chi \) from (V-2.27) gives

\[
\dot{s}^2 = -\frac{k}{s^2} + \frac{2}{a^3 s^3} \left\{ m + (1-ka^2) a^2 c \right\} + \frac{B(m^2-c^2)^{3/2}}{a^3 s^3} \left[ 1 - \frac{m}{as} + \sqrt{1 - \frac{2m}{as} + \frac{c^2}{a^2 s^2}} \right]^{-3} \quad (V-2.28)
\]

This equation contains two arbitrary constants \( k \) and \( B \). One can choose two constants in such a way that \( \dot{s}^2 \) will vanish for \( s = 1 \) and \( s = \xi \) (say). One can assume without loss of generality that \( \xi \) is less than 1 and that \( \dot{s} \) does not vanish between 1 and \( \xi \). The constants \( k \) and \( B \) so determined are
\[ k = \frac{2}{a^3} \left\{ m + (1-ka^2)a^2c \right\} \left[ 1 + \frac{1-\xi}{\alpha} \right] \left( 1 - \frac{1}{\alpha} \sqrt{1 - \frac{2m}{a} + \frac{c^2}{a^2}} \right)^3 \]

and
\[ B = 2(1-\xi) \left\{ m^2 + (1-ka^2)a^2c \right\} \left( 1 - \frac{m}{\alpha} + \frac{1}{\alpha} \sqrt{1 - \frac{2m}{a} + \frac{c^2}{a^2}} \right)^3 \]

Substituting these in (V-2.28) we find that
\[ \dot{s}^2 = 2\left\{ m + (1-ka^2)a^2c \right\} \frac{f(s)}{a^3 s^3} \quad (V-2.29) \]

where
\[ f(s) = 1 - S - (\xi^3 1 - \frac{m}{a^3} + \frac{1}{\alpha} \sqrt{1 - \frac{2m}{a} + \frac{c^2}{a^2}})^3 \left\{ \left( 1 - \frac{m}{a^3} + \frac{1}{\alpha} \sqrt{1 - \frac{2m}{a} + \frac{c^2}{a^2}} \right)^3 \right\} \]

It is clear that \( f(1) = 0, f(\xi) = 0 \). In order to see that \( f(s) \) is positive for \( \xi < s < 1 \) we have only to check that \( \frac{df}{ds} > 0 \) at \( s = \xi \) and \( \frac{df}{ds} < 0 \) at \( s = 1 \). From the equation (V-2.30) we find that
In order to get an idea of the magnitudes of these constants, we assume that

1. the fluid is uncharged i.e., \( C = 0 \)
2. \( \frac{\Delta \phi}{\alpha} \ll 1 \) and
3. \( \frac{m}{a} \) is small.

In that case it is possible to work-out the values of these constants up to any desired power of \( \frac{m}{a} \). Working up to the third power, one finds

\[
\left( \frac{df}{ds} \right)_{s=\kappa} = -1 - \frac{1}{\kappa} \left( 1 - \frac{2m}{a} + \frac{c^2}{a^2} + \frac{c^2}{a^2} \right)^3 \left( 1 - \frac{m}{a} + \frac{c^2}{a^2} \right) \left( 1 - \frac{2m}{a} + \frac{c^2}{a^2} \right)^3 \]
radius $r_0$ is supposed to start radiating at time $t_0$. As the particle continues to radiate, the zone of radiation increases in thickness and its outer surface becomes $r_1$ at a later instant $t_1$. For $r_0 < r < r_1$, $t_0 < t < t_1$, we assume the line-element to be of the form

$$ds^2 = -e^\lambda dr^2 - r^2 d\Omega^2 + e^\nu dt^2$$  \hspace{1cm} (A-2.1)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$

$$\lambda = \lambda (r,t)$$

$$\nu = \nu (r,t)$$

The Einstein's field equations

$$R^k_i - \frac{1}{2} g^k_i S^k - 8\pi T^k_i$$ \hspace{1cm} (A-2.2)

take the forms

$$8\pi T^1_1 = -e^{-\lambda} \left\{ \frac{\nu'}{r} + \frac{1}{r^2} \right\} + \frac{1}{r^2}$$

$$8\pi T^2_2 = 8\pi T^3_3$$ \hspace{1cm} (A-2.3)

$$= -e^{-\lambda} \left\{ \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda \nu'}{2} - \frac{\lambda'}{2r} + \frac{\nu'}{2r} \right\} + e^{-\nu} \left\{ \frac{\dot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\ddot{\nu} \dot{\lambda}}{4} \right\}$$ \hspace{1cm} (A-2.4)

$$8\pi T^4_4 = -e^{-\lambda} \left\{ -\frac{\lambda'}{r} + \frac{1}{r^2} \right\} + \frac{1}{r^2}$$ \hspace{1cm} (A-2.5)
\[ g \pi T^L_1 = e^{-\nu} \frac{\dot{\lambda}}{r} \]  

(A-2.6)

In the space round the radiating charged particle there is an expanding zone of pure radiation which itself is the seat of an electromagnetic field in addition to the Coulomb's field due to the central charge. For this space, the energy-momentum tensor takes the form

\[ T^k_1 = M^k_1 \]  

(A-2.7)

where \( E^k_1 \) is the electromagnetic energy-momentum tensor defined by

\[ 8\pi E^k_i = -F^{ka}_i F^a + \frac{1}{4} \delta^k_i \rho^{ab} F_{ab} \]  

(A-2.8)

where \( F^i_{jk} \) is the electromagnetic field tensor which satisfies the Maxwell's equations

\[ F^i_{jk} = 4\pi J^i_1 \]  

(A-2.9)

and

\[ F_{[ij;k]} = 0 \]  

(A-2.10)

The electromagnetic field round the radiating charged particle can be divided into two fields. One of them is the Coulombian field described by the non-vanishing component \( F^i_{14} \) of \( F^i_{jk} \). The other field is due to the flowing radiation which is described by the components \( F^i_{12}, F^i_{24} \) of \( F^i_{jk} \) with the condition...
The equation (A-2.7) with the condition (A-2.11) gives

\[ T_1^1 = -\frac{1}{8\pi} F_{14}^1 F_{14}^1 - \frac{1}{4\pi} F_{12}^2 F_{12}^2 \]  
\[ T_2^2 = T_3^3 \]
\[ = \frac{1}{8\pi} F_{14}^1 F_{14}^1 \]  
\[ T_4^4 = -\frac{1}{8\pi} F_{14}^1 F_{14}^1 + \frac{1}{4\pi} F_{12}^2 F_{12}^2 \]
\[ T_1^2 = \frac{1}{4\pi} F_{24}^2 F_{12}^2 \]

Eliminating \( F_{12}^1 \), \( F_{24}^1 \), and \( F_{14}^1 \) from the equations (A-2.12) to (A-2.15) and using the condition (A-2.11), we get the following three equations

\[ \frac{-\lambda + \nu}{T_1^1 + T_2^2 - e^2} T_4^4 = 0 \]  
\[ \frac{-\lambda + \nu}{T_1^1 - T_4^4 - 2e^2} T_1^1 = 0 \]  
\[ \frac{-\lambda + \nu}{T_2^2 + T_4^4 + e^2} T_1^1 = 0 \]

Also we have one more equation which we get from the Maxwell's equations (A-2.10). This equation is
The use of the condition (A-2.11) in (A-2.19) gives

\[ \frac{-\lambda + \nu}{2} F_{12,4} + \frac{-\lambda + \nu}{2} F_{12,1} = 0 \]  
(A-2.20)

Substituting the value of \( F_{12} \) in terms of \( T^1 \) from (A-2.15)

we get

\[ \frac{-\lambda + \nu}{2} T^1_{1,4} + \frac{-\lambda + \nu}{2} T^1_{1,1} + \frac{-\lambda + \nu}{2} \left[ \frac{2}{r} + \frac{-\lambda + 3\nu}{2} \right] T^1 = 0 \]  
(A-2.21)

Also the law of conservation \( T^k_{i;k} = 0 \) gives the identity

\[ \frac{\partial}{\partial r} (T^1) + \frac{\partial}{\partial t} (T^1) - \frac{v'}{2} (T^4 - T^1) + \frac{2}{r} (T^1 - T^2) + T^4 \left( \frac{\lambda + \nu}{2} \right) = 0 \]  
(A-2.22)

Substituting the values of the components \( T^1, T^2, T^4 \) in terms of \( T^1 \) from the equations (A-2.16) to (A-2.18), we get

\[ \frac{-\lambda + \nu}{2} T^1_{1,4} + \frac{-\lambda + \nu}{2} T^1_{1,1} + \frac{-\lambda + \nu}{2} \left[ \frac{2}{r} + \frac{-\lambda + 3\nu}{2} \right] T^1 = 0 \]  
(A-2.22')

To satisfy Maxwell's equation (A-2.21), we must have
\[ T_{2,1}^2 + \frac{4}{r} T_2^2 = 0 \]  \hspace{1cm} (A-2.23)

The integration of this equation gives

\[ T_2^2 = \frac{\varphi(t)}{r^4} \]  \hspace{1cm} (A-2.24)

where \( \varphi = \varphi(t) \) is an undetermined function of time \( t \).

Now putting the value of \( T_2^2 \) from (A-2.24) and the values of \( T_1^1 \), \( T_4^4 \), and \( T_1^4 \) in terms of \( g_{1k} \) and their derivatives from (A-2.3), (A-2.5) and (A-2.6) in (A-2.16) to (A-2.18) we get the following three differential equations

\[ e^{-\lambda} \left[ \frac{v'}{r} + \frac{1}{r^2} \right] - \frac{1}{r^2} + e^2 \frac{\dot{\lambda}}{r} = \frac{\varphi(t)}{r^4} \]  \hspace{1cm} (A-2.25)

\[ e^{-\frac{\lambda}{2}} \left[ v' + \lambda' \right] + 2 \dot{\varphi} e^{-\lambda/2} = 0 \]  \hspace{1cm} (A-2.26)

\[ e^{-\lambda} \left[ \frac{\dot{\lambda}'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} + e^2 \frac{\ddot{\lambda}}{r} = - \frac{\varphi(t)}{r^4} \]  \hspace{1cm} (A-2.27)

Any one of the three differential equations (A-2.25) to (A-2.27) can be derived from the remaining two. Therefore if we solve any two of them, the third will automatically be satisfied.

3. A solution of the field equations

Let us assume that \( e^\lambda \) takes the form
where \( M = M(r,t) \) is an undetermined function of \( r \) and \( t \).

Substituting this value of \( \lambda \) in (A-2.27) we get

\[
2e^{-\lambda} \frac{M'}{r^2} + 2e \frac{\lambda - \varphi}{r^2} \frac{M}{r^4} = - \frac{\varphi(t)}{r^4} e^{-\lambda}
\]

or this can also be written as

\[
e^{-\lambda/2} \left[ M' + \frac{\varphi(t)}{2r^2} \right] + e^{-\varphi/2} M = 0
\]

i.e., we can write

\[
e^{-\lambda/2} \left[ M - \frac{\varphi(t)}{2r} \right]' + e^{-\varphi/2} \left[ M - \frac{\varphi(t)}{2r} \right] + e^{-\varphi/2} \frac{\varphi(t)}{2r} = 0
\]

There will be no loss of generality if we choose \( \varphi = \text{constant} \).

Equation (A-3.2) then reduces to the form

\[
e^{-\lambda/2} \left[ M - \frac{\varphi}{2r} \right]' + e^{-\varphi/2} \left[ M - \frac{\varphi}{2r} \right]^* = 0
\]

This is same as

\[
\frac{d}{d\tau} \left[ M - \frac{\varphi}{2r} \right] = 0
\]

where the operator \( \frac{d}{d\tau} \), defined by
expresses the rate of change along the directed flow of radiation, or it differentiates following the lines of flow. We mean the lines of flow as a distribution of electromagnetic energy such that a local observer at any point of the region of the space under consideration finds one and only one direction in which the radiant energy is flowing at the point.

The vector \( v^i \) used in (A-3.4) is given by

\[
v^i v^i = 0
\]  

(A-3.5)

and since the flow of radiation is along the radial direction

\[
v^2 = v^3 = 0
\]  

(A-3.6)

If we put the constant \( \phi = 0 \) in (A-3.3), we get the equation which is obtained by Vaidya (1) for conservation of mass along the flow of radiation. Therefore let us assume that

\[
M - \frac{\phi}{2r} = m , \quad m = m(r,t)
\]  

(A-3.7)

Equation (A-3.3) then becomes

\[
\frac{dm}{d\tau} = 0
\]

or

\[
e^{-\lambda/2} m' + e^{-\nu/2} m = 0
\]  

(A-3.8)
This gives
\[ e^{\mp \sqrt{2} / 2} = -\frac{\dot{m}}{m} e^{\lambda/2} \]
\[ e^{\eta} = \frac{m^2}{m^2} e^{\lambda} \]  \hspace{1cm} (A-3.9)

The use of \((A-3.7)\) in \((A-3.1)\) gives
\[ e^{-\lambda} = (1 - \frac{2m}{r} - \frac{\phi}{r^2}) \]  \hspace{1cm} (A-3.10)

The effect of \(\phi\) in \((A-3.10)\) is due to the presence of the Goulombian field described by the electric field tensor \(F_{14}\). Therefore we can choose \(\phi = -\epsilon^2\). Thus we get Nordstrom's form
\[ e^{-\lambda} = (1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}) \]  \hspace{1cm} (A-3.10')

Now we can take the equation \((A-2.26)\). Substituting the values of \(\phi\) and \(\lambda\) from \((A-3.9)\) and \((A-3.10')\) we get
\[ e^{-\lambda} \left[ \frac{\dot{m}'}{m'} - \frac{m''}{m'} \right] = \frac{2m'}{r^2} - \frac{2\epsilon^2}{r^3} \]  \hspace{1cm} (A-3.11)

The integration of this equation gives
\[ m'(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}) = f(m) \]  \hspace{1cm} (A-3.12)

where \(f = f(m)\) is an arbitrary function of \(m\). Equation \((A-3.12)\) is a differential equation to be solved for \(m\).
Now using the equation (A-2.26) in equation (A-2.21) we get

$$\frac{d}{d\tau} \left\{ r^2 e^\frac{3\lambda + \nu}{2} T^4_1 \right\} = 0 \quad (A-3.13)$$

Substituting the value of $T^4_1$ in terms of $g_{ik}$ from (A-2.6) and using the values of $g_{ik}$ given by (A-3.9) and (A-3.10'), we get the same differential equation (A-3.12)

i.e.

$$m'(1 - \frac{2m}{r} + \frac{e^2}{r^2}) = f(m)$$

The values of $\nu$ and $\lambda$ given by (A-3.9) and (A-3.10) with $\phi = -\frac{e^2}{r}$ satisfy the differential equation (A-2.25).

Thus we have solved all the field equations. The line-element describing the external field of a radiating charged particle is

$$ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1} dr^2 - r^2 d\Omega^2 + \frac{m^2}{f^2} \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2$$

(A-3.14)

for $r_o \leq r \leq r_1, \quad t_o \leq t \leq t_1$

where $f(m) = m' (1 - \frac{2m}{r} + \frac{e^2}{r^2})$

From equations (A-2.11) to (A-2.15) we get the
values of the surviving components of $T^k_1$ and $F_{1k}$ as

\[
T^1_1 = \frac{\epsilon^2}{8 \pi r^4} - \frac{m'}{4 \pi r^2}, \quad T^2_2 = T^3_3 = - \frac{\epsilon^2}{8 \pi r^4}
\]

\[
T^4_4 = \frac{\epsilon^2}{8 \pi r^4} + \frac{m'}{4 \pi r^2}, \quad \frac{m'}{m'} T^4_1 = \frac{m'}{4 \pi r^2} = - \frac{m'}{m} T^1_4
\]

\[
F_{14} = -\frac{\epsilon}{r^2} \frac{m'}{m'} \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^{-1},
\]

\[
F_{12} = \sqrt{f(m)} \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} = -\frac{m'}{m} F_{24}.
\]

The radiation density $\varrho$ which is measured by an observer at rest in $(r, \theta, \phi, t)$ is defined by

\[
\varrho = v_i v^k T_{ik} - \frac{\epsilon^2}{8 \pi r^4}
\]

with \( v^1 = v^2 = v^3 = 0, \quad v^4 = \sqrt{g_{44}} \)

We thus find

\[
\varrho = \frac{m'}{4 \pi r^2}
\]

As $r$ increases the strength of both fields diminishes and at large distances from the radiating charged particle, the gravitational effect of the Coulombian field vanishes first and then that due to the field of flowing radiation vanishes.
The retarded time $u(r,t)$ is an undetermined function of $m$. Choosing $u(r,t)$ as the new coordinate defined by

$$ u' e^{-\lambda/2} + u e^{-\phi/2} = 0 $$

We can put the line-element (A-3.14) into the simple form

$$ ds^2 = (1 - \frac{2m}{r} + \frac{\mathcal{C}^2}{r^2}) \, du^2 + 2dudr - r^2 d\Omega^2 \quad (A-3.15) $$

where $\mathcal{C} = \text{constant}$. 

The above line-element (A-3.15) with $\mathcal{C} = \mathcal{C}(u)$ has been derived by Plebanski and Stachel\(^{3}\) from a purely geometrical consideration of the classification of $R_{ik}$ in spherically symmetrical fields. However it may be noted that for a radiating Reissner-Nordstrom's metric to satisfy Maxwell's equations $\frac{d\mathcal{C}}{du}$ must vanish.