CHAPTER - 2
NUMERICAL TECHNIQUE

Some of the contents of this chapter are published in
2.1 Orthogonal Collocation Method

Literally, collocation means a bunch of points. Orthogonal collocation method (OCM) means a bunch of points satisfying the orthogonality condition. It was proposed by Villadsen and Stewart (1967). OCM is based on the principle of orthogonality, not of the residual function, but of the polynomial, which vanishes at the collocation points and the approximating function is expressed in terms of series of orthogonal polynomials. The collocation points are chosen to be the zeros of orthogonal polynomials, e.g., Jacobi, Legendre, Chebyshev etc. The orthogonality property of the polynomials ensures that the zeros are real and distinct. The choice of the weight function depends upon the type of orthogonal polynomial to be chosen.

Consider an unknown trial function \( y(x) \) for linear or non linear differential equation
\[
L'(y) = 0 \quad \text{in } V \quad \text{and with linear or non linear boundary} \quad L^B(y) = 0 \quad \text{on } B.
\]

General approximation of trial function is given by
\[
y^{(n)} = y_0 + \sum_{j=1}^{N} a_j y_j
\]
where, \( y_0 \) satisfies the boundary conditions. The choice of trial functions \( y_j \) is free as long as it satisfies the given differential equation and is linearly independent.

2.1.1 Categories of OCM

Orthogonal collocation method can be classified in three categories, viz:

*Interior collocation method*: Trial function satisfies the boundary conditions identically and the function is adjusted to satisfy the differential equation at \( n \) points in \( V \).

*Boundary collocation method*: Trial function satisfies the differential equation identically and the function is adjusted to satisfy the boundary conditions at \( n \) points in \( B \).
**Mixed collocation method:** Trial function satisfies neither the differential equation nor the boundary conditions and is adjusted to satisfy both.

Certain steps of interior collocation principle for BVPs are explained hereunder, while the details are available elsewhere (Finlayson, 1980; Arora Ph.D. Thesis, 2007).

\[ y(x) = \sum_{j=1}^{N+2} b_j P_{j-1}(x) \]  
\[ (2.1) \]

where \( b_j \)'s are to be determined using the BVPs and \( P_j(x) \)'s are a class of orthogonal polynomials. For simplicity Eq. (2.1) can be written as:

\[ y(x) = \sum_{j=1}^{N+2} d_j x^{j-1} = d_1 + d_2 x + \ldots + d_{N+2} x^{N+1} \]  
\[ (2.2) \]

There are \( N + 2 \) unknown coefficients \( d_j \)'s, for their determination \( N + 2 \) conditions are required. The \( N \) conditions are obtained by making residual, \( R(x_j) \) zero at \( N \) distinct collocation points, inside the interval \( 0 < x < 1 \) and remaining two conditions are obtained by satisfying boundary conditions at \( x = 0 \) and \( x = 1 \).

Equation (2.2) and its first and second order derivatives at the collocation points can be written in the matrix form as:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_{N+2}
\end{bmatrix} = \begin{bmatrix}
  1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^{N+1} \\
  1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^{N+1} \\
  \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
  1 & x_{N+2} & x_{N+2}^2 & x_{N+2}^3 & \ldots & x_{N+2}^{N+1}
\end{bmatrix} \begin{bmatrix}
  d_1 \\
  d_2 \\
  \vdots \\
  d_{N+2}
\end{bmatrix} 
\]  
\[ (2.3) \]

\[
\begin{bmatrix}
  y_1' \\
  y_2' \\
  \vdots \\
  y_{N+2}'
\end{bmatrix} = \begin{bmatrix}
  0 & 1 & 2x_1 & 3x_1^2 & \ldots & (N+1)x_1^{N} \\
  0 & 1 & 2x_2 & 3x_2^2 & \ldots & (N+1)x_2^{N} \\
  \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
  0 & 1 & 2x_{N+2} & 3x_{N+2}^2 & \ldots & (N+1)x_{N+2}^{N}
\end{bmatrix} \begin{bmatrix}
  d_1 \\
  d_2 \\
  \vdots \\
  d_{N+2}
\end{bmatrix} 
\]  
\[ (2.4) \]
\[ y'' = Dd \sim \begin{bmatrix} y_1'' \\ y_2'' \\ \vdots \\ y_{N+2}'' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 6x_1 & \ldots & N(N + 1)x_1^{N-1} \\ 0 & 0 & 2 & 6x_2 & \ldots & N(N + 1)x_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2 & 6x_{N+2} & \ldots & N(N + 1)x_{N+2}^{N-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N+2} \end{bmatrix} \quad (2.5) \]

The matrices \( Q, C \) and \( D \) are of order \((N + 2) \times (N + 2)\). The derivatives \( y' \) and \( y'' \) can be expressed in terms of \( y \) by substituting \( d = Q^{-1}y \), as follows:

\[ y' = \left( CQ^{-1} \right)y \equiv A \ y \quad (2.6) \]

\[ y'' = \left( DQ^{-1} \right)y \equiv B \ y \quad (2.7) \]

These expressions can be written at \( j^{th} \) collocation point as:

\[ y'_j = A_{1j}y_1 + A_{2j}y_2 + \ldots + A_{N+2j}y_{N+2} = \sum_{j=1}^{N+2} A_{jj}y_j \quad (2.8) \]

\[ y''_j = B_{1j}y_1 + B_{2j}y_2 + \ldots + B_{N+2j}y_{N+2} = \sum_{j=1}^{N+2} B_{jj}y_j \quad (2.9) \]

Interior collocation points can be taken as the roots of orthogonal polynomials like Jacobi, Legendre, Chebyshev etc. The other two points are 0 and 1. For example, shifted Legendre polynomial for \( N = 2 \), can be written as:

\[ P_2 = 0 \Rightarrow 6x^2 - 6x + 1 = 0 \quad (2.10) \]

Collocation points are: \( x = 0.2113248654, \ 0.7886751346 \).

Accordingly, the discretization matrices can be given as:

\[
A = \begin{bmatrix}
-7.000000 & 8.1961524 & -2.1961524 & 1.000000 \\
-2.7320508 & 1.7320508 & 1.7320508 & -0.7320508 \\
0.7320508 & -1.7320508 & -1.7320508 & 2.7320508 \\
-1.000000 & 2.1961524 & -8.1961524 & 7.000000
\end{bmatrix}
\]
The differential equation to be solved is discretized by replacing the derivatives in terms of $A$ and $B$. The differential equation is reduced into a system of algebraic equations (AEs) or differential algebraic equations (DAEs). The resulting system of AEs can be solved by Gauss Seidel, Gauss elimination or multivariable Newton Raphson methods while the system of DAEs can be solved using MATLAB, MATHEMATICA, C++, GEAR or DASSL package etc.

2.1.2 Practical Difficulty with OCM

With the increase in number of collocation points near to 10 or more, the numerical value of collocation points becomes very small, as these points are lying within the interval 0 to 1, this results in the singularity of matrix $y$. In such a situation orthogonal collocation method fails because the matrices $A$ and $B$ can not be determined. To overcome this problem, Prenter (1975) has proposed that trial function can be discretized using Hermite interpolation polynomial.

de Boor and Schwartz (1973) and de Boor (2001) pointed out that orthogonal collocation yields optimal order of accuracy for the error. They have shown that the use of piecewise polynomials is more effective in representing the solutions of the differential equations than pure polynomials, i.e., it is more effective to fit polynomials to smaller segments of the underlying intervals / regions of the solution than to fit polynomials to the entire domain of the solution.

The rapid development of Hermite polynomials is primarily due to their great usefulness in applications. Hermite polynomials possess many nice structural properties as well as excellent approximation powers. Hermite functions have many applications in the
numerical solution of a variety of problems in applied mathematics and engineering. Some of them are data fitting, function approximation, integro-differential equations, optimal control problems, computer-aided geometric design, wavelets etc. Programs based on Hermite functions have found their way in computer applications.

This study will focus on solution of differential equations using cubic Hermite collocation method, in which orthogonal collocation method is associated with the finite element method and cubic Hermite is used as a basis function. The coefficients of the basis functions are easily chosen so that the solution and its first derivative are automatically continuous at the boundary of the elements and in addition to being smoother. This fact reduces the number of equations nearly by one third (Finlayson, 1980) and consequently computational cost is reduced.

2.2 Description of the Cubic Hermite Collocation Method

The cubic Hermite interpolant of the function $f$ relative to the partition $b_1 = x_1 < x_2 < ... < x_{N+1} = b_2$ is a function $s$ that satisfies:

- on each subinterval $[x_j, x_{j+1}]$, $s$ coincides with a cubic polynomial $s_j(x)$,
- $s$ interpolates $f$ and $f'$ at $x_1, x_2, ..., x_{N+1}$,
- $s$ is continuous on $[b_1, b_2]$,
- $s'$ is continuous on $[b_1, b_2]$.

The cubic Hermite Interpolation of $f$ and its first derivative at $x = x_j$ requires that:

$$s_j(x_j) = f(x_j) \text{ and } s'_j(x_j) = f'(x_j).$$

Combining the continuity of $s$ and $s'$ at $x = x_{j+1}$ with interpolation of $f$ and its first derivative at $x = x_{j+1}$, one gets:

$$s_j(x_{j+1}) = s_{j+1}(x_{j+1}) = f(x_{j+1}) \text{ and } s'_j(x_{j+1}) = s'_{j+1}(x_{j+1}) = f'(x_{j+1}).$$
Hence, \(s_j(x)\) is a third degree polynomial that interpolates both \(f\) and \(f'\) at \(x = x_j\) and at \(x = x_{j+1}\). The four constraints imposed on the structure of a cubic Hermite polynomial give rise to four fundamental polynomials, which guarantees its existence and are defined below:

\[
s_j(x) = \sum_{j=1}^{N+1} \left[ H_{2j-1}(x)f(x_j) + H_{2j}(x)f'(x_j) \right]
\]

(2.11)

where,

\[
H_{2j-1}(x) = \begin{cases} 
\left( \frac{x-x_{j-1}}{h_{j-1}} \right)^2 \left( 3 - \frac{2(x-x_{j-1})}{h_{j-1}} \right); & x \in [x_{j-1}, x_j] \\
\left( 1 - \frac{x-x_j}{h_j} \right)^2 \left( 1 - \frac{2(x-x_j)}{h_j} \right); & x \in [x_j, x_{j+1}] \\
0; & \text{otherwise}
\end{cases}
\]

(2.12)

\[
H_{2j}(x) = \begin{cases} 
-h_{j-1} \left( \frac{x-x_{j-1}}{h_{j-1}} \right)^2 \left( 1 - \frac{x-x_{j-1}}{h_{j-1}} \right); & x \in [x_{j-1}, x_j] \\
h_j \left( 1 - \frac{x-x_j}{h_j} \right)^2 \left( \frac{x-x_j}{h_j} \right); & x \in [x_j, x_{j+1}] \\
0; & \text{otherwise}
\end{cases}
\]

(2.13)

The polynomials and their first derivatives assumes the value either 0 or 1 at the prescribed grid points \(x_j\)'s. The grid points \(x_j\)'s are often called the ‘knots’ of the piecewise polynomial since they are points where polynomials are ‘tied together’. The graphical appearance of \(H\)'s are given in Figure 2.1.

Similarly, the Hermite interpolant in 2-D is defined by \(H_j'(u,v) = H_i(u)H_j(v)\), where \(H_i(u)\) and \(H_j(v)\) are Hermite basis functions in 1-D.
The global variable $x$ varies in the $k^{th}$ interval, where $k = 1,2,\ldots,N$, as shown in Figure 2.2. Before applying the orthogonal collocation with in $k^{th}$ interval, a new variable $u = (x - x_k) / h_k$ is introduced in such a way that as $x$ varies from $x_k$ to $x_{k+1}$, $u$ varies from 0 to 1. Orthogonal collocation is applied on the variable $u$ within the $k^{th}$ interval, as shown in Figure 2.2. The double calculation at the mesh points is avoided by using the principle of continuity.

Figure 2.2: Subdivision of mesh points on the global domain. The four coefficients in each $k$ element are estimated by using four collocation points; two boundary points ($u = 0$ and 1) and two interior collocation points $u_1$ and $u_2$ are the zeros of shifted Chebyshev or shifted Legendre polynomials.
The trial function in the \(k\)th element can be represented as:

\[
C(u, t) = \sum_{j=1}^{N+1} a_{j, 2k-2}(t)H_j(u), \quad k = 1, 2, \ldots, N
\]  

(2.14)

where \(a\)'s are time dependent quantities to be determined.

The trial function and its derivatives at the collocation points \((u_i, s)\) can be written as:

\[
\begin{align*}
C(u, t) &= \sum_{j=1}^{N+1} a_{j, 2k-2}(t)H_j(u) \\
\frac{\partial C}{\partial u}(u, t) &= \frac{1}{h} \sum_{j=1}^{N+1} a_{j, 2k-2}(t)A_{ji} \\
\frac{\partial^2 C}{\partial u^2}(u, t) &= \frac{1}{h^2} \sum_{j=1}^{N+1} a_{j, 2k-2}(t)B_{ji}
\end{align*}
\]

(2.15)

The Hermite polynomials \((H_j)\), its first derivatives \((A_{ji})\) and second derivatives \((B_{ji})\) at the collocation points \((u_i)\) are defined as:

\[
\begin{align*}
H_1(u_i) &= (1 + 2u_i)(1 - u_i)^2, \quad H_2(u_i) = u_i(1 - u_i)^2 h_i, \\
H_3(u_i) &= u_i^2(3 - 2u_i), \quad H_4(u_i) = u_i^2(1 - u_i)h_i, \\
A_1(u_i) &= 6u_i^2 - 6u_i, \quad A_2(u_i) = (1 - 4u_i + 3u_i^2)h_i, \\
A_3(u_i) &= 6u_i - 6u_i^2, \quad A_4(u_i) = (3u_i^2 - 2u_i)h_i, \\
B_1(u_i) &= 12u_i - 6, \quad B_2(u_i) = (6u_i - 4)h_i, \\
B_3(u_i) &= 6 - 12u_i, \quad B_4(u_i) = (6u_i - 2)h_i.
\end{align*}
\]

Alternately, (Brill, 2001), the Hermite polynomials can be defined over the approximating subspace \(H_3(\pi)\) consisting of all the functions \(f(x)\) such that:

- \(f(x)\) is equal to a cubic polynomial in each subinterval,
- \(f(x)\) and \(f'(x)\) are continuous on \([b_1, b_2]\),
- \(f(x)\) satisfies the appropriate boundary conditions.
The set \( \left\{ P_j(x), Q_j(x) \right\}_{j=1}^{N+1} \) conveniently generates the approximating functions, where

\[
P_j(x) = \begin{cases} 
\left(1 + \frac{x-x_j}{h_j} \right)^2 \left(1 - \frac{2(x-x_j)}{h_j} \right) ; & x \in [x_{j-1}, x_j] \\
\left(1 - \frac{x-x_j}{h_{j+1}} \right)^2 \left(1 + \frac{2(x-x_j)}{h_{j+1}} \right) ; & x \in [x_j, x_{j+1}] \\
0; & \text{otherwise}
\end{cases}
\]

(2.16)

\[
Q_j(x) = \begin{cases} 
h_j \left(1 + \frac{x-x_j}{h_j} \right)^2 \left(1 - \frac{x-x_j}{h_j} \right) ; & x \in [x_{j-1}, x_j] \\
h_{j+1} \left(1 - \frac{x-x_j}{h_{j+1}} \right)^2 \left(1 + \frac{x-x_j}{h_{j+1}} \right) ; & x \in [x_j, x_{j+1}] \\
0; & \text{otherwise}
\end{cases}
\]

(2.17)

It is assumed that \( P_1(x) \) and \( Q_1(x) \) vanish to the left of \( x_1 \) and \( P_{N+1}(x) \), \( Q_{N+1}(x) \) vanish to the right of \( x_{N+1} \). From the basis defined above, it is observed that each function has the following properties:

- Each \( P_j(x) \) and \( Q_j(x) \) is continuous together with its first derivative on \([b_1, b_2]\).
- Each \( P_j(x) \) and \( Q_j(x) \) is a cubic polynomial in each subinterval and vanishes outside the subinterval \([x_{j-1}, x_{j+1}]\).
- For \( 1 \leq i, j \leq N+1 \), at the point \( x = x_j \), \[
\begin{align*}
P_i(x_j) &= \delta_j, & P'_i(x_j) &= 0 \\
Q_i(x_j) &= 0, & Q'_i(x_j) &= \delta_j.
\end{align*}
\]

The set \( \left\{ P_j(x), Q_j(x) \right\}_{j=1}^{N+1} \) will form a basis for the set of functions \( H_3(\pi) \). Thus, the trial function and its derivatives can be written as follows:

\[
\]
\[
c(u, t) = \sum_{j=1}^{N+1} \left[ a(u_j, t) P_j(u) + a'(u_j, t) Q_j'(u) \right],
\]
\[
\frac{\partial c(u, t)}{\partial x} = \frac{1}{h} \sum_{j=1}^{N+1} \left[ a(u_j, t) P_j'(u) + a'(u_j, t) Q_j'(u) \right],
\]
\[
\frac{\partial^2 c(u, t)}{\partial x^2} = \frac{1}{h^2} \sum_{j=1}^{N+1} \left[ a(u_j, t) P_j''(u) + a'(u_j, t) Q_j''(u) \right],
\]
\[
\frac{\partial c(u, t)}{\partial t} = \sum_{j=1}^{N+1} \left[ \frac{da(u_j, t)}{dt} P_j(u) + \frac{da'(u_j, t)}{dt} Q_j(u) \right],
\]

where \( h = h_k \) and \( a(u_j, t), a'(u_j, t) \) are unknown coefficients to be determined.

**Derivatives at collocation points:**

Expressions \( P_j(x), Q_j(x) \) and their derivatives at collocation points \( \eta_{j,i} \) can be written as follows:

\[
P_{j-1}(\eta_{j,1}) = P_j(\eta_{j,2}) = (1 - \beta)^2 (1 + 2\beta) = P_{\beta} \]
\[
P_{j-1}(\eta_{j,2}) = P_j(\eta_{j,1}) = (1 - \gamma)^2 (1 + 2\gamma) = P_{\gamma} \]
\[
Q_{j-1}(\eta_{j,1}) = -Q_j(\eta_{j,2}) = h_j \beta (1 - \beta)^2 = h_j Q_{\beta} \]
\[
Q_{j-1}(\eta_{j,2}) = -Q_j(\eta_{j,1}) = h_j \gamma (1 - \gamma)^2 = h_j Q_{\gamma} \]

\[
P_{j-1}'(\eta_{j,1}) = -P_j'(\eta_{j,2}) = -6\beta (1 - \beta) / h_j = -P_{\beta}' / h_j \]
\[
P_{j-1}'(\eta_{j,2}) = -P_j'(\eta_{j,1}) = -6\gamma (1 - \gamma) / h_j = -P_{\gamma}' / h_j \]
\[
Q_{j-1}'(\eta_{j,1}) = Q_j'(\eta_{j,2}) = 1 - 4\beta + 3\beta^2 = Q_{\beta}' \]
\[
Q_{j-1}'(\eta_{j,2}) = Q_j'(\eta_{j,1}) = 1 - 4\gamma + 3\gamma^2 = Q_{\gamma}' \]

\[
P_{j-1}''(\eta_{j,1}) = P_j''(\eta_{j,2}) = -6(1 - 2\beta) / h_j^2 = -P_{\beta}'' / h_j^2 \]
\[
P_{j-1}''(\eta_{j,2}) = P_j''(\eta_{j,1}) = -6(1 - 2\gamma) / h_j^2 = -P_{\gamma}'' / h_j^2 \]
\[
Q_{j-1}''(\eta_{j,1}) = -Q_j''(\eta_{j,2}) = -(4\beta - 6\beta) / h_j = -Q_{\beta}'' / h_j \]
\[
Q_{j-1}''(\eta_{j,2}) = -Q_j''(\eta_{j,1}) = -(4 - 6\gamma) / h_j = -Q_{\gamma}'' / h_j \]

Equations (2.22) to (2.24) are used for discretization purpose.
2.2.1 Selection of Collocation Points

In the present study, zeros of shifted Chebyshev polynomials and shifted Legendre polynomials of order 2 are taken as the collocation points for each subinterval \([x_j, x_{j+1}]\),

\[
\eta_{j,i} = \left(\frac{x_{j-1} + x_j}{2}\right) + (-1)^i \frac{h_j}{2\sqrt{3}}, \quad j = 1, 2, \ldots, N + 1, \quad i = 1, 2.
\]  

(2.25)

By shifted Legendre roots \(\beta = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\) and \(\gamma = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\). The values of \(H_j\)'s and their derivatives at shifted - Legendre roots are shown in Table 2.1.

For shifted Chebyshev polynomial \(\beta = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\) and \(\gamma = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)\).

The placement of collocation points plays an important role for achieving the optimum order of convergence (Douglas and Dupont, 1973). The effect of order of convergence on the choice of roots is discussed in the Chapter 4.

Table 2.1: The values of \(H_j\)'s and their derivatives at shifted Legendre roots

<table>
<thead>
<tr>
<th>(H_j(x_i) - H_{ji})</th>
<th>(i)</th>
<th>(j = 1)</th>
<th>(j = 2)</th>
<th>(j = 3)</th>
<th>(j = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21132486</td>
<td>0.88490018</td>
<td>0.13144585</td>
<td>h(_k)</td>
<td>0.11509982</td>
<td>0.03522081</td>
</tr>
<tr>
<td>0.78867513</td>
<td>0.11509982</td>
<td>0.03522081</td>
<td>h(_k)</td>
<td>0.88490018</td>
<td>-0.13144585</td>
</tr>
<tr>
<td>0.21132486</td>
<td>-1.00000000</td>
<td>0.28867513</td>
<td>h(_k)</td>
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</tr>
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<td>-0.28867513</td>
<td>h(_k)</td>
<td>1.00000000</td>
<td>0.28867513</td>
</tr>
<tr>
<td>0.21132486</td>
<td>-3.46410162</td>
<td>-2.73205081</td>
<td>h(_k)</td>
<td>3.46410162</td>
<td>-0.73205081</td>
</tr>
<tr>
<td>0.78867513</td>
<td>3.46410162</td>
<td>0.73205081</td>
<td>h(_k)</td>
<td>-3.46410162</td>
<td>2.73205081</td>
</tr>
</tbody>
</table>
2.3 Review of Numerical Technique

Douglas and Dupont (1973) analyzed the convergence of quasilinear parabolic equations by the method of collocation in which the solution was approximated by Hermite piecewise cubic polynomial in the space and time domain.

Chawla, et al. (1975) applied collocation method using cubic Hermite splines on a two region nonlinear transient heat conduction problem and compared with the finite difference method. It was demonstrated that only a small number of equations are needed to produce the desirable accuracy. The method has the desirable characteristic of an analytical method.

Prenter and Russell (1976) presented an $O(h^4)$ collocation method for solving elliptic partial differential equations on a unit square and proof of convergence was also given. The method was shown to be comparable with the Ritz-Galerkin method.

Archer (1977) proposed a modified version of the usual cubic spline collocation method and analyzed the quasilinear parabolic problems. Continuous time estimates of order $O(h^4)$ were obtained, via arguments based on certain discrete inner products, for a uniform mesh and sufficiently smooth problems.

Archer and Diaz (1978) proposed and analyzed the modified collocation methods for second order two point boundary value problems. The methods require that the approximate solution, a piecewise Hermite cubic polynomial on a uniform mesh, satisfy certain perturbation of the differential equation at the collocation points. Optimal rates of convergence were obtained and some super convergence estimates for the derivatives were established. Posteriori corrections dependent upon the computed solution and super convergent approximations to the derivatives were also studied.


Jumarhon, Amini and Chen (2000) carried out various numerical experiments to study the convergence of the Hermite collocation method by using high order polyharmonic splines and Wendland’s radial basis functions.

Edoh, Russell and Sun (2000) applied Hermite collocation method to solve nonlinear partial differential equations with periodic boundary which arise in investigating invariant tori for dynamical systems. The numerical scheme was tested on two linear problems for stability and convergence analysis.

Dyksen and Lynch (2000) presented decoupling technique for solving the linear systems arising from Hermite cubic collocation solutions of boundary value problems for both Dirichlet and Neumann boundary conditions. The technique decouples the discretized system into two parts, one with a tridiagonal system and the other with the identity matrix. The theoretical work was validated with a number of experimental results.

Brill (2001) presented the numerical solution of PDEs in two spatial dimensions via Hermite collocation. The resulting system was solved by Krylov subspace method, for two preconditioners based on Gauss-Seidel red-black (RBGS) ordering and other based on block incomplete LU factorization (ILU). It was shown that RBGS preconditioner is superior than ILU preconditioner.

Bialecki and Fairweather (2001) has formulated, analysed and implemented orthogonal spline collocation, at Gauss points, for the numerical solution of parabolic, hyperbolic and Schrodinger-type equations.
Lang and Sloan (2002) presented the Hermite collocation solution of near-singular problems using numerical coordinate transformations based on adaptivity. Numerical results were presented for steady and unsteady problems.

Leao and Rodrigues (2004) studied transient and steady-state models for simulated moving bed processes in which Hermite cubic polynomials were used for the discretization of the spatial variable.

Danumjaya and Pani (2005) formulated second-order splitting with orthogonal cubic Hermite collocation method and analysed the extended Fisher–Kolmogorov equation. With the help of Lyapunov functional, a bound for maximum norm was derived for the semidiscrete solution. Optimal error estimates were established for semidiscrete case.

Baroth et al. (2006) proposed a stochastic finite element method for nonlinear mechanical systems whose uncertain parameters were modeled as random variables. The method was based on a Gaussian standardization of the problem and on an Hilbertian approximation of the nonlinear mechanical function using Hermite polynomials.

Rocca and Power (2008) applied double boundary collocation Hermitian approach for the solution of steady convection diffusion problems. The results obtained with this method were characterized by a higher precision especially for the prediction of the fluxes at the boundaries.

Krajnc (2009) studied geometric Hermite interpolation. Three data points and three tangent directions were interpolated per polynomial segment and sufficient conditions for the existence of such a $G^1$ spline were determined. The existence requirements were based only upon geometric properties of data.

Peirce (2010) presented a novel algorithm based on cubic Hermite interpolation to solve the coupled integro-partial differential equations governing a hydraulic fracture
propagating in a plane strain. The influence functions needed to implement the algorithm were evaluated explicitly and were provided for completeness. The convergence rate of the cubic Hermite scheme was determined by asymptotic expansion as $O(h^4)$.

Sewell (2010) solved PDEs in non-rectangular 3D regions using a collocation finite element method, with tricubic Hermite basis functions and an automatic global coordinate transformation.


Orsini, Power and Lees (2011) used Hermite radial basis function for multi-zones problems. Hermite interpolation was exploited to apply multiple flux continuities for the cases where more than two sub-domains were converging at the same point.

Chen and Xiang (2011) applied wavelets based on Hermite cubic splines for solving singularly perturbed convection-dominated diffusion equation. The advantages of the method were explained. To improve the accuracy of singular areas, wavelets were configured hierarchically for solving algebraic equations.

Lamata et al. (2011) studied in-silico continuum simulations of organ and tissue scale physiology using cubic Hermite meshes because it provide a smooth representation of anatomy that is well-suited for simulating large deformation mechanics.

Kazem, Rad and Parand (2012) used two methods based on radial basis functions to approximate the solution of Fokker–Planck equation. The first was based on the Kansas approach and the other one on the Hermite interpolation. The errors show that the Hermite collocation approach gave more accurate results than the Kansas approach.

Luo and Du (2013) presented a two-level method using cubic Hermite interpolation, for the numerical solutions of one-dimensional telegraph equations. The accuracy of the
scheme was found to be of order four. Scheme was also proved to be unconditionally stable. Numerical experiments were carried out to illustrate efficiency of the method.