Chapter 4

A New Mathematical Approach for the Solution of Optimization Problem with Cubic Objective Function

4.1 Introduction

This Chapter presents a dynamic programming based approach for optimization problem having cubic objective function with equality and inequality constraints. In this approach, computations in dynamic programming are done recursively, that is, the optimum solution of one sub-problem is used as an input to the next sub-problem. The feasibility of the constraints are accounted when we move from one sub-problem to next sub-problem.

As competition increases in the electricity supply industry, generating companies try to further improve the operating efficiency of their power plants. While the application of mathematical optimization techniques has a long history in power plant operation, significant improvements can still be achieved through a more realistic optimization modelling and the application of most solution techniques. The practical optimization problems of economic power dispatch in thermal plants and chiller plants are realistically modeled as the cubic equation. Generation costs are affected by the accuracy of cost functions. Most conventional methods

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in the literature used quadratic equations to express their characteristic curves (Bard, 1988). Instead of a second order approximate polynomial, a more accurate third order cost function is assigned by adding one more coefficient. The existing conventional methods fail to impose generation limits in economic power generation scheduling problem and find difficulty in dealing with cubic cost function which reflects nonlinearity of the actual generator response. Due to the complexity of solving optimization with cubic optimization problem, the problem is approximated to quadratic optimization problem. This rough approximation of the optimization problem formulation makes the solution deviated from the optimality.

Theerathamalai and Maheswarapu (2010) have developed Lambda logic based algorithm for the solution of economic dispatch of generators with cubic function. This method has two stages, at first stage pre-prepared power demand data is to be prepared and at second stage power generation of each generator is calculated using quadratic interpolation for the specified load demand. Chang et al., (2010) have implemented gradient search method for the solution of economic dispatch of chiller plant. Gradient search techniques always begin with a feasible solution, and then follow the largest path of changing gradient of objective function to seek the optimal solution. In the process of establishing new operating point, objective function is monotonically changing, thus, the new solution is sought to be better than the former one. After the required number of iterations, the optimal solution is obtained. But the high iteration steps are flaws to gradient method.

In this Chapter, we have formulated a model to study the optimization problem with cubic optimization problem. Since the cubic equation model has higher accuracy, this study is used to establish the solution for economic power dispatch of thermal plant and the power consumption model of chiller plant.
4.2 Problem Formulation

The mathematical formulation of the optimization problem with cubic objective function is given by

\[
\text{Minimize } f(x) = \sum_{i=1}^{n} a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i \tag{4.1}
\]

subject to

\[
\sum_{i=1}^{n} x_i = X_T, \tag{4.2}
\]

\[
x_{i}^{\text{min}} \leq x_i \leq x_{i}^{\text{max}}, \tag{4.3}
\]

\[
x_i \geq 0. \tag{4.4}
\]

where, \(a_i, b_i, c_i\) and \(d_i\) are the coefficients of the variable \(x_i\), \(X_T\) is the sum of the decision variables, \(x_{i}^{\text{min}}\) and \(x_{i}^{\text{max}}\) are the lower and upper bounds of the decision variable \(x_i\).

4.3 Method of Solution

A serial multistage decision process is diagrammatically represented in Figure 3.1. In this process, a number of single stage decision processes are connected in series such that the output of one stage is given to the input of next stage. The stages 1, 2, ..., \(n\) are connected in decreasing order.

The output of the stage 1 is equal to 0, i.e., \(S_1 - x_1 = 0\). The optimum value of objective function for the first sub-problem is given by

\[
f_1^*(S_1) = \min(a_1 x_1^3 + b_1 x_1^2 + c_1 x_1 + d_1) \tag{4.5}
\]

Now, consider the second sub-problem by combining the last two stages together i.e, stage 1 and stage 2. Let \(f_2^*(S_2)\) represents the optimum value of objective function for second sub-problem with the specified value of the input \(S_2\),

\[
f_2^*(S_2) = \min(a_2 x_2^3 + b_2 x_2^2 + c_2 x_2 + d_2 + f_1^*(S_1)) \tag{4.6}
\]
Substituting equation (4.5) in equation (4.6) gives

\[ f_2^*(S_2) = \min(a_2x_2^3 + b_2x_2^2 + c_2x_2 + d_2 + a_1x_1^3 + b_1x_1^2 + c_1x_1 + d_1) \] (4.7)

The output of the stage 2 is given by

\[ S_1 = S_2 - x_2 \]

here \( S_1 = x_1 \),

\[ \therefore x_1 = S_2 - x_2 \] (4.8)

Substituting equation (4.8) in equation (4.7), we get

\[ f_2^*(S_2) = \min(a_2x_2^3 + b_2x_2^2 + c_2x_2 + d_2 + a_1(S_2 - x_2)^3 + b_1(S_2 - x_2)^2 + c_1(S_2 - x_2) + d_1) \] (4.9)

Differentiating the equation (4.9) with respect to \( x_2 \) and equating to zero, we get

\[ 3a_2x_2^2 + 2b_2x_2 + c_2 - 3a_1(S_2 - x_2)^2 - 2b_1(S_2 - x_2) - c_1 = 0 \]

Solving, we get

\[ x_2 = \frac{-(b_2 + b_1 + 3a_1S_2)}{3(a_2 - a_1)} \pm \frac{\sqrt{9a_1a_2S_2^2 + 6(a_1b_2 + a_2b_1)S_2 + (b_1 + b_2)^2 - 3(a_2 - a_1)(c_2 - c_1)}}{3(a_2 - a_1)} \] (4.10)

Substituting equation (4.10) in equation (4.8) gives

\[ x_1 = \frac{(b_1 + b_2 + 3a_2S_2)}{3(a_2 - a_1)} \mp \frac{\sqrt{9a_1a_2S_2^2 + 6(a_1b_2 + a_2b_1)S_2 + (b_1 + b_2)^2 - 3(a_2 - a_1)(c_2 - c_1)}}{3(a_2 - a_1)} \] (4.11)

Substituting equations (4.10) and (4.11) in equation (4.7) and then simplifying, we get

\[
\begin{align*}
f_2^*(S_2) &= \min\left(\frac{S_2^3a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2} + \frac{S_2^2(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} + \frac{S_2^2}{3a_2 - (a_1)^2} \left[ \frac{2(b_1 + b_2)(a_1b_2 + a_2b_1)}{3a_2 - (a_1)^2} + \frac{a_2c_1 - a_1c_2}{(a_2 - a_1)} \right] + \frac{2(b_1 + b_2)^3}{27(a_2 - a_1)^2} + \frac{2(b_1 + b_2)(c_1 - c_2)}{3(a_2 - a_1)} + \frac{d_1 + d_2}{d_2} \right. \\
&\left. + \frac{2[9a_1a_2S_2^2 + 6(a_1b_2 + a_2b_1)S_2 + F_1]^{3/2}}{27(a_2 - a_1)^2} \right)
\end{align*}
\] (4.12)
where, \( F_1 = (b_1 + b_2)^2 - 3(a_2 - a_1)(c_2 - c_1) \).

The third sub-problem is obtained by combining the last three stages together i.e., stage 1 to stage 3. The optimum value of objective function for third sub-problem is given by

\[
f_3^*(S_3) = \min(a_3x_3^3 + b_3x_3^2 + c_3x_3 + d_3 + f_2^*(S_2))
\]

where,

\[
f_3^*(S_3) = \min(a_3x_3^3 + b_3x_3^2 + c_3x_3 + d_3 + \frac{S_3^3a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2} + \frac{S_2^3(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} +
\]

\[
S_2 \left[ \frac{2(b_1 + b_2)(a_1b_2 + a_2b_1)}{3(a_2 - a_1)^2} + \frac{a_2c_1 - a_1c_2}{(a_2 - a_1)} \right] + \frac{2(b_1 + b_2)^3}{27(a_2 - a_1)^2} +
\]

\[
\frac{(b_1 + b_2)(c_1 - c_2)}{3(a_2 - a_1)} + d_1 + d_2 \equiv \frac{2[9a_1a_2S_2^2 + 6(a_1b_2 + a_2b_1)S_2 + F_1]^{3/2}}{27(a_2 - a_1)^2}
\]

(4.14)

From the output of stage 3, becomes

\[
S_2 = S_3 - x_3
\]

(4.15)

Substituting equation (4.15) in (4.14), we get

\[
f_3^*(S_3) = \min(a_3x_3^3 + b_3x_3^2 + c_3x_3 + d_3 + \frac{(S_3 - x_3)^3a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2} +
\]

\[
\frac{(S_3 - x_3)^2(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} +
\]

\[
(S_3 - x_3) \left[ \frac{2(b_1 + b_2)(a_1b_2 + a_2b_1)}{3(a_2 - a_1)^2} + \frac{a_2c_1 - a_1c_2}{(a_2 - a_1)} \right] +
\]

\[
\frac{2(b_1 + b_2)^3}{27(a_2 - a_1)^2} + \frac{(b_1 + b_2)(c_1 - c_2)}{3(a_2 - a_1)} + d_1 + d_2
\]

\[
\equiv \frac{2[9a_1a_2(S_3 - x_3)^2 + 6(a_1b_2 + a_2b_1)(S_3 - x_3) + F_1]^{3/2}}{27(a_2 - a_1)^2}
\]

(4.16)

Differentiating the equation (4.16) with respect to \( x_3 \) and equating to zero, we get

\[
3a_3x_3^2 + 2b_3x_3 + c_3 = \frac{3(S_3 - x_3)^2a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2} - \frac{2(S_3 - x_3)(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} -
\]

\[
\left[ \frac{2(b_1 + b_2)(a_1b_2 + a_2b_1)}{3(a_2 - a_1)^2} + \frac{a_2c_1 - a_1c_2}{(a_2 - a_1)} \right] \equiv \frac{1/9[-18a_1a_2(S_3 - x_3) - 6(a_1b_2 + a_2b_1)]}{(a_2 - a_1)^2} \times
\]

\[
[9a_1a_2(s_3 - x_3)^2 + 6(a_1b_2 + a_2b_1)(S_3 - x_3) + F_1]^{1/2} = 0
\]

(4.17)
By expanding the root function in equation (4.17) and neglecting higher order terms gives

\[\pm \left[ \frac{-2a_1a_2(S_3 - x_3) - 2/3(a_1b_2 - a_2b_1)}{(a_2 - a_1)^2} \right] \times \right] \]

\[F_1^{1/2}(1 + 1/2) \left[ \frac{9a_1a_2(S_3 - x_3)^2 + 6(a_1b_2 + a_2b_1)(S_3 - x_3)}{F_1} \right] - \]

\[1/8 \left[ \frac{9a_1a_2(S_3 - x_3)^2 + 6(a_1b_2 + a_2b_1)(S_3 - x_3)}{F_1} \right]^2 = 0 \] (4.18)

Simplifying, we get

\[\pm \frac{2a_1a_2(S_3 - x_3)F_1^{1/2}}{(a_2 - a_1)^2} \pm \frac{2(a_1b_2 + a_2b_1)F_1^{1/2}}{3(a_2 - a_1)^2} \pm \frac{3(a_1b_2 + a_2b_1)^3(S_3 - x_3)^2}{F_1^{3/2}(a_2 - a_1)^2} \pm \]

\[\frac{9a_1a_2(a_1b_2 + a_2b_1)(S_3 - x_3)^2 + 2(a_1b_2 + a_2b_1)^2(S_3 - x_3)}{F_1^{1/2}(a_2 - a_1)^2} = 0 \] (4.18)

Substituting equation (4.18) in equation (4.17), we get

\[3a_3x_3^2 + 2b_3x_3 + c_3 - 3(S_3^2 + x_3^2 - 2S_3x_3) \left[ \frac{a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2} \right] - \]

\[2(S_3 - x_3) \left[ \frac{(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_1 - a_2)^2} \right] - \frac{2}{3} \left[ \frac{(b_1 + b_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} \right] \]

\[- \left[ \frac{a_2c_1 - a_1c_2}{a_2 - a_1} \right] \pm \frac{2a_1a_2(S_3 - x_3)F_1^{1/2}}{(a_2 - a_1)^2} \pm \frac{2}{3} \frac{(a_1b_2 + a_2b_1)F_1^{1/2}}{(a_2 - a_1)^2} \pm \]

\[\frac{1}{F_1^{1/2}(a_2 - a_1)^2} \left[ 9(a_1a_2)(a_1b_2 + a_2b_1)(S_3^2 + x_3^2 - 2S_3x_3) + 2(a_1b_2 + a_2b_1)^2(S_3 - x_3) \right] \pm \]

\[\frac{1}{F_1^{3/2}(a_2 - a_1)^2} \left[ 3(a_1b_2 + a_2b_1)^3(S_3^2 + x_3^2 - 2S_3x_3) \right] \]
From the equation (4.19), the expression for \( x_3 \) is given by

\[
x_3 = \frac{-\beta_3 \pm \sqrt{\beta_3^2 - 4\alpha_3 \gamma_3}}{2\alpha_3}
\]  (4.20)

where,

\[
\alpha_3 = 3(a_3 - \frac{a_1a_2(a_1 + a_2)}{(a_2 - a_2)^2}) \pm \frac{3a_1a_2(a_1b_2 + a_2b_1)}{F_{1/2}^1(a_2 - a_1)^2} \mp \frac{(a_1b_2 + a_2b_1)^3}{F_{3/2}^1(a_2 - a_1)^2}
\]

\[
\beta_3 = 2(b_3 + \frac{3S_3a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2}) + \frac{(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} \pm \frac{a_1a_2F_{1/2}^1}{(a_2 - a_1)^2} \mp \frac{9a_1a_2(a_1b_2 + a_2b_1)S_3 + (a_1b_2 + a_2b_1)^2}{F_{1/2}^1(a_2 - a_1)^2} \pm \frac{3S_3^2(a_1b_2 + a_2b_1)^3}{F_{3/2}^1(a_2 - a_1)^2}
\]

\[
\gamma_3 = \frac{3S_3^2a_1a_2(a_1 + a_2)}{(a_2 - a_1)^2} - \frac{2S_3(a_1 + a_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} \pm \frac{2(b_1 + b_2)(a_1b_2 + a_2b_1)}{(a_2 - a_1)^2} \mp \frac{(a_2c_1 - a_1c_2)}{(a_2 - a_1)^2} \pm \frac{[2a_1a_2S_3 + 2(a_1b_2 + a_2b_1)]}{F_{1/2}^1(a_2 - a_1)^2} \mp \frac{9a_1a_2(a_1b_2 + a_2b_1)S_3^2 + 2(a_1b_2 + a_2b_1)^2S_3}{F_{1/2}^1(a_2 - a_1)^2} \pm \frac{3S_3^2(a_1b_2 + a_2b_1)^3}{F_{3/2}^1(a_2 - a_1)^2}
\]

Simplifying the equation (4.20), we get

\[
x_3 = \frac{-(b_3 + Q_1 + 3S_3P_1) \pm \sqrt{9S_3^2a_3P_1 + 6S_3(b_3P_1 + a_3Q_1) + F_2}}{3(a_3 - P_1)}
\]  (4.21)
where,
\[
P_1 = \frac{a_1 a_2(a_1 + a_2)}{(a_2 - a_1)^2} + 3a_1 a_2(a_1 b_2 + a_2 b_1) + \frac{(a_1 b_2 + a_2 b_1)^3}{F_1^{1/2}(a_2 - a_1)^2}
\]
\[
Q_1 = \frac{(a_1 + a_2)(a_1 b_2 + a_2 b_1)}{(a_2 - a_1)^2} + \frac{a_1 a_2 F_1^{1/2}}{(a_2 - a_1)^2} + \frac{(a_1 b_2 + a_2 b_1)^2}{F_1^{1/2}(a_2 - a_1)^2}
\]
\[
F_2 = (b_3 + Q_1)^2 - 3(a_3 - P_1)(c_3 - R_1)
\]

here
\[
R_1 = \frac{2(b_1 + b_2)(a_1 b_2 + a_2 b_1)}{3(a_2 - a_1)^2} + \frac{a_2 c_1 - c_2 a_1}{(a_2 - a_1)} + \frac{2(a_1 b_2 + a_2 b_1) F_1^{1/2}}{3(a_2 - a_1)^2}
\]

The fourth sub-problem is formulated by combining the last four stages together.
The optimum value of objective function for the fourth sub-problem is given by
\[
f^*_4(S_4) = \min(a_4 x_4^3 + b_4 x_4^2 + c_4 x_4 + d_4 + f^*_3(S_3))
\]

Substituting equations (4.21) and (4.15) in equation (4.14) gives
\[
f^*_3(S_3) = \min(S_3^3)\left[\frac{2P_1^3}{(a_3 - P_1)^2} + \frac{3P_1^2}{(a_3 - P_1)} + P_1\right] +
\]
\[
S_3^2\left[\frac{2P_1^2(b_3 + Q_1)}{(a_3 - P_1)^2} + \frac{P_1(b_3 + Q_1)}{(a_3 - P_1)} + \frac{2P_1 Q_1}{(a_3 - P_1)} + Q_1\right] +
\]
\[
S_3\left[\frac{2(b_3 + Q_1)^2 P_1}{3(a_3 - P_1)^2} + \frac{2Q_1(b_3 + Q_1)}{3(a_3 - P_1)} - \frac{P_1(c_3 - R_1)}{3(a_3 - P_1)} + R_1\right] +
\]
\[
\frac{(2/27)(9S_3^2 a_3 P_1 + 6S_3(b_3 P_1 + a_3 Q_1) + F_2)^{3/2}}{3(a_3 - P_1)^2} + \frac{2(b_3 + Q_1)^2}{27(a_3 - P_1)^2} -
\]
\[
\frac{(b_3 + Q_1)(c_3 - R_1)}{3(a_3 - P_1)} + d_1 + d_2 + d_3
\]

Substituting equation (4.23) in equation (4.22) and replacing \(S_3\) by \(S_4 - x_4\), we get
\[
f^*_4(S_4) = \min(a_4 x_4^3 + b_4 x_4^2 + c_4 x_4 + d_4 + (S_4 - x_4)^3)\left[\frac{2P_1^3}{(a_3 - P_1)^2} + \frac{3P_1^2}{(a_3 - P_1)} + P_1\right] +
\]
\[
(S_4 - x_4)^2\left[\frac{2P_1^2(b_3 + Q_1)}{(a_3 - P_1)^2} + \frac{P_1(b_3 + Q_1)}{(a_3 - P_1)} + \frac{2P_1 Q_1}{(a_3 - P_1)} + Q_1\right] +
\]
\[
(S_4 - x_4)\left[\frac{2(b_3 + Q_1)^2 P_1}{3(a_3 - P_1)^2} + \frac{2Q_1(b_3 + Q_1)}{3(a_3 - P_1)} - \frac{P_1(c_3 - R_1)}{3(a_3 - P_1)} + R_1\right] +
\]
\[
\frac{(2/27)(9S_3^2 a_3 P_1 + 6S_3(b_3 P_1 + a_3 Q_1) + F_2)^{3/2}}{3(a_3 - P_1)^2} + \frac{2(b_3 + Q_1)^2}{27(a_3 - P_1)^2} -
\]
\[
\frac{(b_3 + Q_1)(c_3 - R_1)}{3(a_3 - P_1)} + d_1 + d_2 + d_3
\]

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Differentiating the equation (4.24) with respect to \( x_4 \) and equating to zero, we get

\[
3a_4x_4^2 + 2b_4x_4 + c_4 - 3(S_4 - x_4)^2 \left[ \frac{2P_1^3}{(a_3 - P_1)^2} + \frac{3P_1^2}{(a_3 - P_1)} + P_1 \right] - \\
2(S_4 - x_4) \left[ \frac{2P_1^2(b_3 + Q_1)}{(a_3 - P_1)^2} + \frac{P_1(b_3 + Q_1)}{(a_3 - P_1)} + \frac{2P_1Q_1}{(a_3 - P_1)} + Q_1 \right] - \\
\left[ \frac{2(b_3 + Q)^2P_1}{3(a_3 - P_1)^2} + \frac{2Q_1(b_3 + Q_1)}{3(a_3 - P_1)} - \frac{P_1(c_3 - R_1)}{3(a_3 - P_1)} + R_1 \right] = 0 \\
(1/9)[-18a_3P_1(S_4-x_4)-6(b_3P_1+a_3Q_1)]\left[a_3P_1(S_4-x_4)^2+6(b_3P_1+a_3Q_1)(S_4-x_4)+F_2\right]^{1/2} = 0
\] (4.25)

By simplifying the last term of equation (4.25) and neglecting the higher order terms, we have,

\[
\pm \frac{2a_3P_1(S_4 - x_4)F_2^{1/2}}{(a_3 - P_1)^2} \pm \frac{2(a_3Q_1 + b_3P_1)F_2^{1/2}}{3(a_3 - P_1)^2} \pm \frac{3(b_3P_1 + a_3Q_1)^3(S_4 - x_4)^2}{F_2^{3/2}(a_3 - P_1)^2} + \\
\left[ \frac{a_3R_1 - P_1c_3}{(a_3 - P_1)} \right] \pm \left[ \frac{2a_3P_1(S_4 - x_4)F_2^{1/2}}{(a_3 - P_1)^2} \right] \pm \left[ \frac{2(a_3Q_1 + b_3P_1)F_2^{1/2}}{3(a_3 - P_1)^2} \right] \pm \\
\frac{9a_3P_1(b_3P_1 + a_3Q_1)(S_4^2 + x_4^2 - 2S_4x_4)}{F_2^{3/2}(a_3 - P_1)^2} + \frac{2(b_3P_1 + a_3Q_1)^2(S_4 - x_4)}{F_2^{1/2}(a_3 - P_1)^2} \pm \\
\frac{3(b_3P_1 + a_3Q_1)^3(S_4^2 + x_4^2 - 2S_4x_4)}{F_2^{3/2}(a_3 - P_1)^2} = 0
\] (4.26)

Substituting equation (4.26) in equation (4.25) and simplifying yields,
therefore,

\[
3x_4^2(a_4 - \left[ \frac{P_1a_3(a_3 + P_1)}{(a_3 - P_1)^2} \right] \pm \left[ \frac{3a_3P_1(b_3P_1 + a_3Q_1)}{F_2^{1/2}(a_3 - P_1)^2} \right] \mp \left[ \frac{(b_3P_1 + a_3Q_1)^3}{F_2^{3/2}(a_3 - P_1)^2} \right]) + \\
2x_4(b_4 + 3S_4 \left[ \frac{P_1a_3(a_3 + P_1)}{(a_3 - P_1)^2} \right] + \left[ \frac{(a_3 + P_1)(b_3P_1 + a_3Q_1)}{(a_3 - P_1)^2} \right] \pm \left[ \frac{a_3P_1F_2^{1/2}}{(a_3 - P_1)^2} \right]) + \\
c_4 - 3S_4^2 \left[ \frac{P_1a_3(a_3 + P_1)}{(a_3 - P_1)^2} \right] - 2S_4 \left[ \frac{(a_3 + P_1)(b_3P_1 + a_3Q_1)}{(a_3 - P_1)^2} \right] - \frac{2F_2^{1/2}}{(a_3 - P_1)^2} \left[ a_3P_1S_4 + \frac{(b_3P_1 + a_3Q_1)}{3} \right] \pm \\
\left[ \frac{9S_4(a_3P_1)(b_3P_1 + a_3Q_1) + (b_3P_1 + a_3Q_1)^2}{F_2^{1/2}(a_3 - P_1)^2} \right] \pm \left[ \frac{3(b_3P_1 + a_3Q_1)^3S_4}{F_2^{3/2}(a_3 - P_1)^2} \right] = 0 \ (4.27)
\]

From the equation (4.27), the expression for \( x_4 \) is given by

\[
x_4 = \frac{-\beta_4 \pm \sqrt{\beta_4^2 - 4\alpha_4\gamma_4}}{2\alpha_4} \quad (4.28)
\]

where,

\[
\alpha_4 = 3(a_4 - \left[ \frac{P_1a_3(a_3 + P_1)}{(a_3 - P_1)^2} \right] \pm \left[ \frac{3a_3P_1(b_3P_1 + a_3Q_1)}{F_2^{1/2}(a_3 - P_1)^2} \right] \mp \left[ \frac{(b_3P_1 + a_3Q_1)^3}{F_2^{3/2}(a_3 - P_1)^2} \right])
\]

\[
\beta_4 = 2(b_4 + 3S_4 \left[ \frac{P_1a_3(a_3 + P_1)}{(a_3 - P_1)^2} \right] + \left[ \frac{(a_3 + P_1)(b_3P_1 + a_3Q_1)}{(a_3 - P_1)^2} \right] \pm \left[ \frac{a_3P_1F_2^{1/2}}{(a_3 - P_1)^2} \right]) + \\
\left[ \frac{9S_4(a_3P_1)(b_3P_1 + a_3Q_1) + (b_3P_1 + a_3Q_1)^2}{F_2^{1/2}(a_3 - P_1)^2} \right] \pm \left[ \frac{3(b_3P_1 + a_3Q_1)^3S_4}{F_2^{3/2}(a_3 - P_1)^2} \right)
\]

\[
\gamma_4 = c_4 - 3S_4^2 \left[ \frac{P_1a_3(a_3 + P_1)}{(a_3 - P_1)^2} \right] - 2S_4 \left[ \frac{(a_3 + P_1)(b_3P_1 + a_3Q_1)}{(a_3 - P_1)^2} \right] - \\
\frac{2}{3} \left[ \frac{(b_3 + Q_1)(b_3P_1 + a_3Q_1)}{(a_3 - P_1)^2} \right] - \frac{a_3R_1 - P_1c_3}{(a_3 - P_1)^2} \pm \frac{2F_2^{1/2}}{(a_3 - P_1)^2} \left[ a_3P_1S_4 + \frac{(b_3P_1 + a_3Q_1)}{3} \right] \pm \\
\left[ \frac{9a_3P_1(b_3P_1 + a_3Q_1)S_4^2 + 2S_4(b_3P_1 + a_3Q_1)^2}{F_2^{1/2}(a_3 - P_1)^2} \right] \pm \left[ \frac{3(b_3P_1 + a_3Q_1)^3S_4^2}{F_2^{3/2}(a_3 - P_1)^2} \right]
\]

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Simplifying the equation (4.28) gives

\[ x_4 = \frac{-(b_4 + Q_2 + 3S_4P_2) \pm \sqrt{(9S_2^2a_4P_2 + 6S_4(b_4P_2 + a_4Q_2) + F_3)} }{3(a_4 - P_2)} \quad (4.29) \]

where,

\[ P_2 = \frac{P_1a_3(P_1 + a_3)}{(a_3 - P_1)^2} + \frac{3a_3P_1(b_3P_1 + a_3Q_1)}{F_2^{1/2}(a_3 - P_1)^2} + \frac{(b_3P_1 + a_3Q_1)^3}{F_2^{3/2}(a_3 - P_1)^2} \]

\[ Q_2 = \frac{(a_3 + P_1)(b_3P_1 + a_3Q_1)}{(a_3 - P_1)^2} + \frac{a_3P_1F_2^{1/2}}{(a_3 - P_1)^2} + \frac{(b_3P_1 + a_3Q_1)^2}{F_2^{3/2}(a_3 - P_1)^2} \]

\[ F_3 = (b_4 + Q_2)^2 - 3(a_4 - P_2)(c_4 - R_2) \]

Here

\[ R_2 = \frac{2(b_3 + Q_1)(b_4P_1 + a_3Q_1)}{3(a_3 - P_1)^2} + \frac{(a_3R_1 - P_1c_3)}{(a_3 - P_1)} + \frac{2(b_3P_1 + a_3Q_1)}{3(a_3 - P_1)^2} \]

Similarly, the optimal solution of stage 5 is given by

\[ x_5 = \frac{-(b_5 + Q_3 + 3S_5P_3) \pm \sqrt{(9S_2^2a_5P_3 + 6S_5(b_5P_3 + a_5Q_3) + F_4)} }{3(a_5 - P_3)} \quad (4.30) \]

where,

\[ P_3 = \frac{P_2a_4(P_1 + a_4)}{(a_4 - P_2)^2} + \frac{3a_4P_2(b_4P_2 + a_4Q_2)}{F_3^{1/2}(a_4 - P_2)^2} + \frac{(b_4P_2 + a_4Q_2)^3}{F_3^{3/2}(a_4 - P_2)^2} \]

\[ Q_3 = \frac{(a_4 + P_2)(b_4P_2 + a_4Q_2)}{(a_4 - P_2)^2} + \frac{a_4P_2F_3^{1/2}}{(a_4 - P_2)^2} + \frac{(b_4P_2 + a_4Q_2)^2}{F_3^{3/2}(a_4 - P_2)^2} \]

\[ F_4 = (b_5 + Q_3)^2 - 3(a_5 - P_3)(c_5 - R_3) \]

Here

\[ R_3 = \frac{2(b_4 + Q_2)(b_5P_3 + a_4Q_2)}{3(a_4 - P_2)^2} + \frac{(a_4R_2 - P_2c_4)}{(a_4 - P_2)} + \frac{2(b_5P_3 + a_4Q_2)}{3(a_4 - P_2)^2} \]

In general, the solution of \( i^{th} \) sub-problem is given by

\[ x_i = \frac{-(b_i + Q_{i-2} + 3S_iP_{i-2}) \pm \sqrt{9S_2^2a_iP_{i-2} + 6S_i(b_iP_{i-2} + a_iQ_{i-2}) + F_{i-1}} }{3(a_i - P_{i-2})} \quad for \ i = 3, 4, ..., n \quad (4.31) \]

where, \( P_0 = a_1, Q_0 = b_1 \) and \( R_0 = c_1 \)

\[ P_{i-2} = \frac{P_{i-3}a_{i-1}(P_{i-3} + a_{i-1})}{(a_{i-1} - P_{i-3})^2} + \frac{3a_{i-1}P_{i-3}(b_{i-1}P_{i-3} + a_{i-1}Q_{i-3})}{F_{i-2}^{1/2}(a_{i-1} - P_{i-3})^2} + \frac{(b_{i-1}P_{i-3} + a_{i-1}Q_{i-3})^3}{F_{i-2}^{3/2}(a_{i-1} - P_{i-3})^2} \]
\[ Q_{i-2} = \frac{(a_{i-1} + P_{i-3})(b_{i-1}P_{i-3} + a_{i-1}Q_{i-3})}{(a_{i-1} - P_{i-3})^2} \pm \frac{a_{i-1}P_{i-3}F_{i-2}^{1/2}}{(a_{i-1} - P_{i-3})^2} + \frac{(b_{i-1}P_{i-3} + a_{i-1}Q_{i-3})^2}{F_{i-2}^{1/2}(a_{i-1} - P_{i-3})^2} \]

\[ F_{i-1} = (b_i + Q_{i-2})^2 - 3(a_i - P_{i-2})(c_i - R_{i-2}) \]

here

\[ R_{i-2} = \frac{2(b_{i-1} + Q_{i-3})(b_{i-1}P_{i-3} + a_{i-1}Q_{i-3})}{3(a_{i-1} - P_{i-3})^2} + \frac{(a_{i-1}R_{i-3} - P_{i-3}C_{i-1})}{(a_{i-1} - P_{i-3})} \pm \frac{2(b_{i-1}P_{i-3} + a_{i-1}Q_{i-3})}{3(a_{i-1} - P_{i-3})^2}F_{i-2}^{1/2} \]

4.3.1 Computational Procedure

The procedure for implementing the proposed analytical approach for the solution of cubic objective function with equality and inequality constraints is given in the following steps:

Step 1: Read the coefficients \( a, b, c \) and \( d \), lower and upper limits of each variable, and \( X_T \) of the given problem.

Step 2: Treat the given optimization problem as a multistage decision problem. The problem is solved by breaking the original problem into a number of single stage problems. The number of single stage problem is equal to the number of decision variables.

Step 3: Set \( S_n = X_T \) and determine the optimal value of decision variable \( x_n \) for stage \( n \) using the generalized recursive equation (4.31). Calculate the state variable \( S_{n-1} \) using the relation \( S_{n-1} = S_n - x_n \).

Step 4: Calculate the optimal values of the decision variables \( x_{n-1} \) to \( x_3 \) using the generalized recursive equation.

Step 5: Compute the optimal values of \( x_2 \) and \( x_1 \) using equations (4.10) and (4.11), respectively.

Step 6: During the recursive procedure, if any decision variable violates their upper or lower limit then that variable is fixed at the corresponding violated limit...
and this sub-problem is eliminated from the recursive procedure and this variable value is subtracted from the equality constraint. Now start the recursive procedure from stage n-1 to stage 1.

Step 7: Calculate the objective function using the optimal solution obtained through the recursive procedure.

A generalized recursive equation for the cubic function derived in this Chapter has the following salient features:

1. The proposed method does not require any initial assumption.

2. A generalized recursive equation derived in this section directly gives optimal solution without need of iterative steps.

3. This mathematical approach gives the optimal solution with less computational effort.

4. It can also be implemented for larger problem consisting of more number of decision variables.

The flowchart of the proposed recursive approach is shown in Figure 4.1.

4.4 Computational Studies

The practical application of the proposed method is demonstrated through economic power dispatch problem of thermal power plant and economic dispatch of chiller plant problem. The program is developed using MATLAB 6.5 and has been executed in Intel core i3 CPU, 2.4 GHz with a 4GB RAM.

4.4.1 Economic Power Dispatch Problem in a Thermal Power Plant

In the economic power dispatch problem, the cost function of each thermal generating unit has been realistically represented by a cubic function, where the power balance constraint is also considered apart from the generation capacity limits
Input the coefficients \( a, b, c \) and \( d \), no. of decision variables \( n \) and their limits, Input to the first stage \( X_T \).

Set \( S_1 = X_T \)
\[ S_0 = 0 \]

Set \( i = n \)

Remove the corresponding single stage problem and solve the \( n \)-subproblems

Determine the optimal value of decision variable \( x \) for the \( i \)th subproblem using the generalised recursive equation (4.31)

\[ x_i < x_{i, \text{min}} \]
Yes
Set \( x_i = x_{i, \text{min}} \)
\[ S_n = S_n \cdot x_i \]
\[ n = n -1 \]

No

\[ x_i > x_{i, \text{max}} \]
Yes
Set \( x_i = x_{i, \text{max}} \)
\[ S_n = S_n \cdot x_i \]
\[ n = n -1 \]

No

Set \( i = n - 1 \)

If \( i > 2 \)

Yes

Calculate the optimal values of variables \( x_3 \) and \( x_1 \) using equations (4.10) and (4.11)

No

Calculate the objective function

End

Figure 4.1 Flowchart of the proposed analytical method
of the thermal generating units. It is an optimization problem and its objective is to reduce the total generation cost while satisfying constraints. The problem is mathematically stated as follows:

\[
\text{Minimize } f(x) = \sum_{i=1}^{n} a_i P_i^3 + b_i P_i^2 + c_i P_i + d_i
\]

where, \( f(x) \) is the total fuel cost, \( a_i, b_i, c_i \) and \( d_i \) are the fuel cost coefficients of \( i^{th} \) generating unit and \( P_i \) is the power output of \( i^{th} \) generating unit.

subject to

(i) Power balance constraint:

At a particular dispatch interval (usually 1hr), the total generation of committed units must be such that the system demand \( P_D \) is satisfied.

\[
\sum_{i=1}^{n} P_i = P_D
\]

(ii) Generator capacity constraints:

The physical restrictions on the power output of generating units constitute the following constraints

\[
P_i^{min} \leq P_i \leq P_i^{max}
\]

The proposed methodology has been tested on three-unit and five-unit test systems (Theerthamalai and Maheswarapu, 2010). The fuel cost, operating limits of the generators for three and five unit test systems are given in Table 4.1 and Table 4.2, respectively.

### Table 4.1 Fuel cost coefficients and operating limits of 3-unit system

<table>
<thead>
<tr>
<th>Generating unit</th>
<th>( a_i )</th>
<th>( b_i )</th>
<th>( c_i )</th>
<th>( d_i )</th>
<th>( P_i^{min} ) MW</th>
<th>( P_i^{max} ) MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.27e-7</td>
<td>9.68e-4</td>
<td>6.95</td>
<td>749.55</td>
<td>320</td>
<td>800</td>
</tr>
<tr>
<td>2</td>
<td>6.453e-8</td>
<td>7.375e-4</td>
<td>7.05</td>
<td>1285</td>
<td>300</td>
<td>1200</td>
</tr>
<tr>
<td>3</td>
<td>9.98e-8</td>
<td>1.04e-3</td>
<td>6.531</td>
<td>1531</td>
<td>275</td>
<td>1100</td>
</tr>
</tbody>
</table>
Table 4.2 Fuel cost coefficients and operating limits of 5-unit system

<table>
<thead>
<tr>
<th>Generating unit</th>
<th>(a_i)</th>
<th>(b_i)</th>
<th>(c_i)</th>
<th>(d_i)</th>
<th>(P_{i_{\min}}) MW</th>
<th>(P_{i_{\max}}) MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.27e-7</td>
<td>9.68e-4</td>
<td>6.95</td>
<td>749.55</td>
<td>320</td>
<td>800</td>
</tr>
<tr>
<td>2</td>
<td>6.453e-8</td>
<td>7.375e-4</td>
<td>7.05</td>
<td>1285</td>
<td>300</td>
<td>1200</td>
</tr>
<tr>
<td>3</td>
<td>9.98e-8</td>
<td>1.04e-3</td>
<td>6.531</td>
<td>1531</td>
<td>275</td>
<td>1100</td>
</tr>
<tr>
<td>4</td>
<td>1.27e-7</td>
<td>9.68e-4</td>
<td>6.95</td>
<td>749.55</td>
<td>320</td>
<td>800</td>
</tr>
<tr>
<td>5</td>
<td>6.453e-8</td>
<td>7.375e-4</td>
<td>7.05</td>
<td>1285</td>
<td>300</td>
<td>1200</td>
</tr>
</tbody>
</table>

4.4.2 Economic Dispatch of Chiller Plant

The power consumption of \(i^{th}\) chiller plant is expressed as

\[ P_i = a_i Q_i^3 + b_i Q_i^2 + c_i Q_i + d_i \]

where, \(Q_i\) is the output of \(i^{th}\) chiller plant, \(a_i, b_i, c_i\) and \(d_i\) are the coefficients of \(i^{th}\) chiller plant. The objective of economic dispatch of chiller plant problem is to determine a set of chiller output which minimize the total power consumption subject to satisfying the equality and inequality constraints,

\[
\text{Minimize } J = \sum_{i=1}^{n} P_i
\]

subject to

\[
\sum_{i=1}^{n} Q_i = CL
\]

and

\[
0 \leq Q_i \leq Q_i^{max}
\]

where, CL is the system cooling load and \(Q_i^{max}\) is the maximum operating limit of the \(i^{th}\) chiller. The sample plant considered in this Chapter has three 800 RT (capacity) units and chillers data is given in Table 4.3.


Table 4.3 Chillers data

<table>
<thead>
<tr>
<th>Chiller unit</th>
<th>$a_i$</th>
<th>$b_i$</th>
<th>$c_i$</th>
<th>$d_i$</th>
<th>$P_{min}$</th>
<th>RT</th>
<th>$P_{max}$</th>
<th>RT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.54e-6</td>
<td>-1.52e-3</td>
<td>1.023</td>
<td>100.95</td>
<td>0</td>
<td>800</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.39e-7</td>
<td>-5.947e-4</td>
<td>0.7579</td>
<td>66.598</td>
<td>0</td>
<td>800</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.95e-7</td>
<td>2.25e-5</td>
<td>0.3806</td>
<td>130.09</td>
<td>0</td>
<td>800</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4 Optimal power generation results for 3-unit system

<table>
<thead>
<tr>
<th>Method</th>
<th>Load demand (MW)</th>
<th>Unit 1 (MW)</th>
<th>Unit 2 (MW)</th>
<th>Unit 3 (MW)</th>
<th>Total cost ($/hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lambda logic based method</td>
<td>1100</td>
<td>320.00</td>
<td>321.10</td>
<td>457.89</td>
<td>11460.42</td>
</tr>
<tr>
<td></td>
<td>1300</td>
<td>361.21</td>
<td>416.56</td>
<td>522.28</td>
<td>12986.54</td>
</tr>
<tr>
<td></td>
<td>3000</td>
<td>800.00</td>
<td>1159.04</td>
<td>1040.90</td>
<td>27110.56</td>
</tr>
<tr>
<td>Proposed method</td>
<td>1100</td>
<td>320.00</td>
<td>322.36</td>
<td>457.64</td>
<td>11460.42</td>
</tr>
<tr>
<td></td>
<td>1300</td>
<td>361.01</td>
<td>416.89</td>
<td>522.10</td>
<td>12986.16</td>
</tr>
<tr>
<td></td>
<td>3000</td>
<td>800.00</td>
<td>1159.25</td>
<td>1040.75</td>
<td>27110.56</td>
</tr>
</tbody>
</table>

4.5 Results and Discussion

Tables 4.4 & 4.5 shows the comparison of optimum generation schedule and total fuel cost of thermal generating units for various system demand obtained through the proposed dynamic programming approach and Lambda logic based algorithm reported in the literature.

In Lambda logic based method, equality constraint is not satisfied for 5000MW load demand. From the comparison, it is clear that the proposed method provides optimum solution for economic power dispatch problems.

The optimal chiller loadings for different system cooling loads obtained through Lagrangian Multiplier Method (LMM), Gradient Method (GM) and proposed method are compared in Table 4.6. The method satisfies the constraints and provides quality solution for all the demands. The optimal chiller loadings for different system cooling loads are shown in Figures 4.2-4.5.

The drawback of Lagrangian multiplier method is that the solution is not
Table 4.5 Optimal power generation results for 5-unit system

<table>
<thead>
<tr>
<th>Method</th>
<th>Load demand (MW)</th>
<th>Unit 1 (MW)</th>
<th>Unit 2 (MW)</th>
<th>Unit 3 (MW)</th>
<th>Unit 4 (MW)</th>
<th>Unit 5 (MW)</th>
<th>Total cost ($/hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lambda method</td>
<td>5000</td>
<td>800.00</td>
<td>1159.04</td>
<td>1040.96</td>
<td>800.00</td>
<td>1174.15</td>
<td>44788.58*</td>
</tr>
<tr>
<td></td>
<td>2500</td>
<td>436.27</td>
<td>518.08</td>
<td>591.94</td>
<td>436.27</td>
<td>518.08</td>
<td>24023.77</td>
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<tr>
<td></td>
<td>1800</td>
<td>320.00</td>
<td>343.71</td>
<td>472.58</td>
<td>320.00</td>
<td>343.71</td>
<td>18609.70</td>
</tr>
<tr>
<td>Proposed method</td>
<td>5000</td>
<td>800.00</td>
<td>1173.79</td>
<td>1051.10</td>
<td>800.00</td>
<td>1175.11</td>
<td>45022.15</td>
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<tr>
<td></td>
<td>2500</td>
<td>432.86</td>
<td>514.07</td>
<td>588.77</td>
<td>441.74</td>
<td>522.56</td>
<td>24018.72</td>
</tr>
<tr>
<td></td>
<td>1800</td>
<td>320.00</td>
<td>343.83</td>
<td>472.24</td>
<td>320.00</td>
<td>343.92</td>
<td>18609.69</td>
</tr>
</tbody>
</table>

Table 4.6 Comparison of results

<table>
<thead>
<tr>
<th>Load (RT)</th>
<th>LMM (MW)</th>
<th>GM (MW)</th>
<th>Proposed method (MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>960 (40%)</td>
<td>-</td>
<td>841.44</td>
<td>840.36</td>
</tr>
<tr>
<td>1200 (50%)</td>
<td>-</td>
<td>1102.31</td>
<td>1102.31</td>
</tr>
<tr>
<td>1440 (60%)</td>
<td>1102.31</td>
<td>1102.31</td>
<td>1102.31</td>
</tr>
<tr>
<td>1680 (70%)</td>
<td>1244.40</td>
<td>1244.40</td>
<td>1244.40</td>
</tr>
<tr>
<td>1920 (80%)</td>
<td>1403.31</td>
<td>1403.31</td>
<td>1403.31</td>
</tr>
<tr>
<td>2160 (90%)</td>
<td>1583.97</td>
<td>1583.97</td>
<td>1583.97</td>
</tr>
</tbody>
</table>

converged at low demands and the major flaw of gradient method is that more number of iterations is needed for the convergence (Chang et al., 2010). The proposed method overcomes these flaws and directly gives the optimal solution for the problem. From the comparison, it is clear that the proposed method provides an optimal solution for all load demands.
Figure 4.2 Optimal chiller loadings for cooling load of 960 RT

Figure 4.3 Optimal chiller loadings for cooling load of 1200 RT
Figure 4.4 Optimal chiller loadings for cooling load of 1680 RT

Figure 4.5 Optimal chiller loadings for cooling load of 2160 RT
4.6 Conclusion

A new analytical approach for the solution of cubic objective function with equality and inequality constraints is developed in this Chapter. The generalized equation derived in this Chapter simplifies the task of evaluation of optimal solution for the optimization problem with cubic objective function. The proposed approach requires less computational efforts. The feasibility of the method for solving optimization problem with cubic function is demonstrated using three test problems. The comparison of results obtained through the proposed method with other reported methods show the merits of the developed method to obtain global optimal solution.