CHAPTER 3

MATHEMATICAL MODELING OF INDUCTION MOTORS

To start with, a well-known technique called the SVPWM technique is discussed as this forms the basis of the mathematical modeling of IMs. Furthermore, the *d–q* dynamic model **[7]** using Park's transformation, Kron's dynamic model and the Stanley dynamic model is also presented. The mathematical model presented in this chapter is further used to design variable speed IM drive with various types of controllers, such as the PI controller, the Mamdani-Fuzzy controller, the Takagi-Sugeno FL controller and the ANFIS controller, which are discussed in further chapters.

3.1 INTRODUCTION

Design of a controller is based on transfer function, which is further used to control any parameter of the system, e.g. speed, torque, flux, etc. The mathematical model can be obtained by various methods, viz., from first principles, system identification methods, etc. This mathematical model may be a linear/non-linear differential equation or a transfer function (in *s*- or *z*-domain) or in state space form **[5]**.

In general, the mathematical model of any IM can be modeled by various methods, viz., space vector phase theory or the two-axis theory of electrical machines **[70]**. The model used in our work consists of SVPWM voltage source inverter, IM, direct flux, torque speed control **[16]**. Main drawback of the coupling effect in the control of SCIMs is that it gives a highly over damped plant response, thus making the system very slow and sluggish. Moreover, since the order of the system is very high, the system suffers from instability as for this higher order system, a controller of larger order should be developed, which increases the implementation cost.

The previously mentioned problem can be solved by making use of either vector control/field-oriented control or using any other type of control methods. These types of control strategies can control an IM like DC machine in separately excited condition **[7]**. Of course, the control of AC drives can exhibit better performance. Thus, due to the above-mentioned reasons, an IM model was developed using the concept of rotating *d–q* field reference frame concept **[16]**. The power circuit of the 3Φ induction motor is shown in Fig. 3.1.

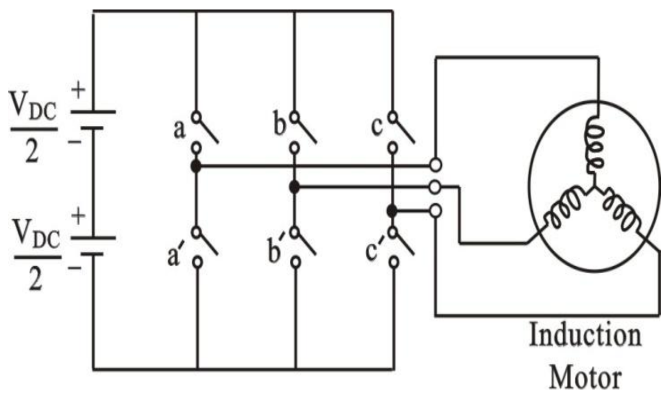

Fig 3.1: Power circuit connection diagram for the IM

Table 3.1: SCIM specifications

HP	50
Speed	1800 rpm
Voltage	460 V
Frequency	50 Hz
Phase	3
Poles	2
Type	Squirrel Cage Type IM

The specifications of the Squirrel Cage Induction Motor (SCIM) used in our work, as in Table 3.1. The motor's mathematical model, certain assumptions have been considered. The motor is symmetrical 2-pole having 3-phase windings.

The slotting effects and the iron losses are neglected. Note that in the design and implementation of AC drives, a dynamic model of the IM is required to design a controller to control the various parameters. A very good control requires an approximate model of the real time system. All dynamic effects of transient/steady state conditions are considered in the model. Furthermore, the dynamic model used in the research work is valid for any change in the inverter's supply

3.2 REVIEW OF THE SVPWM TECHNIQUE

Pulse width modulation (PWM) is used in various power electronics applications. SVPWM is a popular technique used in control of AC motor drives. The triangular carrier wave is modulated by sine wave and the switching points of inverter are determined by the point of intersection.

The main application of PWM is in power electronics, for example in motor control. When generating analogue signals, the disadvantage is that the PWM resolution rapidly decreases with the required signal bandwidth. Hence, to avoid the drawbacks of this method, SVPWM could be used for more sophisticated control of AC motor drives, as described in the further sub-sections.

3.2.1 Operational Principle of PWM

Fig. 3.2 shows the circuit model of a 1Φ inverter with a centre-tapped grounded DC bus along with the principle of operation of PWM.

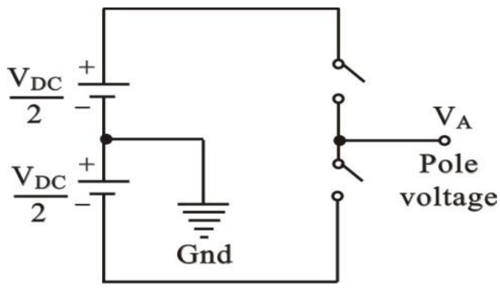

Fig. 3.2: The single-phase circuit model of inverter

The inverter output voltage is shown in Fig. 3.3 and is obtained as follows:

1. When the control voltage v_{cont} is $> V_{tr}$, $V_{AO} = \frac{V_{DC}}{2}$

2. When the control voltage v_{cont} is $< V_{tr}$, $V_{AO} = -\frac{V_{DC}}{2}$

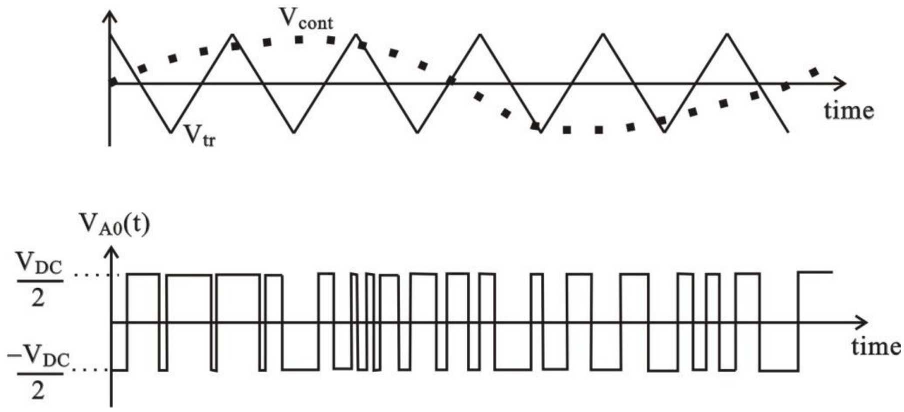

Fig. 3.3: PWM graphical concept

The output voltage of the 1Φ inverter has a number of features: PWM frequency is the same as that of the frequency of V_{tr} , the amplitude is controlled by the peak value of the voltage of v_{contr} and the fundamental frequency is controlled by the frequency of v_{contr} . Furthermore, the modulation index '*m*' of the inverter is given by

$$m = \frac{V_{contr}}{V_{tr}} = \frac{\text{Peak value of } (V_{AO})}{V_{DC} / 2} \quad (3.1)$$

where $(V_{A0})_1$ is the fundamental frequency component of V_{A0}.

3.2.2 Generation of Space Vectors in PWM Control

The circuit diagram of a 3Φ voltage source PWM inverter is shown diagrammatically in Fig. 3.4. Here, S_1 to S_6 are 6 power switches that will determine the shape of the output and are controlled by the switching variables a, a', b, b', c and c' respectively. When one of the transistors in the upper half of the inverter is in the on condition, i.e., when *a, b* or *c* is in logic 1 state, the corresponding transistors in lower half of the inverter will be in off condition (logic 0 state). Thus, the on and off states of the transistors S_1, S_3, S_5 in the upper half of the inverter can be used to determine the output voltage of the inverter.

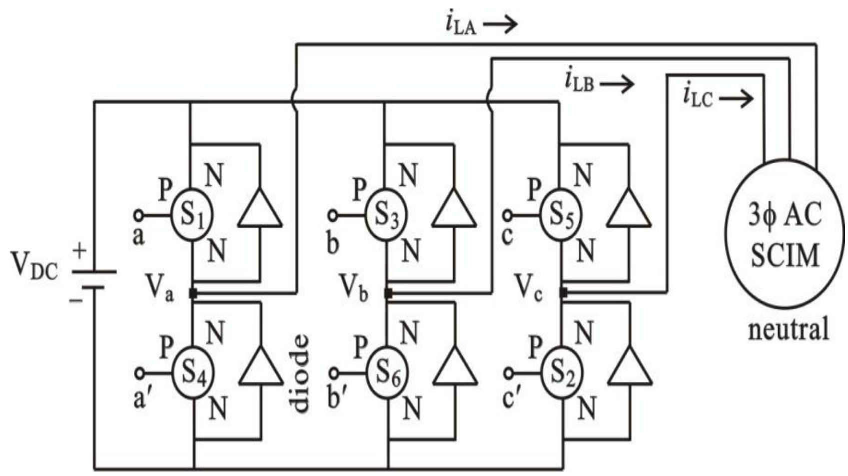

Fig. 3.4: Diagrammatic view of a 3 Φ AC voltage source PWM inverter

The line-line (phase–phase) voltages and line-neutral voltages relationships can be expressed in the form of a matrix. The first relationship between switching variable vector and line-to-line voltage vector is as follows:

$$\begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix} = V_{DC} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad (3.2)$$

Furthermore, the relationship between the switching variable vector and the phase voltage vector is given as follows:

$$\begin{bmatrix} V_{an} \\ V_{bn} \\ V_{cn} \end{bmatrix} = \frac{V_{DC}}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3.3}$$

The eight possible combinations of the on and off patterns for the 3 switches in the upper part of the PWM inverter is as in Fig. 3.4. The on and off states of transistors in lower part of inverter bridge are opposite to that of the switches in the upper part of the inverter bridge. According to Equations (3.2) and (3.3), the 8 switching vectors, output line to neutral voltage and the output line to line voltages in terms of V_{DC} are given in Table 3.2.

Table 3.2: Switching vectors with their voltage levels

Voltage	Switching vectors			V_{LN}			V_{LL}		
Vectors	*a*	*b*	*c*	V_{an}	V_{bn}	V_{cn}	V_{ab}	V_{bc}	V_{ca}
V_0	0	0	0	0	0	0	0	0	0
V_1	1	0	0	$2/3$	$-1/3$	$-1/3$	1	0	-1
V_2	1	1	0	$1/3$	$1/3$	$-2/3$	0	1	-1
V_3	0	1	0	$-1/3$	$2/3$	$-1/3$	-1	1	0
V_4	0	1	1	$-2/3$	$1/3$	$1/3$	- 1	0	1
V_5	0	0	1	$-1/3$	$-1/3$	$2/3$	0	- 1	1
V_6	1	0	1	$1/3$	$-2/3$	$1/3$	1	- 1	0
V_7	1	1	1	0	0	0	0	0	0

SVPWM generates less harmonic distortion in the output voltages and/or currents, which are applied to the three phases of an IM. Also, it provides more efficient use of the supply voltages to the IM compared with the other methods. To implement the SVPWM, the output voltage equations in *abc* reference frame are transformed into the stationary *d–q* reference frame that consists of 2 axes: the *d*-axis and the *q*-axes as shown in Fig. 3.5.

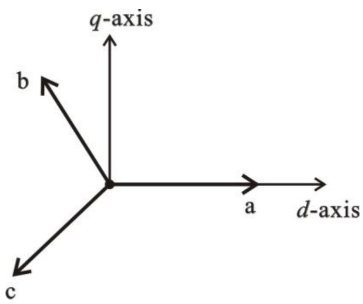

Fig. 3.5: The relationship b/w *abc* ref and stationary *d–q* frame

The relationship between the two reference frames can be obtained as follows:

$$f_{dqo} = K_s f_{abc} \tag{3.4}$$

where

$$K_s = \frac{2}{3}\begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},\ f_{dq0} = \begin{bmatrix} f_d \\ f_q \\ f_0 \end{bmatrix},\ f_{abc} = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix} \tag{3.5}$$

Here, f is considered a current variable or a voltage variable. From Fig. 3.5, the transformation given in Equation (3.5) can be considered equivalent to an orthogonal projection of the vector $[\mathrm{a\ b\ c}]^T$ onto the 2-dimensional perpendicular unit vector $[1\ \ 1\ \ 1]^T$, which is in the equivalent *d–q* plane in a V_{s6} 3-dimensional system. Because of this transformation, 6 non-zero vectors and 2 zero vectors could be generated, which in turn results in 8 permissible switching states. The 6 non-zero vectors V_{s1} to V_{s6} which can be shown as the axes of a circle as in Fig. 3.6. Thus, this feeds the electric power to the loads. The angle between any two adjacent non-zero vectors is 60^0. The two zero vectors V_{s1} to V_{s6} are placed at the origin of the hexagonal (circle) grid of vectors. These eight basic vectors in the grid are denoted by V_{s0}, V_{s1}, V_{s2}, V_{s3}, V_{s4}, V_{s5}, V_{s6}, and V_{s7}.

The operator transformation given in Equation (3.5) is used to obtain the desired reference voltage V_{ref} in the *d–q* plane. The main objective of SVPWM technique used to develop the mathematical model of the IM is to approximate the reference voltage vector V_{ref} using the eight possible switching states. In general, the SVPWM concept can be implemented in 3 steps:

Step 1: Find the voltages V_d, V_q, V_{ref} and the angle α.

Step 2: Find the time periods (duration) T_1, T_2 and T_0.

Step 3: Find the switching times of each power transistor (s_1 *to* s_6).

The eight permissible switching states are shown in Fig. 3.6.

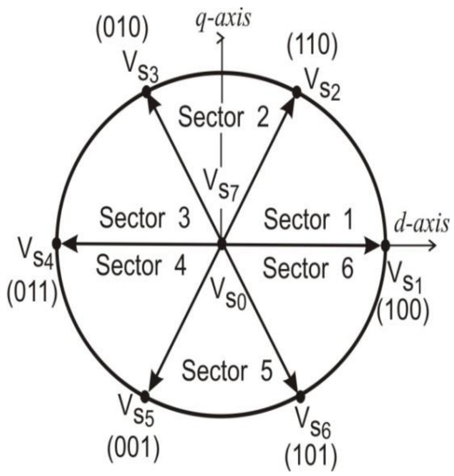

Fig. 3.6: Diagrammatic representation of the sequence of the space vectors

3.3 THE DYNAMIC D-Q MODEL USING PARK'S TRANSFORMATION

In the previous modeling methods, per phase equivalent circuit of the machine was considered. This was valid only in the steady-state condition. The equivalent circuit that was used for obtaining the mathematical model of the IM is shown in Figs. 3.7 and 3.8, respectively **[7]**.

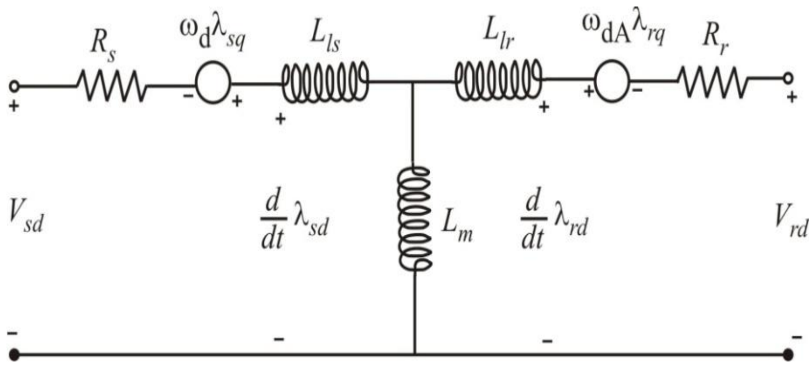

Fig. 3.7: IM Equivalent circuit in the *d*-axis frame

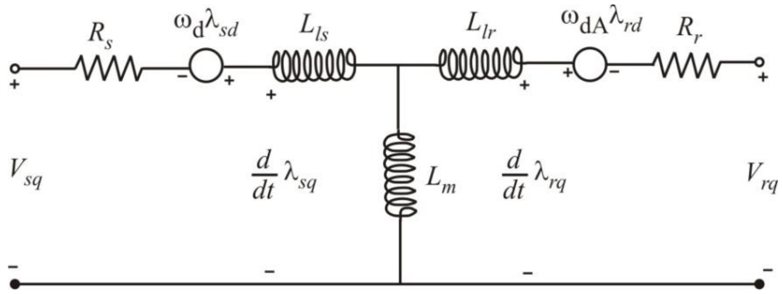

Fig. 3.8: IM Equivalent circuit in the *q*-axis frame

In the current methods of modeling and control of SCIMs, especially one with an adjustable AC drive, the machine should have certain feedback loops. Therefore, the transient behavior should be taken into consideration in the machine model. Furthermore, high-performance drive control requires a better understanding of the vector/field-oriented control. This section provides a better understanding of the concepts relating to the development of the *d–q* theory. In the IM, the 3-phase rotor windings move with respect to the 3-phase stator windings. Generally, any machine model could be best described by a set of non-linear differential equations with time-varying mutual inductances. However, such a model tends to be very complex and the controller design becomes further complex **[7]**.

The mathematical model of the IMs (the 3-phase IM) could be represented by an equivalent 2-phase, where d^s, q^s, d^r *and* q^r correspond to the stator, rotor, direct and quadrature axes, respectively. Although this model looks quite simple, the problem of time-varying parameters still remains. Hence, to solve this problem, R. H. Park, in 1920, developed the transformation technique to solve the problem of time-varying parameters. The stator variables are referred with respect to the rotor reference frame, which rotates at a synchronous speed. The time-varying inductances that occur due to the interaction between the electric and magnetic circuits can be removed using this Park transformation.

Later, in 1930, H.C. Stanley demonstrated that the time-varying inductances that appear in the *v*–*i* equations of the IM due to electric and magnetic effects can be removed by transforming the variables with respect to the fictitious stationary windings. In this case, the rotor variables are transformed to the stator reference frame. In this thesis, a dynamic machine model in synchronously rotating and stationary references frame is presented, which is further used to develop sophisticated controllers to control the speed of the IM. The stator voltage equations formulated from stationary reference frame **[7]** are as follows:

$$V_{sA}(t) = R_s i_{sA}(t) + \frac{d\psi_{sA}(t)}{dt} \tag{3.6}$$

$$V_{sB}(t) = R_s i_{sB}(t) + \frac{d\psi_{sB}(t)}{dt} \tag{3.7}$$

$$V_{sC}(t) = R_s i_{sC}(t) + \frac{d\psi_{sC}(t)}{dt} \tag{3.8}$$

The rotor voltage equations formulated to the rotating frame fixed to the rotor are as follows **[7]**:

$$V_{sa}(t) = R_r i_{ra}(t) + \frac{d\psi_{ra}(t)}{dt} \tag{3.9}$$

$$V_{sb}(t) = R_r i_{rb}(t) + \frac{d\psi_{rb}(t)}{dt} \tag{3.10}$$

$$V_{sc}(t) = R_r i_{rc}(t) + \frac{d\psi_{rc}(t)}{dt} \tag{3.11}$$

where the flux linkages related to the stator and rotor windings are given as **[7]**

$$\begin{aligned}\psi_{sA} = {} & \overline{L_s} i_{sA} + \overline{M_s} i_{sB} + \overline{M_s} i_{sC} + \overline{M_{sr}} \cos(\theta_m) i_{ra} \\ & + \overline{M_{sr}} \cos\left(\theta_m + {}^{2\pi}\!/_{3}\right) i_{rb} + \overline{M_{sr}} \cos\left(\theta_m + {}^{4\pi}\!/_{3}\right) i_{rc}\end{aligned} \tag{3.12}$$

$$\begin{aligned}\psi_{sB} = {} & \overline{M_s} i_{sA} + \overline{L_s} i_{sB} + \overline{M_s} i_{sC} + \overline{M_{sr}} \cos\left(\theta_m + {}^{4\pi}\!/_{3}\right) i_{ra} \\ & + \overline{M_{sr}} \cos(\theta_m) i_{rb} + \overline{M_{sr}} \cos\left(\theta_m + {}^{2\pi}\!/_{3}\right) i_{rc}\end{aligned} \tag{3.13}$$

$$\begin{aligned}\psi_{sC} = {} & \overline{M_s} i_{sA} + \overline{M_s} i_{sB} + \overline{L_s} i_{sC} + \overline{M_{sr}} \cos\left(\theta_m + {}^{2\pi}\!/_{3}\right) i_{ra} \\ & + \overline{M_{sr}} \cos\left(\theta_m + {}^{4\pi}\!/_{3}\right) i_{rb} + \overline{M_{sr}} \cos(\theta_m) i_{rc}\end{aligned} \tag{3.14}$$

$$\begin{aligned}\psi_{ra} = {} & \overline{M_{sr}} \cos(-\theta_m) i_{sA} + \overline{M_{sr}} \cos\left(-\theta_m + {}^{2\pi}\!/_{3}\right) i_{sB} \\ & + \overline{M_{sr}} \cos\left(-\theta_m + {}^{4\pi}\!/_{3}\right) i_{sC} + \overline{L_r} i_{ra} + \overline{M_r} i_{rb} + \overline{M_r} i_{rc}\end{aligned} \tag{3.15}$$

$$\psi_{rb} = \overline{M_{sr}}\cos\left(-\theta_m + \tfrac{4\pi}{3}\right)i_{sA} + \overline{M_{sr}}\cos\left(-\theta_m\right)i_{sB} + \overline{M_{sr}}\cos\left(-\theta_m + \tfrac{2\pi}{3}\right)i_{sC} + \overline{M_r}i_{ra} + \overline{L_r}i_{rb} + \overline{M_r}i_{rc} \tag{3.16}$$

$$\psi_{rc} = \overline{M_{sr}}\cos\left(-\theta_m + \tfrac{2\pi}{3}\right)i_{sA} + \overline{M_{sr}}\cos\left(-\theta_m + \tfrac{4\pi}{3}\right)i_{sB} + \overline{M_{sr}}\cos\left(-\theta_m\right)i_{sC} + \overline{M_r}i_{ra} + \overline{L_r}i_{rb} + \overline{M_r}i_{rc} \tag{3.17}$$

Note that *L, M and i* are, respectively, the self-inductance, mutual inductance and the currents referred to the stator and rotor windings. Substituting Equations (3.12)–(3.17) in Equations (3.6)–(3.11) and further simplifying, the equations for the stator and rotor can be written in the vector-matrix notation form as follows:

$$\begin{bmatrix} V_{sA} \\ V_{sB} \\ V_{sC} \\ V_{ra} \\ V_{rb} \\ V_{rc} \end{bmatrix} = \begin{bmatrix} R_s + p\overline{L}_s & p\overline{M}_s & p\overline{M}_s & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} \\ p\overline{M}_s & R_s + p\overline{L}_s & p\overline{M}_s & p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} \\ p\overline{M}_s & p\overline{M}_s & R_s + p\overline{L}_s & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m \\ p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} & R_r + p\overline{L}_r & p\overline{M}_r & p\overline{M}_r \\ p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_r & R_r + p\overline{L}_r & p\overline{M}_r \\ p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_r & p\overline{M}_r & R_r + p\overline{L}_r \end{bmatrix} \begin{bmatrix} i_{sA} \\ i_{sB} \\ i_{sC} \\ i_{ra} \\ i_{rb} \\ i_{rc} \end{bmatrix} \tag{3.18}$$

The 3-phase stator and rotor voltage equations written in vector-matrix can be further transformed into 2-phase stator and rotor voltage equations using the well-known Park's transformation. To obtain this, a 3-phase SCIM with stationary axis a_s-b_s-c_s 120^0 apart is considered.

The 3-phase stationary reference frame variables a_s-b_s-c_s are transformed into 2-phase stationary reference frame variables (d^s–q^s). Furthermore, these 2-phase variables are transformed into synchronously rotating reference frame variables (d^e–q^e) and vice-versa. Let us assume that (d^S–q^s) axes are oriented at an angle of θ. The direct axis voltage v^s_{ds} and quadrature axis voltage v^s_{qs} are further resolved into another type of component, viz., a_s-b_s-c_s, and finally, writing them in the vector-matrix notation form, we obtain

$$\begin{bmatrix} v_{as} \\ v_{bs} \\ v_{cs} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 1 \\ \cos(\theta - 120^0) & \sin(\theta - 120^0) & 1 \\ \cos(\theta + 120^0) & \sin(\theta + 120^0) & 1 \end{bmatrix} \begin{bmatrix} v^s_{qs} \\ v^s_{ds} \\ v^s_{os} \end{bmatrix} \qquad (3.19)$$

Taking the inverse of the above, we obtain the following:

$$\begin{bmatrix} v^s_{qs} \\ v^s_{ds} \\ v^s_{os} \end{bmatrix} = \frac{2}{3} \begin{bmatrix} \cos(\theta) & \cos(\theta - 120^0) & \cos(\theta + 120^0) \\ \sin(\theta) & \sin(\theta - 120^0) & \sin(\theta + 120^0) \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} v_{as} \\ v_{bs} \\ v_{cs} \end{bmatrix} \qquad (3.20)$$

where v^s_{os} is added as the zero sequence component, which may or may not be present. Note that in the above equations, voltage was considered as the variable. Similarly, the current and flux linkage equations can also be transformed into similar equations. Note that if θ is set to zero, the q^s-axis will be aligned with the a_s-axis. Once the zero sequence components are ignored, the transformation equations can be simplified as

$$v_{as} = v_{qs}^s \tag{3.21}$$

$$v_{bs} = \frac{1}{2}v_{qs}^s - \frac{\sqrt{3}}{2}v_{ds}^s \tag{3.22}$$

$$v_{cs} = -\frac{1}{2}v_{qs}^s + \frac{\sqrt{3}}{2}v_{ds}^s \tag{3.23}$$

The inverse equations are obtained as

$$v_{qs}^s = \frac{2}{3}v_{as} - \frac{1}{3}v_{bs} - \frac{1}{3}v_{cs} = v_{as} \tag{3.24}$$

$$v_{ds}^s = \frac{-1}{\sqrt{3}}v_{bs} + \frac{1}{\sqrt{3}}v_{cs} \tag{3.25}$$

The synchronously rotating d^e–q^e axes rotate at synchronous speed ω_e with respect to the d^s–q^s axes and the angle θ_e is equal to ω_e. The 2-phase d^s–q^s windings are transformed into the hypothetical windings mounted on the d^e–q^e axes. The d^s–q^s axes voltages can be converted or resolved into the d^e–q^e frame as follows:

$$v_{qs} = v_{qs}^s \cos\theta_e - v_{ds}^s \sin\theta_e \tag{3.26}$$

$$v_{ds} = v_{qs}^{s} \sin\theta_e + v_{ds}^{s} \cos\theta_e \tag{3.27}$$

The superscript *'e'* has been dropped henceforth from the synchronously rotating frame parameters. Resolving the rotating frame parameters into a stationary frame, Equations (3.26) and (3.27) can be written as

$$v_{qs}^{s} = v_{qs} \cos\theta_e + v_{ds} \sin\theta_e \tag{3.28}$$

$$v_{ds}^{s} = -v_{qs} \sin\theta_e + v_{ds} \cos\theta_e \tag{3.29}$$

Let us assume that the 3-phase stator voltages are balanced and are given by

$$v_{as} = V_m \cos(\omega_e t + \phi) \tag{3.30}$$

$$v_{bs} = V_m \cos(\omega_e t - 2\pi/3 + \phi) \tag{3.31}$$

$$v_{cs} = V_m \cos(\omega_e t + 2\pi/3 + \phi) \tag{3.32}$$

Substituting equations (3.30)–(3.32) in (3.24)–(3.25) yields

$$v_{qs}^{s} = V_m \cos(\omega_e t + \phi) \tag{3.33}$$

$$v_{ds}^{s} = -V_m \sin(\omega_e t + \phi) \tag{3.34}$$

Substituting equations (3.26)–(3.27) in (3.33)–(3.34) yields

$$v_{qs} = V_m \cos\phi \tag{3.35}$$

$$v_{ds} = -V_m \sin\phi \tag{3.36}$$

Equations (3.33) and (3.34) show that $\upsilon_{qs}{}^s$ and $\upsilon_{ds}{}^s$ are balanced 2-phase voltages of equal peak values and the latter is at 90^0 angle phase lead with respect to the other component. Equations (3.35) and (3.36) verify that sinusoidal variables in a stationary frame appear as DC quantities in a synchronously rotating reference frame. Note that the stator variables are not necessarily balanced sinusoidal waves; in fact, they can be any arbitrary time functions also. The variables in a reference frame can be combined and represented in a complex space vector or phase as

$$\overline{V} = v_{qds}^{s} = v_{qs}^{s} - jv_{ds}^{s},$$

$$= V_m\left[\cos\left(\omega_e t + \phi\right) + j\sin\left(\omega_e t + \phi\right)\right],$$

$$= \hat{V}_m e^{j\phi} e^{j\omega_e t},$$

$$= \sqrt{2}V_s e^{j(\theta_r + \phi)}. \tag{3.37}$$

which indicates that the vector $\overline{V}$ rotates counter-clockwise at a speed of ω_e from the initial ($t = 0$) angle of Φ to the q^e -axis. Equation (3.37) also indicates that for a sinusoidal variable, the vector magnitude is the peak value $\hat{V}_m$ which is $\sqrt{2}$ times the rms phase magnitude of V^s. The $q^e - d^e$ components can also be combined into a vector form as follows:

$$
\begin{aligned}
v_{qds}^{s} &= v_{qs} - jv_{ds}, \\
&= \left(v_{qs}^{s}\cos\theta_e - v_{ds}^{s}\sin\theta_e\right) - j\left(v_{qs}^{s}\sin\theta_e + v_{ds}^{s}cos\theta_e\right) \\
&= \left(v_{qs}^{s} - jv_{ds}^{s}\right)e^{-j\theta_e}, \\
&= \overline{V}e^{-j\theta_e}.
\end{aligned}
\quad (3.38)
$$

Equation (3.38) can further be written using inverse notations as

$$
\overline{V} = v_{qs}^{s} - jv_{ds}^{s} = \left(v_{qs} - jv_{ds}\right)e^{+j\theta_e} \quad (3.39)
$$

From the concepts of vector algebra, the vector magnitudes in the stationary and rotating frames will be equal. In other words, Equation (3.39) can be re-written as

$$
\left|\overline{V}\right| = \hat{V}_m = \sqrt{\left(v_{qs}^{s}\right)^2 + \left(v_{ds}^{s}\right)^2} = \sqrt{\left(v_{qs}\right)^2 + \left(v_{ds}\right)^2} \quad (3.40)
$$

The factor $e^{j\theta}$ can be interpreted as a vector rotational operator (defined as a vector rotator or unit vector) that converts the rotating frame variables into stationary frame variables. $cos\theta_e$ and $sin\theta_e$ are the Cartesian components of the unit vector. In Equation (3.38), $e^{-j\theta}$ is defined as the inverse vector rotator that converts the d^s–q^s variables into d^e–q^e variables. Vector V and its components are further projected on the rotating and stationary axes. The a_S-b_S-c_S variables can also be expressed in the vector form. Substituting equations (3.24) and (3.25) into Equation (3.37), we get

$$\begin{aligned}\overline{V} &= v_{qs}^{s} - jv_{ds}^{s} \\ &= \left(\frac{2}{3}v_{as} - \frac{1}{3}v_{bs} - \frac{1}{3}v_{cs}\right) - j\left(-\frac{1}{\sqrt{3}}v_{bs} + \frac{1}{\sqrt{3}}v_{cs}\right) \\ &= \frac{2}{3}\left[v_{as} + \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)v_{bs} + \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)v_{cs}\right] \\ &= \frac{2}{3}\left[v_{as} + av_{bs} + a^{2}v_{cs}\right]\end{aligned} \tag{3.41}$$

where $a = e^{j\frac{2}{3}\pi}$ & $a^2 = e^{-j\frac{2}{3}\pi}$ the two parameters *a and* a^2 are interpreted as unit vectors aligned to the respective 'b_s' *and* 'c_s' axes of the machines. The reference axis corresponds to the v_{as}-axis. Note that all the above transformations done for the stator variables can also be carried out for the rotor circuit variables.

3.4 DYNAMIC KRON EQUATION MACHINE MODEL

For a 2-phase machine, we need to represent both d_s–q_s and d_r–q_r circuits and their variables in a synchronously rotating d^e–q^e frame. The stator circuit equations can be modeled as follows:

$$v_{qs}^{s} = R_s i_{qs}^{s} + \frac{d}{dt}\Psi_{qs}^{s} \tag{3.42}$$

$$v_{ds}^{s} = R_s i_{ds}^{s} + \frac{d}{dt}\Psi_{ds}^{s} \tag{3.43}$$

where Ψ_{qs}^{s} and Ψ_{ds}^{s} are the *q*-axis and *d*-axis stator flux linkages, respectively. Equations (3.42) and (3.43) are further converted into d^e–q^e frames as

$$v_{qs} = R_s i_{qs} + \frac{d}{dt}\Psi_{qs} + \omega_e \Psi_{ds} \tag{3.44}$$

$$v_{ds} = R_s i_{ds} + \frac{d}{dt}\Psi_{ds} - \omega_e \Psi_{qs} \tag{3.45}$$

Note that in the above equations, all the variables are in rotating form. The last terms in equations (3.44) and (3.45) are the speed emfs due to the rotation of the axes, i.e., when $\omega_e=0$, the equations revert back to the stationary form. The flux linkages in the d^e and q^e axes induce emfs in the q^e and d^e axes with a leading angle of 90^0. If the rotor is not moving, i.e., $\omega_r=0$, then the rotor equations for a doubly fed wound rotor IM will be similar to the stator equations and are given by

$$v_{qr} = R_r i_{qr} + \frac{d\Psi_{qr}}{dt} + \omega_e \Psi_{dr} \tag{3.46}$$

$$v_{dr} = R_r i_{dr} + \frac{d\Psi_{dr}}{dt} - \omega_e \Psi_{qr} \tag{3.47}$$

where all the variables and parameters are referred to the stator. Since the rotor moves at a speed of $\omega_r = 0$, the d–q axes fixed on the rotor moves at a speed of ω_e–ω_r relative to the synchronously rotating frame. Therefore, in d^e–q^e frame, the rotor equations can be rewritten as follows:

$$v_{qr} = R_r i_{qr} + \frac{d\Psi_{qr}}{dt} + \left(\omega_e - \omega_r\right)\Psi_{dr} \tag{3.48}$$

$$v_{dr} = R_r i_{dr} + \frac{d\Psi_{dr}}{dt} - (\omega_e - \omega_r)\Psi_{qr} \quad (3.49)$$

The flux linkage expressions in terms of the currents can be written similar to that in equations (3.50)–(3.55).Using equations (3.50)–(3.55) in equations (3.44)–(3.49), the electrical transient model of the IM in terms of v and i is given in matrix form as in Equation (3.56); in the inverse form, the i–v matrix could be written as in Equation (3.57).

$$\psi_{qs} = L_{ls} i_{qs} + L_m (i_{qs} + i_{qr}) \quad (3.50)$$

$$\psi_{qr} = L_{lr} i_{qr} + L_m (i_{qs} + i_{qr}) \quad (3.51)$$

$$\psi_{qm} = L_m (i_{qs} + i_{qr}) \quad (3.52)$$

$$\psi_{ds} = L_{ls} i_{ds} + L_m (i_{ds} + i_{dr}) \quad (3.53)$$

$$\psi_{dr} = L_{lr} i_{dr} + L_m (i_{ds} + i_{dr}) \quad (3.54)$$

$$\psi_{dm} = L_m (i_{ds} + i_{dr}) \quad (3.55)$$

$$\begin{bmatrix} v_{qs} \\ v_{dr} \\ v_{qr} \\ v_{dr} \end{bmatrix} = \begin{bmatrix} R_s + sL_s & -\omega_e L_s & sL_m & -\omega_g L_m \\ -\omega_e L_s & R_s + sL_s & -\omega_e L_m & sL_m \\ sL_m & (\omega_e - \omega_r) L_m & R_r + sL_r & (\omega_e - \omega_r) L_r \\ -(\omega_e - \omega_r) & sL_m & -(\omega_e - \omega_r) L_r & R_r + sL_r \end{bmatrix} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qr} \\ i_{dr} \end{bmatrix} \quad (3.56)$$

Or

$$\begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qr} \\ i_{dr} \end{bmatrix} = \begin{bmatrix} R_s + sL_s & -\omega_e L_s & sL_m & -\omega_g L_m \\ -\omega_e L_s & R_s + sL_s & -\omega_e L_m & sL_m \\ sL_m & (\omega_e - \omega_r) L_m & R_r + sL_r & (\omega_e - \omega_r) L_r \\ -(\omega_e - \omega_r) & sL_m & -(\omega_e - \omega_r) L_r & R_r + sL_r \end{bmatrix}^{-1} \begin{bmatrix} v_{qs} \\ v_{dr} \\ v_{qr} \\ v_{dr} \end{bmatrix} \quad (3.57)$$

where‘s’ is the Laplacian operator. For a single phase IM, such as the cage motor, v_{qr} = v_{ds} = *0*. If the speed ω_r is considered to be constant (infinite inertia load), the electrical dynamics of the IM are given by a fourth-order linear system. Then, by deriving the inputs v_q, v_{ds} *and* ω_e, the currents i_{qs}, i_{ds}, i_{qr} *and* i_{dr} can be solved from Equation (3.56). Note that the speed ω_r cannot be treated as a constant and is related to the torques as

$$T_e = T_L + J \frac{d}{dt} \omega_m = T_L + \frac{2}{p} J \frac{d\omega_r}{dt} \quad (3.58)$$

where T_L is the load torque, *J* is the rotor inertia and ω_m is the mechanical speed of the IM. In order to obtain a compact representation of the IM model, the equivalent circuits are expressed in the complex form. Multiplying Equation (3.45) by *-j* and adding it to Equation (3.44) gives

$$v_{qs} - jv_{ds} = R_s \left(i_{qs} - ji_{ds} \right) + \frac{d}{dt} \left(\Psi_{qs} - j\Psi_{ds} \right) + j\omega_e \left(\Psi_{qs} - j\Psi_{ds} \right) \quad (3.59)$$

$$v_{qds} = R_s i_{qds} + \frac{d}{dt} \Psi_{qds} + j\omega_e \Psi_{qds} \quad (3.60)$$

where v_{qds}, i_{qds}, etc. are the complex vectors. Similar to the previous equations, the rotor Equations (3.48)–(3.49) can be combined to represent the rotor equation as

$$v_{qdr} = R_r i_{qdr} + \frac{d}{dt}\Psi_{qdr} + j(\omega_e - \omega_r)\Psi_{qdr} \tag{3.61}$$

Note that the steady-state equations can always be derived by substituting the time derivative components to zero. The final steady-state equations after making certain assumptions can be obtained as

$$v_s = R_s I_s + j\omega_e \Psi_s \tag{3.62}$$

$$0 = \frac{R_s}{r} R_r + j\omega_e \Psi_r \tag{3.63}$$

where the complex vectors have been substituted by the corresponding rms phasors. These equations satisfy the steady-state equivalent circuits shown in Figs. 3.7 and 3.8, respectively, if the parameter R_m is neglected.

The development of torque is also very important in the modeling of IMs. Here, it will be expressed in a more general form, relating the d–q components of the variables. From Equation (3.58), the torque can be generally expressed in the vector form as

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\overline{\Psi}_m \times \overline{I}_r \tag{3.64}$$

Resolving the variables into d^e–q^e components, we obtain

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{dm} i_{qr} - \Psi_{qm} i_{dr}\right) \tag{3.65}$$

Several other torque expressions can be derived from the above torque relations as follows:

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{dm} i_{qs} - \Psi_{qm} i_{ds}\right) \tag{3.66}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{ds} i_{qs} - \Psi_{qs} i_{ds}\right) \tag{3.67}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right) L_m \left(i_{qs} i_{dr} - i_{ds} i_{qr}\right) \tag{3.68}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{dr} i_{qr} - \Psi_{qr} i_{dr}\right) \tag{3.69}$$

Equations (3.57), (3.58) and (3.65) give the complete mathematical model of the electro-mechanical dynamics of an IM in the synchronous frame. The composite system is of the fifth order and non-linearity of the model is evident.

3.5 DYNAMIC STANLEY EQUATION MACHINE MODEL

The dynamic machine model in stationary frame can be derived simply by substituting $\omega_e = 0$ in Equation (3.57) or in Equations (3.44), (3.45), (3.48) and (3.49). The corresponding stationary frame equations are given as follows:

$$v_{qs}^{s} = R_s i_{qs}^{s} + \frac{d}{dt}\Psi_{qs}^{s} \tag{3.70}$$

$$v_{ds}^{s} = R_s i_{ds}^{s} + \frac{d}{dt}\Psi_{ds}^{s} \tag{3.71}$$

$$0 = R_r i_{qr}^{s} + \frac{d}{dt}\Psi_{qr}^{s} - \omega_r \Psi_{dr}^{s} \tag{3.72}$$

$$0 = R_r i_{dr}^{s} + \frac{d}{dt}\Psi_{dr}^{s} + \omega_r \Psi_{qr}^{s}. \tag{3.73}$$

where $\upsilon_{qr} = \upsilon_{dr} = 0$. The torque Equations (3.64)–(3.69) can also be written with the corresponding variables in the stationary frame as follows:

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{dm}^{s} i_{qr}^{s} - \Psi_{qm}^{s} i_{dr}^{s}\right) \tag{3.74}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{dm}^{s} i_{qs}^{s} - \Psi_{qm}^{s} i_{ds}^{s}\right) \tag{3.75}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{ds}^{s} i_{qs}^{s} - \Psi_{qs}^{s} i_{ds}^{s}\right) \tag{3.76}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)L_m\left(i_{qs}^{s} i_{dr}^{s} - i_{ds}^{s} i_{qr}^{s}\right) \tag{3.77}$$

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)\left(\Psi_{dr}^{s} i_{qr}^{s} - \Psi_{qr}^{s} i_{dr}^{s}\right) \tag{3.78}$$

which give rise to the Stanley machine model Equations (3.42)–(3.43) and (3.72)-(3.73) can easily combined to derive the complex model as

$$v^s_{qds} = R_s i^s_{qds} + \frac{d}{dt}\Psi^s_{qds} \tag{3.79}$$

$$0 = R_r i^s_{qds} + \frac{d}{dt}\Psi^s_{qdr} - j\omega_r \Psi^s_{qds} \tag{3.80}$$

where $\upsilon^s_{qds} = \upsilon^s_{qs} - j\upsilon^s_{ds}$, $\psi^s_{qds} = \psi^s_{qs} - j\psi^s_{ds}$, $i^s_{qds} = i^s_{qs} - ji^s_{ds}$ and $\psi^s_{qdr} = \psi^s_{qr} - j\psi^s_{dr}$. The complex equivalent circuit in stationary frame can also be developed further and is not shown here for the sake of convenience. Often, a per-phase equivalent circuit with $\left(\omega_r \overline{\Psi_r}\right)$ and sinusoidal variables can be obtained, which leads to Figs. 3.7 and 3.8 by omitting the parameter L_m of all the various models explained in the different sections so far, the equations of the dynamic *d-q* model depicted in section 3.3 is used in this research work.