CHAPTER VIII

COMMON FIXED POINT THEOREMS

ON SAKS SPACE

154 - 164
8.1 In this chapter some common fixed point theorems on Saks space are given. For the details of Saks spaces we refer Alexiewicz ([1], [2]), Alexiewicz and Semandi ([3], [4]), Orlicz ([108], [109]), Orlicz and Ptak [110], Wiweger [164], and Cho and Singh ([29], [30]).

We need the following definitions to make clear the Saks spaces.

Definition (1): A real valued function $N$ defined on a linear space $X$ is called a B-norm if it satisfies the following conditions:

(i) $N(x) = 0$ if and only if $x = 0$,

(ii) $N(x+y) \leq N(x) + N(y)$,

(iii) $N(ax) = |a| N(x)$, where $a$ is any real numbers.

Definition (2): A real valued function $N$ defined on a linear space $X$ will be called an F-norm if it satisfies the following conditions:

(i) $N(x) = 0$ if and only if $x = 0$, 

(ii) $N(x+y) \leq N(x) + N(y)$,

(iii) $N(ax) = |a| N(x)$, where $a$ is any real numbers.
(ii) $N(x + y) \leq N(x) + N(y)$,

(iii) If $\{a_n\}$ be a sequence of real numbers converges to a real number $a$ and

$$N(x_n - x) \to 0, \text{ as } n \to \infty,$$

then

$$N(a_n x_n - ax) \to 0, \text{ as } n \to \infty.$$

**Definition (3):** A two-norm space is a linear space $X$ with two norms, a $B$-norm $N_1$ and $F$-norm $N_2$ and is defined by $(X, N_1, N_2)$.

**Definition (4):** Let $N_1$ and $N_2$ be two-norms defined on $X$, then $N_1$ is called non-weaker than $N_2$ in $X$ (that is $N_2 \sim N_1$), if

$$N_1(x_n) \to 0, \text{ as } n \to \infty \Rightarrow N_2(x_n) \to 0, \text{ as } n \to \infty$$

Where $x_n \in X$.

We note here that the two-norms $N_1$ and $N_2$ are equivalent if $N_1 \leq N_2$ and $N_2 \leq N_1$.

**Definition (5):** Let $(X, N_1, N_2)$ be a two-norms space, then a sequence $\{x_n\}$ of $X$ said to be $\tau$-convergent to a point $x$ in $X$, if

$$\sup_n N_1(x_n) < \infty \text{ and } \lim_n N_2(x_n - x) = 0.$$
Definition (6): Let \((X, N_1, N_2)\) be a two-norm space, then a sequence \(\{x_n\}\) of \(X\) is a \(Y\)-cauchy sequence, if
\[
(x_{p_n} - x_{q_n}) \to 0 \text{ as } p_n, q_n \to \infty.
\]

Definition (7): A two-norm space \((X, N_1, N_2)\) is called \(Y\)-complete, if every \(Y\)-cauchy sequence \(\{x_n\}\) in two-norm space, there exists a point \(x\) in \(X\) such that \(x_n \to x\).

Let \(X\) be a linear set and suppose that \(N_1\) and \(N_2\) are \(B\)-norm and \(F\)-norm on \(X\) respectively.

Let \(X_S = \{x \in X, N_1(x) < 1\}\) and define
\[
d(x, y) = N_2(x-y)
\]
for all \(x, y \in X_S\).

Then \(d\) is a metric on \(X_S\) and the metric space \((X_S, d)\) will be called a saks set.

Definition (8): Let \((X_S, d)\) be a saks set, then it is called saks space, if it is complete. The saks space is denoted by \((X, N_1, N_2)\).

8.2 In this section some fixed point theorems on saks space are proved. We introduce the concept of "weakly
uniformly contraction" pair of mappings in Saks spaces.
Which is introduced by Pathak [120] in metric spaces.

We recall this definition in Saks space as follows:

**Definition (9):** Let \((X, N_1, N_2) = (X, d, d)\) be a Saks space
and \(N_1\) is equivalent to \(N_2\) on \(X\). Let \(f\) and \(g\) are self
mappings of \(X\), then \(\{f, g\}\) is called "weakly uniformly
contraction" pair of mappings if

(i) \(N_2(fgx - ggx) \leq N_2(fx - gx)\) and

(ii) \(N_2(ffx - gfx) \leq N_2(fx - gx)\) for every \(x\) in \(X\).

We need the following lemmas for our main theorems.

**Lemma (1)** ([108]): Let \((X, d, d) = (X, N_1, N_2)\) be a Saks
space. Then the following statements are equivalent

(i) \(N_1\) is equivalent to \(N_2\) on \(X\).

(ii) \((X, N_1)\) is a Banach space and \(N_1 \leq N_2\) on \(X\).

(iii) \((X, N_2)\) is a Fréchet space and \(N_2 \leq N_1\) on \(X\).

**Lemma (2)** ([152]): For every \(t > 0\), \(\gamma(t) < t\) if and only
if

\[
\lim_{n \to \infty} \gamma^n(t) = 0.
\]
Where \(\gamma^n\) denotes the \(n\)-times composition of \(\gamma\).
Theorem (1) Let $(X, d) = (X, N_1, N_2)$ be a saks space and $N_1$ is equivalent to $N_2$ on $X$. Let $E, F$ and $T$ are self mappings of $X$ satisfying the following conditions:

(8.2.1) $E(X) \subseteq T(X)$ and $F(X) \subseteq T(X)$,

(8.2.2) $\{T, E\}$ and $\{T, F\}$ are weakly uniformly contraction pair of mappings,

(8.2.3) $T(X)$ is a closed sub space of $X$ with respect to $N_1$,

(8.2.4) For every $x, y \in X$,

$$N_2(Ex-Fy) \leq k \cdot \phi(\max(N_2(Tx-Ty), N_2(Tx-Ex), N_2(Ty-Fy)),$$

$$k[N_2(Tx-Fy) + N_2(Ty-Ex)]),$$

$0 \leq k < 1$ and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-decreasing, continuous function with (i) $\phi(t) < t$ for every $t > 0$ and

(ii) $t - \phi(t) \rightarrow 0$ as $t \rightarrow \infty$.

Then $E, F$ and $T$ have a unique common fixed point in $X$.

Proof : Let $x_0$ be an arbitrary point in $X$. Since $E(X) \subseteq T(X)$, there exists $x_1$ in $X$ such that $Tx_1 = Ex_0$. Again since $F(X) \subseteq T(X)$, there exists $x_2$ in $X$ such that $Tx_2 = Ex_1$. In general there exists $x_{2n+1}$ and $x_{2n+2}$ in $X$ such that

$$Tx_n = \begin{cases} Ex_{n-1}, & \text{if } n \text{ is odd} \\ Fx_{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

Now from (8.2.4) and (8.2.5), we have

\[ N_2(Tx_{2n+1} - Tx_{2n+2}) = N_2(Fx_{2n} - Fx_{2n+1}) \]

\[ \leq k \phi (\max\{N_2(Tx_{2n} - Tx_{2n+1}), N_2(Tx_{2n} - Ex_{2n})\}) \]

\[ N_2(Tx_{2n+1} - Fx_{2n+1}) + h(N_2(Tx_{2n+1} - Ex_{2n}) + N_2(Tx_{2n} - Fx_{2n+1})) \]

\[ = k \phi (\max\{N_2(Tx_{2n} - Tx_{2n+1}), N_2(Tx_{2n} - Tx_{2n+1})\}) \]

\[ N_2(Tx_{2n+1} - Tx_{2n+2}), \phi(Tx_{2n} - Tx_{2n+1})) \]

if \( N_2(Tx_{2n+1} - Tx_{2n+2}) \geq N_2(Tx_{2n} - Tx_{2n+1}) \), then we arrive at a contradiction. Therefore

\[ N_2(Tx_{2n+1} - Tx_{2n+2}) \leq k \phi (N_2(Tx_{2n} - Tx_{2n+1})) \]

\[ N_2(Tx_{2n+1} - Tx_{2n+2}) \leq k \cdot N_2(Tx_{2n} - Tx_{2n+1}) (\Rightarrow \phi(t) < t). \]

Similarly, \( N_2(Tx_{2n} - Tx_{2n+1}) \leq k \cdot N_2(Tx_{2n+1} - Tx_{2n}). \)

Proceeding in this way

\[ N_2(Tx_{2n+1} - Tx_{2n+2}) < (k)^{2n} \cdot N_2(Tx_1 - Tx_2). \]

Since \( 0 < k < 1 \), \( N_2(Tx_{2n+1} - Tx_{2n+2}) \to 0 \) as \( n \to \infty \).

By Lemma (1), \( N_1(Tx_{2n+1} - Tx_{2n+2}) \to 0 \) as \( n \to \infty \).

The above condition shows that \( \{Tx_n\} \) is a Cauchy sequence with respect to \( N_1 \). Again from Lemma (1), \((X, N_1)\) is a Ban space, so the sequence \( \{Tx_n\} \) converges to a point \( p \) in \( T(X) \).
Since \( \{E_{2n}\} \) and \( \{F_{2n+1}\} \) are sub-sequences of \( \{T_{n}\} \) so these sub-sequences are also converges to a point \( p \). Hence, there exists \( z \) in \( X \) such that \( Tz = p \).

Now putting \( x = x_{2n} \) and \( y = z \) in condition (8.2.4) and letting \( n \to \infty \), then we have

\[
N_2(E_{2n} - Fz) \leq k \cdot \phi \left( \max\{N_2(T_{2n} - Tz), N_2(T_{2n} - E_{2n})\}, N_2(Tz - Fz), \frac{1}{2}[N_2(Tz - E_{2n}) + N_2(T_{2n} - Fz)]\right).
\]

This yields

\[
N_2(Tz - Fz) \leq k \cdot \phi \left( \max\{0, 0, N_2(Tz - Fz), 0\}\right),
\]

\[
N_2(Tz - Fz) \leq k \cdot \phi \left( N_2(Tz - Fz)\right)
\]

implies that \( N_2(Tz - Fz) < k \cdot N_2(Tz - Fz) \).

Hence, \( Tz = Fz \). Similarly, \( Tz = Ez \).

Thus \( Tz = Ez = Fz \).

Since \( \{T, E\} \) be a weakly uniformly contraction pair of mappings, \( N_2(ITz - TTz) = 0 \) implies \( ETz = TTz \) and \( N_2(TEz - EEz) = 0 \) implies \( TEz = EEz \).

Therefore, \( TTz = ETz = EEz = TEz \).

Similarly \( TTz = FTz = FFz = TFz \) as \( \{T, F\} \) is a weakly
uniformly contraction pair of mappings.

Again substituting $x = Ez$ and $y = z$ in (8.2.4), then

$$N_2(EEz-Fz) \leq k \cdot \phi \left( \max \{ N_2(TEz-Tz), N_2(TEz-EEz), N_2(Tz-Fz), \frac{1}{k} [N_2(TEz-Fz) + N_2(Tz-EEz)] \} \right),$$

thus we have $EEz = Ez$. Similarly $FFz = Fz$ and $TTz = Tz$.

Therefore, $Tz = Ez = Fz$ as a common fixed point of $E, F$ and $T$.

For uniqueness of common fixed point:

Suppose $q$ be another common fixed point of $E, F$ and $T$, then again from (8.2.4). We have

$$N_2(p-q) = N_2(Ep-Fq) \leq k \cdot \phi \left( \max \{ N_2(Tp-Tq), N_2(Tp-Ep), N_2(Tq-Fq), \frac{1}{k} [N_2(Tq-Fq) + N_2(Tq-Ep)] \} \right),$$

$$= k \cdot \phi \left( \max \{ N_2(p-q), 0, 0, N_2(p-q) \} \right),$$

$$= k \cdot \phi \left( N_2(p-q) \right),$$

$$N_2(p-q) < N_2(p-q)$$
a contradiction. Hence, $p$ is a unique common fixed point of $E, F$ and $T$.

This completes the proof.
Theorem (2) [102] Let \((X, d) = (X, N_1, N_2)\) be a saks space and \(N_1\) is equivalent to \(N_2\) on \(X\). Let \(E, F\) and \(T\) are self mappings of \(X\) satisfying (8.2.1), (8.2.2), (8.2.3) and there exists \(\phi \in F, x, y \in X\);

\(8.2.6\) \(N_2^2(Ex-Fy) \leq 4 \phi (N_2^2(Tx-Ty))\),

\[N_2(Tx-Ex), N_2(Ty-Fy),\]

\[N_2(Tx-Fy), N_2(Ty-Ex),\]

\[N_2(Tx-Fy), N_2(Ty-Fy)\]

for any \(t > 0, \phi (t, t, 0, at, 0) \leq bt, \phi (t, t, 0, 0, at) \leq bt\)

where \(b = 1\) for \(a = 2\) and \(b < 1\) for \(a < 2\) and

\[\gamma(t) = \phi (t, t, a_1 t, a_2 t, a_3 t) < t,\]

where \(a_1 + a_2 + a_3 = 4\) and \(\gamma : \mathbb{R} \rightarrow \mathbb{R}^+\).

Then \(E, F\) and \(T\) have a unique common fixed point.

Proof : Pick up a point \(x_0\) in \(X\) and consider the iteration (8.2.5). Now using (8.2.6) & (8.2.5), we have

\[N_2^2(Tx_{2n+1} - Tx_{2n+2}) = N_2^2(Ex_{2n} - Fx_{2n+1})\]

\[\leq 4 \phi (N_2^2(Tx_{2n} - Tx_{2n+1})),\]

\[N_2(Tx_{2n} - Ex_{2n}), N_2(Tx_{2n+1} - Fx_{2n+1})\]

\[N_2(Tx_{2n} - Fx_{2n+1}), N_2(Tx_{2n+1} - Ex_{2n})\]

\[N_2(Tx_{2n} - Ex_{2n}), N_2(Tx_{2n+1} - Ex_{2n})\]

\[N_2(Tx_{2n} - Fx_{2n+1}), N_2(Tx_{2n+1} - Fx_{2n+1})\]
contradiction. Hence, we have

\[ N_2^2 (Tx_{2n} - Tx_{2n+1}) \leq \phi (N_2^2 (Tx_{2n} - Tx_{2n+1}), N_2^2 (Tx_{2n} - Tx_{2n+1}), \ldots, 0, 0, 2N_2^2 (Tx_{2n} - Tx_{2n+1})). \]

Which implies that \( N_2^2 (Tx_{2n+1} - Tx_{2n+2}) \leq N_2^2 (Tx_{2n} - Tx_{2n+1}). \)

Similarly, \( N_2^2 (Tx_{2n} - Tx_{2n+1}) \leq N_2^2 (Tx_{2n-1} - Tx_{2n}). \)

Hence, \( N_2^2 (Tx_{2n+1} - Tx_{2n+2}) \to 0 \) as \( n \to \infty \) and by Lemma (1)

\[ N_1^2 (Tx_{2n} - Tx_{2n+1}) \to 0 \] as \( n \to \infty \). This shows that the sequence \( \{Tx_n\} \) is a cauchy sequence with respect to \( N_1 \).

Again from Lemma (1), \((X, N_1)\) is a Banach space and thus \( \{Tx_n\} \) has a limit \( p \) in \( T(X) \). Since \( \{Ex_{2n}\} \) and \( \{Fx_{2n+1}\} \) are sub-sequences of \( \{Tx_n\} \) so these sub-sequences are also converges to \( p \). Hence, there exists a point \( z \) in \( X \) such that \( Tz = p \).
Putting \( y = z \) and \( x = x_{2n} \) in (8.2.6), then we have

\[
N_2(E_{2n}x_n - Fz) \leq \phi \left( \phi \left( N_2(Tx_{2n} - Tz), \right) \right),
\]

\[
N_2(Tx_{2n} - Ex_{2n}). N_2(Tz - Fz),
\]

\[
N_2(Ex_{2n} - Fz). N_2(Tz - Ex_{2n}),
\]

\[
N_2(Tx_{2n} - Ex_{2n}). N_2(Tz - Ex_{2n}),
\]

\[
N_2(Tx_{2n} - Fz). N_2(Tz - Fz)).
\]

Taking \( n \) tends to infinity in (8.2.7), then

\[
N_2(Tz - Fz) \leq \phi \left( \phi \left( 0, 0, 0, 0, N_2^2(Tz - Fz) \right) \right) \]

\[
\leq \psi \left( N_2^2(Tz - Fz) \right).
\]

By Lemma (2), \( N_2(Tz - Fz) < N_2(Tz - Fz) \) implies \( Tz = Fz \).

Similarly we can show that \( Tz = Ez \).

Hence, \( Tz = Ez = Fz \).

Common fixed point and uniqueness, follows in lines of Theorem (1).

Remark : If we relax the condition (8.2.2) from the Theorem (1) and Theorem (2), then we have coincidence point theorems.