CHAPTER IV

COMPATIBLE MAPPINGS OF TYPE (A) AND
COMMON FIXED POINT THEOREMS ON
2-METRIC SPACE

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COMPATIBLE MAPPINGS OF TYPE (A) AND COMMON FIXED POINT THEOREMS ON 2-METRIC SPACE

4.1 The intent of this chapter is to introduce the concept of compatible mappings and compatible mappings of type (A) on 2-metric spaces and drive some relations between these mappings. Also, we prove a coincidence point theorem and a common fixed point theorem for compatible mappings of type (A) on 2-metric spaces.

4.2 In this section, we introduce the concepts of compatible mappings and compatible mappings of type (A) and show that these mappings are equivalent under some conditions. Throughout this section, \((X,d)\) denotes a 2-metric space with a continuous 2-metric \(d\).

Now, we shall give some definitions:

**Definition** (M) : A 2-metric is a set \(X\) with a real-valued function \(d\) on \(X \times X \times X\) satisfying the following conditions:

(M1) For two distinct points \(x, y\) in \(X\), there exists a point \(z\) in \(X\) such that \(d(x, y, z) \neq 0\),
(M₂) \( d(x,y,z) = 0 \) if at least two of \( x, y, z \) are equal.

(M₃) \( d(x,y,z) = d(x,z,y) = d(y,z,x) \).

(M₄) \( d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z) \) for all \( x, y, z, u \) in \( X \).

The function \( d \) is called a 2-metric for the space \( X \) and \( (X,d) \) is called a 2-metric space. It has been shown by Gähler ([48]) that a 2-metric \( d \) is non-negative and although \( d \) is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric \( d \) which is continuous in all of its arguments will be called continuous.

**Definition (1)**: A sequence \( \{x_n\} \) in a 2-metric space \( (X,d) \) is said to be convergent to a point \( x \) in \( X \), which is denoted by \( \lim_{n \to \infty} x_n = x \), if \( \lim_{n \to \infty} d(x_n, x, z) = 0 \) for all \( z \) in \( X \). Then \( x \) is called the limit of the sequence \( \{x_n\} \) in \( X \).

**Definition (2)**: A sequence \( \{x_n\} \) in a 2-metric space \( (X,d) \) is said to be Cauchy sequence if \( \lim_{m,n \to \infty} d(x_m, x_n, z) = 0 \) for all \( z \) in \( X \).
Definition (3) : A 2-metric space \((X,d)\) is said to be complete if every Cauchy sequence in \(X\) is convergent.

Note that in a 2-metric space \((X,d)\) a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric \(d\) is continuous on \(X\) ([107]).

Definition (4) : Let \(S\) and \(T\) be mappings from a 2-metric space \((X,d)\) into itself. \(S\) and \(T\) are said to be compatible if

\[
\lim_{n \to \infty} d(ST_{x_n}, TS_{x_n}, z) = 0,
\]

for all \(z\) in \(X\), whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \text{ in } X.
\]

Definition (5) : Let \(S\) and \(T\) be mappings from a 2-metric space \((X,d)\) into itself. \(S\) and \(T\) are compatible of type (A) if

\[
\lim_{n \to \infty} d(TS_{x_n}, SS_{x_n}, z) = 0 \text{ and } \lim_{n \to \infty} d(ST_{x_n}, TT_{x_n}, z) = 0,
\]

for all \(z\) in \(X\), whenever \(\{x_n\}\) is a sequence in \(X\) and such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \text{ in } X.
\]
The following propositions show that Definition (4) and (5) are equivalent under some conditions:

**Proposition (1)**: Let \( S \) and \( T \) be continuous mappings of a 2-metric space \((X,d)\) into itself. If \( S \) and \( T \) are compatible, then they are compatible of type (A).

**Proof**: Suppose that \( S \) and \( T \) are compatible. Let \( \{x_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in \( X \). By (M), we have

\[
d(SSx_n, TSx_n, z) \leq d(SSx_n, TSx_n, STx_n) + d(SSx_n, STx_n, z) + d(STx_n, TSx_n, z).
\]

Hence, since \( S \) and \( T \) are compatible and \( S \) is continuous, we have

\[
\lim_{n \to \infty} d(SSx_n, TSx_n, z) = 0 \text{ for all } z \text{ in } X.
\]

Similarly, we have

\[
\lim_{n \to \infty} d(TTx_n, STx_n, z) = 0 \text{ for all } z \text{ in } X.
\]

Therefore, \( S \) and \( T \) are compatible mappings of type (A).

This completes the proof.
Proposition (2) : Let S and T be compatible mappings of type (A) from a 2-metric space \((X,d)\) into itself. If one of \(S\) and \(T\) is continuous, then \(S\) and \(T\) are compatible.

Proof : Assume that, without loss of generality, \(T\) is continuous. Let \(\{x_n\}\) be a sequence in \(X\) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \text{ in } X.
\]
Since, \(T\) is continuous, we have \(\lim_{n \to \infty} TSx_n = Tt\). By, (M4), since we have
\[
d(TSx_n, TTx_n, z) \leq d(TSx_n, TTx_n, Tt) + d(TSx_n, Tt, z) + d(Tt, TTx_n, z).
\]

\[
\lim_{n \to \infty} d(TSx_n, TTx_n, z) = 0 \text{ for all } z \text{ in } X.
\]
Again, by (M4), since we have
\[
d(STx_n, TSx_n, z) \leq d(STx_n, TSx_n, TTx_n) + d(STx_n, TTx_n, z) + d(TTx_n, TSx_n, z),
\]
\[
\lim_{n \to \infty} d(STx_n, TSx_n, z) = 0 \text{ for all } z \text{ in } X.
\]
Therefore, \(S\) and \(T\) are compatible. This completes the proof.
As a direct consequence of Proposition (1) and (2), we have the following:

**Proposition (3)**: Let $S$ and $T$ be continuous mappings from a 2-metric space $(X,d)$ into itself. Then $S$ and $T$ are compatible if and only if they are compatible of type (A).

Next, we give some properties of compatible mappings of type (A) for our main theorems:

**Proposition (4)**: Let $S$ and $T$ be compatible mappings of type (A) from a 2-metric space $(X,d)$ into itself. If $S^t = T^t$ for some $t$ in $X$, then $S^{T^t} = T^{S^t} = T^{S^t} = S^{T^t}$.

**Proof**: Suppose that $\{x_n\}$ is a sequence in $X$ defined by $x_n = t$, $n = 1, 2, \ldots$, and $S^t = T^t$. Then, we have

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = S^t.
\]

Since $S$ and $T$ are compatible mappings of type (A), we have

\[
\lim_{n \to \infty} d(S^{T^t}, T^{S^t}, z) = \lim_{n \to \infty} d(S^{T^t}, T^{S^t}, z) = 0.
\]

Hence, we have $S^{T^t} = T^{S^t}$. Similarly, we have $T^{S^t} = S^{T^t}$. 
But \( Tt = St \) implies \( TTt = TSt \). Therefore \( STt = TTt = TSt = SSSt \).

This completes the proof.

**Proposition (5)**: Let \( S \) and \( T \) be compatible mappings of type (A) from a 2-metric \((X, d)\) into itself. Suppose that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \in X.
\]

Then we have the following:

(i) \( \lim_{n \to \infty} TSx_n = St \) if \( S \) is continuous at \( t \).

(ii) \( STt = TSt \) and \( St = Tt \) if \( S \) and \( T \) are continuous at \( t \).

**Proof**: (i) Suppose that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in \( X \). Since \( S \) is continuous, we have \( \lim_{n \to \infty} STx_n = St \). By

\[(M_4)\), we have

\[
d(TSx_n, St, z) \leq d(TSx_n, St, SSx_n) + d(TSx_n, SSx_n, z) + d(SSx_n, St, z).
\]

Therefore, since \( S \) and \( T \) are compatible mappings of type (A), we have

\[
\lim_{n \to \infty} TSx_n = St.
\]

(ii) Since \( T \) is continuous at \( t \), we have

\[
\lim_{n \to \infty} TSx_n = Tt.
\]
On the other hand, by (i), since $S$ is continuous at $t$, we have

$$\lim_{n \to \infty} T S x_n = S t.$$

Hence, by uniqueness of the limit, we have $S t = T t$ and so, by Proposition (4), $S T t = T S t$. This completes the proof.

4.3 Let $S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself. A point $u$ in $X$ is called a coincidence point of $S$ and $T$ if $S u = T u$.

Let $N$ and $R^+$ be the set of all natural numbers and non-negative real numbers, respectively, and $F$ the family of mappings $\phi$ from $(R^+)^5$ into $R^+$ such that $\phi$ is upper semi-continuous, nondecreasing in each coordinate variable, and for any $t > 0$,

$$\phi(t,t,0,at,0) \leq \beta t \text{ and } \phi(t,t,0,at) \leq \beta t,$$

where $\beta = 1$ for $a = 2$ and $\beta < 1$ for $a < 2$,

$$\gamma(t) = \phi(t,t,a_1 t,a_2 t,a_3 t) < t,$$

where $\gamma : R^+ \to R^+$ is a mapping and $a_1 + a_2 + a_3 = 4$. 
Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself such that

\[(4.3.1) \quad A(X) \cup B(X) \subseteq S(X) \cap T(X),\]

\[(4.3.2) \quad d^2(Ax, By, z) \leq \Phi(d^2(Sx, Ty, z),
\begin{align*}
&d(Sx, Ax, z) \cdot d(Ty, By, z), \\
&d(Sx, By, z) \cdot d(Ty, Ax, z), \\
&d(Sx, Ax, z) \cdot d(Ty, Ax, z), \\
&d(Sx, By, z) \cdot d(Ty, By, z)),
\end{align*}\]

where $\Phi \in \mathbb{P}$. Then, by (4.3.1), since $A(X) \subseteq T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this point $x_1$, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in $X$ such that

\[(4.3.3) \quad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}\]

for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For our main theorems, we need the following Lemmas:
Lemma (1): For any \( t > 0 \), \( \Phi(t) = t \) if and only if \( y^n(t) = 0 \), where \( y^n \) denotes the \( n \)-times composition of \( y \).

Lemma (2): Let \( X \), \( d \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself satisfying the conditions (4.1.1) and (4.1.2). Then we have the following:

(a) for every \( n \in \mathbb{N}_0 \), \( d(y_n, y_{n+1}, y_{n+2}) = 0 \), and
(b) for every \( i, j, k \in \mathbb{N}_0 \), \( d(y_i, y_j, y_k) = 0 \), where \( \{y_n\} \) is the sequence in \( X \) defined by (4.1.3).

Proof: (a) In (4.1.2), taking \( x = x_{2n+2}, y = x_{2n+1} \) and \( z = y_{2n} \), we have

\[
d^2(y_{2n+2}, y_{2n+1}, y_{2n}) = d^2(Ax_{2n+2}, Bx_{2n+1}, y_{2n}) \\
\leq \Phi(d^2(y_{2n+1}, y_{2n}, y_{2n})) \\
d(y_{2n+1}, y_{2n+2}, y_{2n}) + d(y_{2n}, y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n+2}, y_{2n}) \\
d(y_{2n+1}, y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n+2}, y_{2n}) \\
d(y_{2n+1}, y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n+1}, y_{2n}) \\
= \Phi(0, 0, 0, 0, 0) \\
< 0
\]
and so \( d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0 \). Similarly, we have
\[
d(y_{2n+1}, y_{2n+2}, y_{2n+3}) = 0.
\]
Hence, \( d(y_n, y_{n+1}, y_{n+2}) = 0 \) for every \( n \in \mathbb{N}_0 \).

(b) For all \( z \in X \), let \( d_n(z) = d(y_n, y_{n+1}, z), n = 0, 1, 2, \ldots \).

By (a), we have
\[
d(y_n, y_{n+2}, z) \leq d(y_n, y_{n+2}, y_{n+1}) + d(y_{n+1}, y_{n+2}, z) +
\]
\[
d(y_{n+1}, y_{n+2}, z)
\]
\[
= d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z)
\]
\[
= d_n(z) + d_{n+1}(z).
\]
Taking \( x = x_{2n+2} \) and \( y = x_{2n+1} \) in (4.3.2), we have
\[
d_{2n+1}^2(z) = d^2(y_{2n+2}, y_{2n+1}, z)
\]
\[
= d^2(Ax_{2n+2}, Bx_{2n+1}, z)
\]
\[
= \phi(d_n^2(y_{2n+1}, y_{2n}, z), d(y_{2n+1}, y_{2n+2}, z) \cdot d(y_{2n}, y_{2n+1}, z),
\]
\[
d(y_{2n+1}, y_{2n+2}, z) \cdot d(y_{2n}, y_{2n+2}, z),
\]
\[
d(y_{2n+1}, y_{2n+2}, z) \cdot d(y_{2n}, y_{2n+2}, z),
\]
\[
d(y_{2n+1}, y_{2n+2}, z) \cdot d(y_{2n}, y_{2n+1}, z))
\]
\[
= \phi(d_{2n}^2(z), d_{2n+1}(z), d_n(z), 0, d_{2n+1}(z)(d_{2n}(z) + d_{2n+1}(z)), 0).
\]
Now, we shall prove that \( \{d_n(z)\} \) is a non-increasing sequence in \( \mathbb{R}^+ \). In fact, suppose that \( d_{n+1}(z) > d_n(z) \) for some \( n \). Then, for some \( a < 2 \),

\[
d_{n+1}(z) + d_n(z) = a \ d_{n+1}(z).
\]

Since \( \phi \) is non-decreasing in each variable and \( b < 1 \) for some \( a < 2 \), by (4.3.2), we have

\[
d_{2n+1}^2(z) \leq \phi \left( d_{2n+1}^2(z), d_{2n+1}^2(z), 0, ad_{2n+1}^2(z), 0 \right)
\]

\[
\leq b \ d_{2n+1}^2(z)
\]

\[
< d_{2n+1}^2(z)
\]

and

\[
d_{2n+2}^2(z) \leq \phi \left( d_{2n+2}^2(z), d_{2n+2}^2(z), 0, 0, ad_{2n+2}^2(z) \right)
\]

\[
\leq b \ d_{2n+2}^2(z)
\]

\[
< d_{2n+2}^2(z).
\]

Hence, for every \( n \in \mathbb{N} \), \( \frac{d^2}{d_n(z)} \leq \beta \frac{d^2}{d_n(z)} < d_n^2(z) \), which is a contradiction. Therefore, \( \{d_n(z)\} \) is a non-increasing sequence in \( \mathbb{R}^+ \). By using the fact that the sequence \( \{d_n(z)\} \) is non-increasing, we have the following:
(A) \( d_0(y_0) = 0 \implies d_n(y_0) = 0 \) for every \( n \in \mathbb{N} \).

(B) \( d_{m-1}(y_m) = 0 \) for any \( n \in \mathbb{N} \) \( \implies d_n(y_m) = 0 \) for all \( n \geq m - 1 \).

(C) \( d_{m-1}(y_m) = 0 = d_{m-1}(y_n) \) for \( 0 \leq n < m - 1 \) and (M4)

\[ d_n(y_m) \leq d_n(y_{m-1}) \leq d_n(y_{m-2}) \leq \ldots , \]

(D) \( d_n(y_{n+1}) = 0 \implies d_n(y_m) = 0 \) for \( 0 \leq n < m - 1 \).

Thus, we have shown \( d_n(y_n) = 0 \) for all \( m, n = 0, 1, 2, \ldots \).

(E) \( d_{j-1}(y_j) = 0 = d_{j-1}(y_k) \) for any \( i, j, k \in \mathbb{N}_0 \) with \( i < j \)

\[ d(y_i, y_j, y_k) \leq d(y_i, y_{j-1}, y_k) . \]

Therefore, by using the above inequality in (E) repeatedly, we have

\[ d(y_i, y_j, y_k) \leq d(y_i, y_1, y_k) = 0, \]

which means that \( d(y_i, y_j, y_k) = 0 \) for every \( i, j, k \in \mathbb{N}_0 \).

This completes the proof.

**Lemma (3)**: Let \( A, B, S \) and \( T \) be mappings from a 2-metric space \( (X, d) \) into itself satisfying the conditions (4.3.1) and (4.3.2). Then the sequence \( \{y_n\} \) defined by (4.3.3) is a Cauchy sequence in \( X \).
Proof: In the proof of Lemma (2), since \( \{d_n(z)\} \) is a non-decreasing sequence in \( \mathbb{R}^+ \), by (4.3.2), we have

\[
d_1^2(z) = d^2(y_1, y_2, z) = d^2(Bx_1, Ax_2, z) \\ \\
\leq \Phi(d_0^2(z), d_0(z), d_1(z), 0, 0, (d_0(z) + d_1(z))d_1(z)) \\ \\
\leq \Phi(d_0^2(z), d_0^2(z), d_0^2(z), d_0^2(z), 2d_0^2(z)) \\ \\
= \gamma(d_0^2(z)).
\]

In general, we have \( d_n^2(z) \leq \gamma^n(d_0^2(z)) \), which implies that, if \( d_0(z) > 0 \), by Lemma (1), we have

\[
\lim_{n \to \infty} d_n^2(z) \leq \lim_{n \to \infty} \gamma^n(d_0^2(z)) = 0.
\]

Therefore, we have \( \lim_{n \to \infty} d_n(z) = 0 \). For \( d_0(z) = 0 \), since \( \{d_n(z)\} \) is non-increasing, we have clearly \( \lim_{n \to \infty} d_n(z) = 0 \).

Now, we shall prove that \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( \lim_{n \to \infty} d_n(z) = 0 \), it is sufficient to show that a subsequence \( \{y_{2n}\} \) of \( \{y_n\} \) is a Cauchy sequence in \( X \).

Suppose that the sequence \( \{y_{2n}\} \) is not a Cauchy sequence in \( X \). Then there exists a point \( a \in X \), \( \epsilon > 0 \) and strictly increasing sequence \( \{m_k\}, \{n_k\} \) of positive integers such
that $k \leq n_k \leq m_k$.

(4.3.4) \[ d(y_{2n_k}, y_{2m_k}, a) \geq \epsilon \text{ and } d(y_{2n_k}, y_{2m_k}, a) < \epsilon \]

for all $k = 1, 2, \ldots$. By Lemma (2) and $(M_4)$, we have

\[ d(y_{2n_k}, y_{2m_k}, a) \leq d(y_{2m_k-2}, y_{2m_k}, a) \]
\[ \leq d_{2m_k-2} + d_{2m_k-1}(a). \]

Since \( \{d(y_{2n_k}, y_{2m_k}, a) - \epsilon\} \) and \( \{\epsilon - d(y_{2n_k}, y_{2m_k}, a)\} \) are sequences in $\mathbb{R}^+$ and $\lim_{n \to \infty} d_n(a) = 0$, we have

(4.3.5) \[ \lim_{k \to \infty} d(y_{2n_k}, y_{2m_k}, a) = \text{ and } \lim_{k \to \infty} d(y_{2n_k}, y_{2m_k-2}, a) = \epsilon. \]

Note that, by $(M_4)$,

(4.3.6) \[ |d(x, y, a) - d(x, y, a)| \leq d(a, b, x) + d(a, b, y) \]

for all $x, y, a, b \in X$.

Taking $x = y_{2n_k}, y = a, a = y_{2m_k-1}$ and $b = y_{2m_k}$ in (4.3.6) and using Lemma (2) and (4.3.5), we get

(4.3.7) \[ \lim_{k \to \infty} d(y_{2n_k}, y_{2m_k-1}, a) = \epsilon. \]

Once again, by using Lemma (2), (4.3.5) and (4.3.6), we get
(4.3.8) \( \lim_{k \to \infty} d(y_{2n_k+1}, y_{2m_k}, a) = \epsilon \) and 
\( \lim_{k \to \infty} d(y_{2n_k-1}, y_{2m_k-1}, a) = \epsilon \).

By (4.3.2), we have

\[
(4.3.9) \quad d^2(y_{2m_k}, y_{2n_k+1}, a) = d^2(\alpha_{2m_k}, \beta_{2n_k+1}, a) \\
\leq \Phi(d^2(y_{2m_k-1}, y_{2n_k}, a), \\
\quad d(y_{2m_k-1}, y_{2m_k}, a) \cdot d(y_{2n_k}, y_{2n_k+1}, a), \\
\quad d(y_{2m_k-1}, y_{2n_k}, a) \cdot d(y_{2n_k}, y_{2n_k+1}, a), \\
\quad d(y_{2m_k-1}, y_{2n_k+1}, a) \cdot d(y_{2n_k}, y_{2n_k+1}, a)).
\]

Using (4.3.4), (4.3.5), (4.3.6) and (4.3.7), since \( \Phi \in F \), we have

\[
\epsilon^2 \leq \Phi(\epsilon^2, 0, \epsilon^2, 0, 0) \leq \gamma(\epsilon^2) < \epsilon^2
\]

as \( k \to \infty \) in (4.3.9), which is a contraction. Therefore, 
\( \{y_{2n_k}\} \) is a Cauchy sequence in \( X \). This completes the proof.
Now, we are ready to give a coincidence point theorem:

**Theorem (1)**: Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying (4.3.1), (4.3.2) and the following conditions:

(4.3.10) $S(X) \cap T(X)$ is a complete subspace of $X$.

Then (i) $A$ and $S$ have a coincidence point in $X$, and (ii) $B$ and $T$ have a coincidence point in $X$.

**Proof**: By Lemma (3), the sequence $\{y_n\}$ defined by (4.3.3) is a Cauchy sequence in $S(X) \cap T(X)$. Since $S(X) \cap T(X)$ is a complete subspace of $X$, $\{y_n\}$ converges to a point $w$ in $S(X) \cap T(X)$. On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequence in $S(X) \cap T(X)$, they also converges to the same limit $w$. Hence, there exist two points $u, v$ in $X$ such that $Su = w$ and $Tv = w$, respectively. By (4.3.2), we have

$$d^2(Au, Bx_{2n+1}, z) \leq \Phi (d^2(Su, Tx_{2n+1}, z),$$

$$d(Su, Au, z) \cdot d(Tx_{2n+1}, Bx_{2n+1}, z),$$

(eq. continued on next page)
that is,
\[ \begin{align*}
\phi \left( d \left( y_{2n+1}, z \right) \right) & \leq \left( \sum_{i=1}^{n} d \left( y_{2i-1}, z \right) \right) \leq \phi \left( d \left( y_{2n}, z \right) \right), \\
\phi \left( d \left( z \right) \right) & \leq \left( \sum_{i=1}^{n} d \left( y_{2i-1}, z \right) \right) \leq \phi \left( d \left( y_{2n}, z \right) \right),
\end{align*} \]

Since \( \lim_{n \to \infty} d \left( z \right) = 0 \) in the proof of Lemma (2), letting \( n \to \infty \), we have

\[ \begin{align*}
d^2 \left( Au, w, a \right) & \leq \phi \left( 0, 0, 0, d^2 \left( w, Au, z \right), 0 \right),
\end{align*} \]

Which is a contradiction. Hence \( Au = w = Su \), that is, \( u \) is a coincidence point of \( A \) and \( S \). Similarly, we can show that \( v \) is a coincidence point of \( B \) and \( T \). This completes the proof.

As a direct consequence of Theorem (1), we have the following:
Corollary (2) : Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying the conditions (4.3.1) and (4.3.2) with $A = B$ and (4.3.10).

Then (i) $A$ and $S$ have a coincidence point in $X$, and (ii) $A$ and $T$ have a coincidence point in $X$. Indeed, $A, S$ and $T$ have a coincidence point in $X$ if $A$ is one-to-one.

Proof : By Theorem (1), we have the directly proofs of (i) and (ii). As in the proof of Theorem (1), we have $Au = Su = w = Bu = Tv$. Hence, since $A = B$ is one-to-one, $u = v$, that is, $u = v$ is a coincidence point of $A = B$, $S$ and $T$. This completes the proof.

4.4 In this section, by using Theorem (1), we prove a common fixed point theorem for compatible mappings of type (A) on 2-metric spaces.

Theorem (3) : Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying (4.3.1), (4.3.2), (4.3.10) and the following conditions:
(4.4.1) the pairs $A, S$ and $B, T$ are compatible mappings of type (A).

Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof**: By Theorem (1), there exists two points $u, v$ in $X$ such that $Au = Su = w$ and $Bv = Tv = w$, respectively.

Since $A$ and $S$ are compatible mappings of type (A), by Proposition (4), $ASu = SSu = SAu = AAu$, which implies that $Aw = Sw$. Similarly, since $B$ and $T$ are compatible mappings of type (A), we have $Bw = Tw$.

Now, we prove that $Aw = w$. If $Aw \neq w$, then, by (4.3.2), we have

$$d^2(Aw, B_{2n+1}x) \leq \phi(d^2(Sw, T_{2n+1}x)), $$

that is,

$$d^2(Aw, B_{2n+1}x) \leq \phi(d^2(Sw, y_{2n}x)), $$

that is,

$$d^2(Aw, B_{2n+1}x) \leq \phi(d^2(Sw, y_{2n}x)), $$

(eq. continued on next page)
Letting $n \to \infty$, we have

$$d^2(Aw,w,z) \leq \Phi(d^2(Sw,w,z), 0, d^2(Aw,w,z), 0, 0)$$

which is a contradiction. Hence, we have $Aw = w = Sw$.

Similarly, we have $Bw = w = Tw$. This means that $w$ is a common fixed point of $A, B, S$ and $T$. The uniqueness of the fixed point $w$ follows easily from (4.3.2).

Remark: Theorem (3) extends, generalizes and improves a number of fixed point theorems for commuting mappings, weakly commuting mappings and compatible mappings on 2-metric spaces.