CHAPTER - II

FIXED POINT THEOREMS IN COMPLETE METRIC SPACE

2.1 Common fixed point theorems for a pair of mappings in complete metric space

During the past few years, the Banach [2] contraction principle has been generalized and unified in several directions by many authors. Dass and Jaggi [18], Dass and Gupta [17], Fisher ([24],[25]) and others generalized Banach [2] contraction principle using rational inequalities. Later on, Bajaj [3] established some common fixed point theorems for a pair of mappings in complete metric space. In 1978, Fisher and Khan [27] proved the following theorem:

Theorem 2.1. If S and T are the self-mappings of a complete metric space (X, d) and satisfying the inequality

\[ d(Sx, Ty) \leq \frac{bd(x, Sx) d(x, Ty) + cd(y, Sx) d(y, Ty)}{d(x, Ty) + d(y, Sx)} \]

for all \( x, y \) in \( X \) with \( d(x, Ty) + d(y, Sx) \neq 0 \), where \( b, c \geq 0 \) and \( (b+c) < 1 \), then \( S \) and \( T \) have common fixed point.

Further, if \( d(x, Ty) + d(y, Sx) = 0 \) implies \( d(Sx, Ty) = 0 \), then \( S \) and \( T \) have a unique common fixed point.

In this section we establish some common fixed point theorems for a pair of mappings in complete metric space.

space satisfying a new condition, we have used our result for solvability of non-linear equations and some geometrical problems.

We first prove the following theorem which gives theorem A2 as a special case.

Theorem 1. Let S and T be two self mappings of a complete metric space (X,d) satisfying the inequality

\[ d(Sx, Ty) \leq \frac{\alpha_1 d(x, Sx) d(x, Ty) + \alpha_2 d(y, Sx) d(y, Ty)}{d(x, Ty) + d(y, Sx)} + \frac{\beta_1 d(x, Sx) d(y, Sx) + \beta_2 d(x, Ty) d(y, Ty)}{d(x, Sx) + d(x, Ty)} \]

for all \( x, y \) in \( X \) with \( d(x, Ty) + d(y, Sx) \neq 0 \) and \( d(x, Sx) + d(x, Ty) \neq 0 \), where \( \alpha_i, \beta_1 \geq 0 \) (\( i = 1, 2 \)) and \((\alpha_1 + \frac{\beta_2}{\beta_1} + \beta_1) < 1 \), then S and T have a common fixed point. Further, if

\[ d(x, Ty) + d(y, Sx) = 0 \quad \text{or} \quad d(x, Sx) + d(x, y) = 0, \]

then S and T have a unique common fixed point.

Proof. With usual argument we can assume that \( \beta_1 = \beta_2 \). Let \( x_0 \) be an arbitrary point of \( X \). We define a sequence of elements of \( X \) such that

\[ x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1} \quad (n = 0, 1, 2, \ldots) . \]

Then using (2.1.2) we have for \( x_n \neq x_{n+1} (n = 0, 1, 2, \ldots) \),

\[ d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \]

\[ \leq \frac{\alpha_1 d(x_{2n}, x_{2n+1}) d(x_{2n}, x_{2n+2}) + \alpha_2 d(x_{2n+1}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} + \frac{\beta_1 d(x_{2n}, x_{2n+1}) d(x_{2n}, x_{2n+2}) + \beta_2 d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+2})} \]
Evidently \( \{x_n\} \) is a Cauchy sequence, and there exists a point \( z \in X \) such that \( x_n \rightarrow z \).

Now consider \( z \neq Tz \); then we have

\[
d(z, Tz) \leq d(z, x_{2n+1}) + d(Sx_{2n}, Tz).
\]

Using (2.1.2) we have

\[
d(z, Tz) \leq d(z, x_{2n+1}) + \frac{\alpha_2 d(x_{2n}, x_{2n+1}) d(z, Tz)}{d(x_{2n}, Tz) + d(z, x_{2n+1})} + \frac{\beta_1 d(x_{2n}, x_{2n+1}) d(z, x_{2n+1}) + \beta_2 d(x_{2n}, Tz) d(z, Tz)}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tz)}.
\]

On letting \( n \rightarrow \infty \), we have

\[
d(z, Tz) \leq \beta_2 d(z, Tz)
\]

a contradiction. Hence \( z = Tz \); i.e. \( z \) is a fixed point of \( T \).

Similarly, we can prove \( z = Sz \). If possible, we assume \( w (w \neq z) \) is another fixed point of \( S \) and \( T \). In view of (2.1.3), \( z = w \).

The following example support the theorem 1.

Example 1. Let \( X = [0,1] \) be the complete metric space with usual metric \( d \) and \( S \) and \( T \) be the selfmappings of \( X \), such that \( S : [0,1] \rightarrow [0,1] \) be defined by
\[ S(x) = \frac{x}{3}, \ x \in [0, \frac{1}{3}) \]
\[ S(x) = \frac{x}{4}, \ x \in \left[\frac{1}{3}, 1\right]. \]

Let \( T : [0,1] \rightarrow [0,1] \) be defined by \( T(x) = \frac{x}{2} \) for all \( x \).

This example satisfies the theorem 1 for \( x = 0, \ y = 0 \),
\[ 5/18 < \alpha_1, \alpha_2 < 1/6, \ \beta_1 < 1/4, \ \beta_2 < 1/4. \]

Clearly zero is a unique common fixed point of \( S \) and \( T \),
although the mapping \( S \) is discontinuous at \( x = \frac{1}{3} \).

**Example 2.** Let \( X = [0,1] \) be the complete metric space with usual metric \( d \). Define \( T(x) = 0, \ 0 \leq x < 1 \),
\( T(1) = \frac{1}{2} \) and \( S = T \), then for \( x = \frac{1}{2}, \ y = 1 \), we see that \( T \)
does not satisfy condition (2.1.1) of theorem A2. But for \( x = \frac{1}{2}, \ y = 1 \), \( \beta \in \left[\frac{1}{2}, 1\right) \) such that \( (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) < 1 \)
and \( S = T \), we see that our condition (2.1.2) of theorem 1
is satisfied and condition (2.1.3) holds only for \( x = 0 \).
Hence our contractive condition is distinct from that of \( \)
Fisher and Khan [27].

**Remark 1.** In case \( \beta_1 = \beta_2 = 0 \), we obtain theorem A2
of Fisher and Khan [27] as a special case of our theorem 1.

Now we extend the above theorem 1 for a sequence
of mappings in a complete metric space. Here \( I_+ \) stands for
the set of positive integers.

**Theorem 2.** Let \( T_0 \) and \( \{ T_n : n \in I_+ \} \) be
mappings of a non-empty complete metric space \( X \) into itself
satisfying the inequality
\[ d(T_0 x, T_n y) \leq \frac{\alpha_1 d(x, T_0 x) d(x, T_n y) + \alpha_2 d(y, T_0 x) d(y, T_n y)}{d(x, T_n y) + d(y, T_0 x)} + \frac{\beta_1 d(x, T_0 x) d(y, T_n y) + \beta_2 d(x, T_n y) d(y, T_n y)}{d(x, T_0 x) + d(x, T_n y)} \]

for all \( x, y \) in \( X \) and for each \( n = 1, 2, \ldots \) for which

\[ d(x, T_n y) + d(y, T_0 x) \neq 0 \text{ and } d(x, T_0 x) + d(x, T_n y) \neq 0, \]

where \( \alpha_i, \beta_i \geq 0 \) (\( i = 1, 2 \)) and \( (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \leq 1 \);

then there exists a point \( z \in X \) such that \( T_n z = z \), \( n = 0, 1, 2, \ldots \), and for an arbitrary point \( x_0 \), the sequence

\[ x_1 = T_0 x_0, x_2 = T_1 x_1, x_3 = T_0 x_2, \ldots, x_{2n-1} = T_0 x_{2n-2}, \]

\[ x_{2n} = T_n x_{2n-1} = T_0 x_{2n} \]

(n = 1, 2, ...) converges to \( z \). Further, if \( d(x, T_n y) + d(y, T_0 x) = 0 \) or \( d(x, T_0 x) + d(x, T_n y) = 0 \), then \( z \) is a unique fixed point of \( T_n \) for \( n = 0, 1, 2, \ldots \).

Proof. Let \( x_0 \) be an arbitrary point in \( X \). Consider \( \{ x_n \} \) defined in (2.1.5). By (2.1.4), \( x = x_{2n-2} \) and \( y = x_{2n-1} \) we have, when \( x_n \neq x_{n+1} \) for any \( n \),

\[ d(x_{2n-1}, x_{2n}) \leq \frac{(\alpha_1 + \beta_2)}{(1 - \beta_2)} d(x_{2n-2}, x_{2n-1}). \]

Similarly, when \( x_n \neq x_{n+1} \) for any \( n \),

\[ d(x_{2n-2}, x_{2n-1}) \leq \frac{(\alpha + \beta_1)}{(1 - \beta_1)} d(x_{2n-3}, x_{2n-2}). \]

So \( \{ x_n \} \) is a Cauchy sequence and there exists a point \( z \) in \( X \) such that \( x_n \rightarrow z \).
If $T_0 z \neq z$, then applying (2.1.4) to

$$d(z, T_0 z) \leq d(z, x_{2n}) + d(x_{2n}, T_0 z),$$

simplifying and making $n \to \infty$ we have:

$$d(z, T_0 z) \leq \beta d(z, T_0 z)$$

a contradiction, Hence $z = T_0 z$.

Now putting $x = y = z$ in (2.1.4); we obtain

$$d(T_0 z, T_n z) \leq \beta d(T_0 z, T_n z)$$

yielding $z = T_0 z = T_n z$; which is true for $n = 1, 2, \ldots$.

The rest part of the argument follows easily.

For solving non-linear functional equations, we prove the following theorem:

**Theorem 3.** Let $\{f_n\}$ and $\{g_n\}$ be the sequence of elements in complete metric space $(X, d)$. And let $v_n$ be the unique solution of the equations $u = Su = f_n$ and $u - Tu = g_n$, where $S$ and $T$ be the mappings of $X$ into itself satisfying the condition (2.1.2) of theorem 1. If $f_n, g_n \to 0$ as $n \to \infty$, then sequence $\{v_n\}$ converges to the common solution of the equations $u = Su$ and $u = Tu$.

**Proof.** By (2.1.2), we have for $m > n$,

$$d(v_n, v_m) \leq d(v_n, Sv_n) + d(Sv_n, Tv_m) + d(Tv_m, v_m)$$

$$+ \alpha_1 d(v_n, Sv_n) d(v_n, Tv_m) + \alpha_2 d(v_m, Sv_n) d(v_m, Tv_m)$$

$$+ \beta d(v_n, Sv_n) d(v_m, Sv_n) + \beta d(v_n, Tv_m) d(v_m, Tv_m)$$

$$+ d(Tv_m, v_m).$$
Using the fact that $v_n$ is a unique common solution of the equations $u - Su = f_n$ and $u - Tu = g_n$; $f_n$ and $g_n \to 0$ as $n \to \infty$, in above inequality, we have

$$d(v_n, v_m) \leq 0.$$

It follows that $\{v_n\}$ is a Cauchy sequence. Hence it converges to a point $v$ in $X$.

Also from (2.1.2),

$$d(v, Tv) \leq d(v, v_n) + d(v, Sv_n) + d(Sv_n, Tv)$$

$$\leq d(v, v_n) + d(v, Sv_n) + \frac{\alpha_1 d(v_n, Sv_n) d(v_n, Tv) + \alpha_2 d(v, Sv_n) d(v, Tv)}{d(v_n, Tv) + d(v, Sv_n)}$$

$$+ \frac{\beta_1 d(v_n, Sv_n) d(v, Sv_n) + \beta_2 d(v_n, Tv) d(v, Tv)}{d(v_n, Sv_n) + d(v_n, Tv)}.$$

Again using the fact that $v_n$ is a unique common solution of the equations $u - Su = f_n$ and $u - Tu = g_n$, with $f_n, g_n \to 0$ as $n \to \infty$ and that the sequence $\{v_n\}$ converges to $v$ in the above inequality; one has

$$d(v, Tv) \leq 0.$$

Therefore we get $v = Tv$. Similarly it can be proved that $v = Sv$. Thus $v$ is the common solution of the equations $u = Su$ and $u = Tu$.

In the sequence, we prove the following geometrical problem.

**Theorem 4.** Let $X = \mathbb{R}$, be the usual metric space of real numbers and $[a, b] \subset \mathbb{R}$; $S, T : [a, b] \to [a, b]$ be two differentiable functions such that
Further, let the functions \( S \) and \( T \) be such that

\[
\frac{1}{|x-y|} \left[ \frac{|S(x)-S(y)|}{|x-S(x)||x-T(y)|} + \frac{|T(x)-T(y)||y-S(x)||y-T(y)|}{|x-T(y)| + |y-S(x)|} \right]
\]

\[
|x-S(x)| + |x-T(y)|
\]

\[
(2.1.6)
\]

\[
|S(x)-T(y)| = \frac{|T(x)-T(y)-S(x)+S(y)|}{|x-S(x)| + |x-T(y)|}
\]

\[
+ \left[ \frac{|x-S(x)| |y-S(x)| + |x-T(y)| |y-T(y)|}{|S(x)-S(y)||T(x)-T(y)|} \right]
\]

for all \( x, y \in [a,b] \) with \( |x-y| \neq 0 \) \( |S(x)-S(y)| \neq 0 \)

\[
|T(x)-T(y)| \neq 0, \quad |x-T(y)| + |y-S(x)| \neq 0 \quad \text{and} \quad |x-S(x)| + |x-T(y)| \neq 0.
\]

then there exists a unique solution of the equations \( x = Sx = Tx \).

Proof. By Lagrange's mean value theorem, for any \( x, y \in [a,b] \) we have:

\[
\frac{S(x)-S(y)}{x-y} = S'(z), \quad \text{where} \quad y < z_1 < x
\]

and \( \frac{T(x)-T(y)}{x-y} = T'(z), \quad \text{where} \quad y < z_2 < x. \)

Hence \( \frac{|S(x)-S(y)|}{|x-y|} = |S'(z)| \leq \alpha_1 \), where \( 0 < \alpha_1 < 1 \) and

\[
\frac{|T(x)-T(y)|}{|x-y|} = |T'(z)| \leq \alpha_2, \quad \text{where} \quad 0 < \alpha_2 < 1.
\]

Again, by Cauchy mean value theorem, for any \( x, y \in [a,b] \), we have:
\[
\frac{S(x)-S(y)}{T(x)-T(y)} = \frac{S'(z_0)}{T'(z_0)}, \quad \text{where } y < z_0 < x
\]
which implies
\[
\frac{|T(x)-T(y)-S(x)+S(y)|}{|S(x)-S(y)|} = \frac{|T(z_0)-S'(z_0)|}{S'(z_0)} \leq \frac{\alpha_1-\alpha_2}{\alpha_1}
\]
where \(0 < \frac{\alpha_1-\alpha_2}{\alpha_1} < 1\).

Similarly, we have:
\[
\frac{|T(x)-T(y)-S(x)+S(y)|}{|T(x)-T(y)|} \leq \frac{|\alpha_1-\alpha_2|}{\alpha_2} \quad \text{where } 0 < \frac{\alpha_1-\alpha_2}{\alpha_2} < 1.
\]

Thus, \(S\) and \(T\) satisfy condition (2.1.2) of theorem 1 on \([a,b]\) into itself. Hence, by theorem 1, there exists a unique common fixed point \(x^* \in [a,b]\) i.e. \(S(x^*) = T(x^*) = x^*\).

2.2 In 1984, Wang et al [87] proved the following fixed point theorems for expansion mappings:

Theorem A\(_3\): If there exists a real constant \(a > 1\) such that
\[
(2.2.1) \quad d(fx, fy) \geq a \min \{d(x, fx), d(fy, y), d(x, y)\}
\]
for any \(x, y\) in \(X\) and \(f\) is onto continuous, then \(f\) has a fixed point.

Theorem A\(_4\): If there exists a real constant \(a > 1\) such that
\[
(2.2.2) \quad d(fx, fy) \geq a \cdot d(x, y)
\]
for any \(x, y\) in \(X\) and \(f\) is onto, then \(f\) has a unique fixed point.

In 1985, Rhoades [63] generalized the above theorems for a pair of mappings. He proved the following theorems:

Theorem A5. Let \( f, g \) be surjective self maps of a complete metric space \((X, d)\). Suppose there exists a constant \( a > 1 \) such that
\[
(2.2.3) \quad d(fx, gy) \geq a \cdot d(x, y)
\]
for each \( x, y \) in \( X \), then \( f \) and \( g \) have a common fixed point.

Theorem A6. Let \( f, g \) be surjective continuous self maps of a complete metric space \( X \). If there exists a real number \( a > 1 \) such that
\[
(2.2.4) \quad d(fx, gy) \geq a \cdot \min \{d(x, fx), d(y, gy), d(x, y)\}
\]
for each \( x, y \in X \), then \( f \) or \( g \) has a fixed point or \( f \) and \( g \) have a common fixed point.

As an extension of theorem A5, we prove the following theorem:

Theorem 5. Let \( f \) and \( g \) be surjective continuous self maps of a complete metric space \((X, d)\). Suppose there exists a constant \( a > 1 \) such that
\[
(2.2.5) \quad \left[ d(fx, gy) \right]^2 \geq a \cdot \left[ d(x, fx) \cdot d(y, gy) \right]
\]
for each \( x, y \) in \( X \), then \( f \) and \( g \) have a unique common fixed point.

Proof. Let \( x_0 \in X \). Since \( f \) is surjective there exists a point \( x_1 \in f^{-1}x_0 \). Since \( g \) is surjective, there exists a point \( x_2 \in g^{-1}x_1 \). Continuing in this manner (2.2.5) obtains a sequence \( \{x_n\} \) with
Suppose \( x_{2n+1} = x_{2n} \) for some \( n \). From (2.2.5) we have

\[
\left[ d(x_{2n}, x_{2n+1}) \right]^2 = \left[ d(fx_{2n+1}, gx_{2n+2}) \right]^2
\]

\[
\geq a \left[ d(x_{2n+1}, fx_{2n+1}) \right] \left[ d(x_{2n+2}, gx_{2n+2}) \right]
\]

\[
\geq a \left[ d(x_{2n+1}, x_{2n}) \right] \left[ d(x_{2n+2}, x_{2n+1}) \right]
\]

\[ 0 \geq a \left[ d(x_{2n+2}, x_{2n+1}) \right], \]

i.e. \( d(x_{2n+2}, x_{2n+1}) = 0 \). The condition \( x_{2n+1} = x_{2n} \) implies \( x_{2n} \) is a fixed point of \( g \).

Assume \( x_n \neq x_{n+1} \) for each \( n \). Now from (2.2.5) we have

\[
\left[ d(x_{2n}, x_{2n+1}) \right]^2 = \left[ d(fx_{2n+1}, gx_{2n+2}) \right]^2
\]

\[
\geq a \left[ d(x_{2n+1}, fx_{2n+1}) \right] \left[ d(x_{2n+2}, gx_{2n+2}) \right]
\]

\[
\geq a \left[ d(x_{2n+1}, x_{2n}) \right] \left[ d(x_{2n+2}, x_{2n+1}) \right]
\]

or \( d(x_{2n}, x_{2n+1}) \geq a \left[ d(x_{2n+2}, x_{2n+1}) \right] \),

which, since \( a > 1 \), implies that \( \{x_n\} \) converges to a point \( u \in X \). The condition \( x_{2n+1} \in F^{-1}x_{2n} \),

\( x_{2n+2} \in g^{-1}x_{2n+1} \) and the continuity of \( f \) and \( g \) imply that \( u \) is a common fixed point of \( f \) and \( g \). Condition (2.2.5) forces uniqueness of the fixed point.

Example 3. Consider two surjective self maps \( f, g \) in the following manner:

\( f, g : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(x) = 1-x, g(x) = \frac{1+x}{4} \).

We observe that for \( x = -1 \) and \( y = 1 \), condition (2.2.3) of
Theorem 6. of Rhoades [63] gives

\[ \frac{5}{4} \geq 2a, \]

which is a contradiction as \( a > 1 \). But \( \frac{1}{2} \) is the unique common fixed point of \( f \) and \( g \).

Remark 2. For \( x = -1 \), \( y = 1 \) in our condition (2.2.5) of Theorem 5 gives

\[ \frac{25}{16} \geq \frac{39}{4}, \]

which is true for \( 1 < \frac{25a}{12} \). Hence our condition (2.2.5) is distinct from condition (2.2.3) of Rhoades [63].

Our next theorem is a generalization of Theorem A6 of Rhoades [63] for sequence of mappings.

Theorem 6. Let \( \{f_n\} \) \( n = 1, 2, \ldots \) be a sequence of surjective continuous self-mappings of a complete metric space \( X \). If there exists a real number \( a > 1 \) such that for all \( n \in \mathbb{N} \),

\[ (2.2.6) \quad d(f_0 x, f_n y) \geq \min \left\{ d(x, y), d(x, f_0 x), d(y, f_n y) \right\} \]

for each \( x, y \in X \), then there exists a common fixed point of \( f_n \) in \( X \).

Proof. Let \( x_0 \in X \). Since \( f_n \) is surjective, we define a sequence as follows

\[ x_1 \in f_0^{-1}(x_0), \quad x_2 \in f_1^{-1}(x_1), \quad x_3 \in f_0^{-1}(x_2), \quad \ldots \]

\[ x_{2n+1} \in f_0^{-1}(x_{2n}), \quad x_{2n+2} \in f_1^{-1}(x_{2n+1}), \quad \ldots \]

If \( x_n = x_{n+1} \) for any \( n \), then \( f \) or \( f_n \) has a fixed point. Assume \( x_n \neq x_{n+1} \) for each \( n \). Now
\[ d(x_{2n}, x_{2n+1}) = d(f_0(x_{2n+1)}, f_{2n+1}(x_{2n+2})) \]
\[ \geq a \min \left\{ d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, f_0 x_{2n+1}), d(x_{2n+2}, f_{n+1} x_{2n+2}) \right\} \]
\[ \geq a \min \left\{ d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1}) \right\} \]

or \[ d(x_{2n}, x_{2n+1}) \geq a d(x_{2n+1}, x_{2n}) \] does not hold, since \( a > 1 \). Therefore we have \[ d(x_{2n}, x_{2n+1}) \geq a d(x_{2n+1}, x_{2n+2}). \]

Since \( a > 1 \), \( \{x_n\} \) is a Cauchy sequence hence convergent to some \( u \) in \( X \).

The condition \( x_{2n+1} \in f_0^{-1}(x_{2n}), x_{2n+2} \in f_{n+1}^{-1}(x_{2n+1}) \)
and continuity of \( f_0 \) and \( f_n \) where for all \( n \in \mathbb{N} \), implies that \( u \) is a common fixed point of \( f_0 \) and \( f_n \).

Remark 3. Setting \( f_0 = f, f_1 = f_2 = \cdots = f_n = g \)
in theorem 6 yields theorem \( A_6 \) of Rhoades [63].

Remark 4. Setting \( f_0 = f_1 = f_2 = \cdots = f_n = f \) in theorem 6 yields theorem \( A_3 \) of Wang et al [87].

A metric space \( X \) with two metrics \( d \) and \( d_1 \) has been considered by Maia [50] in 1968, he proved the following theorem:

Theorem \( A_7 \). Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \) and \( T \) a self map of \( X \). Let \( X \) satisfy the following conditions:

(2.2.7) \( d_1(x, y) \leq d(x, y) \) for all \( x, y \in X \).

(2.2.8) \( X \) is complete with respect to \( d_1 \).

(2.2.9) \( T \) is continuous with respect to \( d_1 \).

(2.2.10) \( d(Tx, Ty) \leq \beta d(x, y) \)
for every \( x, y \in X \) and \( \alpha \in [0,1] \), then \( T \) has a unique fixed point.

In 1986, an extension of the above theorem has been carried out by Dhange and Dhoble [20]. A brief statement of the theorem is given below:

**Theorem A**. Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \). Let \( T \) be a self map of \( X \), and \( X \) satisfying the following conditions:

\[
(2.2.11) \quad d_1(x,y) \leq a \left[ d(x,Tx) + d(y,Tx) \right] \quad a \geq 1 \quad \text{and for every } x, y \in X,
\]

\[(2.2.12) \quad X \text{ is } T \text{ orbitally complete with respect to } d_1, \]

\[(2.2.13) \quad T \text{ is orbitally continuous with respect to } d_1, \]

\[(2.2.14) \quad \text{there exists a real number } b \text{ such that } \]

\[
\min \left\{ d(Tx,Ty), d(x,Tx), d(y,Tx) \right\} + b \left\{ d(x,Ty), d(y,Tx) \right\} \leq p d(x,Tx) + q d(x,y),
\]

for every \( x, y \in X \), where \( p \) and \( q \) are non-negative constants such that \((p + q) = h \in [0,1]\), then \( T \) has a fixed point.

We establish a fixed point theorem in complete metric space, which satisfies a new condition distinct from Dhange and Dhoble [20].

**Theorem 7**. Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \). Let \( T \) be a self map of \( X \) and \( X \) satisfies the following conditions:

\[
(2.2.15) \quad d_1(x,y) \leq a \left[ d(x,Tx) + d(y,Tx) \right], \quad a \geq 1 \quad \text{and for every } x, y \in X,
\]

\[(2.2.16) \quad X \text{ is } T \text{ orbitally complete with respect to } d_1, \]
(2.2.17) T is orbitally continuous with respect to $d_1$.

(2.2.18) $\min \left\{ d(x, Tx) d(x, Ty) d(Tx, Ty), \frac{1}{2} [d(x, y)]^2 d(x, Ty) \right\}$

\[ \frac{1}{2} d(x, Tx) d(y, Ty) \left[ d(x, Ty) + d(Tx, Ty) \right] - \min \left\{ d(x, Tx) d(x, y) d(Tx, Ty), \right\} \]

\[ \frac{1}{2} d(x, Ty) d(y, Tx) \left[ d(x, Ty) + d(Tx, Ty) \right] \leq \frac{1}{2} q d(x, Tx) d(x, y) \left[ d(x, Ty) + d(Tx, Ty) \right] \]

for all $x, y \in X$, $q \in [0, 1)$, then $T$ has a fixed point.

Proof. Let $x_0$ be an arbitrary point of $X$. We define a sequence $\{x_n\}$ of points of $X$ such that $x_1 = Tx_0$, $x_2 = Tx_1$, \ldots $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$. If for some $n \in N$, $x_{n+1} = x_n$ then the assertion follows immediately. Therefore we assume that $x_n \neq x_{n+1}$ for each $n \in N$.

Now for $x = x_{n-1}$ and $y = x_n$ we have by (2.2.18),

\[ \min \left\{ d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1}) d(x_n, x_{n+1}), \frac{1}{2} [d(x_{n-1}, x_n)]^2 d(x_{n-1}, x_{n+1}) \right\} \]

\[ \frac{1}{2} d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1}) \left[ d(x_{n-1}, x_{n+1}) + d(x_n, x_n) \right] - \min \left\{ \right. \]

\[ d(x_{n-1}, x_n) d(x_{n-1}, x_n) d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1}) d(x_n, x_n) \left[ d(x_{n-1}, x_{n+1}) + d(x_n, x_n) \right] \]

\[ \leq \frac{1}{2} q d(x_{n-1}, x_n) d(x_{n-1}, x_n) \left[ d(x_{n-1}, x_{n+1}) + d(x_n, x_n) \right] \]

or $\min \left\{ d(x_{n-1}, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_n), \frac{1}{2} d(x_n, x_{n+1}) \right\} \leq \frac{1}{2} q d(x_{n-1}, x_n)$.

Since $d(x_{n-1}, x_n) \leq q d(x_{n-1}, x_n)$ is impossible, as $q < 1$.

We, therefore, have

\[ d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n). \]

Proceeding in the same manner, we get

\[ d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n) \leq q^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq q^n d(x_0, x_1). \]

In the following few para, we have shown that $\{T^n x\}$ is Cauchy sequence with respect to $d_1$. 
\[ d(x_n, x_{n+p}) \leq q^n d(x_0, x_1) + q^{n+p} d(x_0, x_1) \]

Since \( d_1(x, y) \leq q^n d(x, y) \), we have

\[ d_1(x_n, x_{n+p}) \leq q^n d(x_0, x_1) + q^{n+p} d(x_0, x_1) \]

\[ \leq q^n \frac{d(x_0, x_1)}{1-q} \to 0 \quad \text{as} \quad n \to \infty \]

This shows that \( \{ T^n x \} \) is a Cauchy sequence with respect to \( d_1 \). The metric space \( X \) being \( T \) orbitally complete with respect to \( d_1 \) and there exists a point \( u \in X \) such that

\[ \lim_{n \to \infty} x_n = u. \]

By \( T \) orbitally continuity of \( T \), we get

\[ u = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} T x_n = T \lim_{n \to \infty} x_n = T u. \]

Therefore, \( T \) has a fixed point.

Finally, we will furnish few examples to discuss the validity of theorem 7.

Example 4. Let \( \mathbb{R} \) denote the set of all real numbers and \( T : \mathbb{R} \to \mathbb{R} \) be a mapping defined by

\[ T(x) = x - 2x^3 \] for all \( x \in \mathbb{R} \), where \( (\mathbb{R}, d) \) is metric space with metric \( d \). Therefore for \( x = 1 \) and \( y = -1 \) by Dhange and Dhobale [20] i.e. from the condition \( (2.2.14) \) of Theorem \( A_\delta \), we get

\[ 2 \leq 2(p + q). \]
which is a contradiction, since \((p+q) < 1\) and, hence, condition (2.2.14) of Dhanqe and Dhobale [20] fails to prove the existence of the fixed point of the map \(T\).

Remark 5. In the above example, we observe that, the condition of Dhanqe and Dhobale [20] fails but satisfies our condition (2.2.14) of theorem 7. This evidently shows that the condition of our theorem 7 has a different origin from Dhanqe and Dhobale [20].

The following example shows that advantage of theorem 7 over theorem A_8.

Example 5. Consider \(X = [0,1]\) with two metrics \(d\) and \(d_1\) defined by

\[
d(x,y) = |x-y| \quad \text{and} \quad d_1(x,y) = \frac{3d(x,y)}{1+d(x,y)}
\]

for all \(x,y \in X\).

Let \(T\) be a self map of \(X\) defined by \(T(x) = x/3\) for all \(x \in X\).

Then for \(x = 0\) and \(y = 1/3\) the hypothesis (2.2.7) of theorem A_7 fails to hold and hence the theorem A_7 does not guarantee the existence of fixed point of \(T\). But in hypothesis (2.2.15) of our theorem 7, if we choose \(a = 3\), then we see that for above two metrics the hypothesis remain true. This shows that \(T\) has a fixed point zero.

2.3 Common fixed point theorems for three mapping in complete metric space.

Let \((X,d)\) be a metric space. A mapping \(T : X \rightarrow X\) is known as a contraction mapping if \(d(Tx,Ty) \leq g d(x,y)\) for all \(x,y \in X\) and \(g \in [0,1)\). Many authors have extended
this result in different spaces. Jaggi and Das [38] extended this result of Banach [2] and proved the following theorem:

**Theorem A.** Let $f$ be a self map defined on a metric space $(X,d)$ satisfying following conditions:

1. For some $\alpha, \beta \in [0,1]$ with $(\alpha + \beta) < 1$

\[
\frac{\alpha d(x,fx) + \beta d(x,y)}{d(x,y) < \frac{1}{2} d(x,fx) + d(y,fx) + d(x,y)}
\]

for all $x,y \in X$, $x \neq y$.

2. There exists $x_0 \in X$ such that

\[
\left\{ f^n x_0 \right\} \supset \left\{ f^n y_0 \right\}
\]

with $\lim_{k \to \infty} f^n x_0 \in X$.

Then $f$ has a unique fixed point $u = \lim_{k \to \infty} f^n x_0$.

In 1988, Paliwal [53] extended the above result and proved the following theorem:

**Theorem A.** Let $T_1$ and $T_2$ be two continuous self mappings of a metric space $(X,d)$ such that

\[
(2.3.3) \quad d(T_1^r x, T_2^s y) \leq \frac{\alpha d(x,T_1^r x) + \beta d(y,T_2^s y)}{d(x,T_2^s y) + d(y,T_1^r s) + d(x,y)}
\]

for all $x,y \in X$, $x \neq y$, where $r > 0$, $s > 0$ are integers and $\alpha, \beta$ are non-negative real numbers such that $(\alpha + \beta) < 1$. If for some $x_0 \in X$, the sequence $\{ x_n \}$ consisting of points $x_{n+1} = T_1^r x_{2n}$, $x_{n+2} = T_2^s x_{2n+1}$ has a

subsequence \( \{x_{n_k}\} \) converges to a point \( u \); then \( T_1 \) and \( T_2 \) have unique common fixed point \( u \).

We shall prove some unique common fixed point theorems on three mappings in complete metric space which generalize the results of Jaggi and Dass [38], Paliwal [59] and Edelstein [22]. In fact we prove the following theorems:

Theorem 8: Let \( E, F \) and \( T \) be the three continuous self-mappings of a complete metric space \((X,d)\) and satisfy the following conditions:

\[
(2.3.4) \quad ET = TE, FT = TF, E(X) \subseteq T(X) \text{ and } F(X) \subseteq T(X)
\]

\[
(2.3.5) \quad d(Ex, Fy) \leq \alpha \frac{d(Tx, Ex) \cdot d(Ty, Fy)}{d(Tx, Fy) + d(Ty, Ex) + d(Tx, Ty)} + \beta d(Tx, Ty)
\]

for all \( x, y \) in \( X \), in \( Tx \neq Ty \), \( \alpha, \beta \geq 0 \) and \( (\alpha + \beta) < 1 \), then \( E, F \) and \( T \) have a unique common fixed point.

Proof. Let \( x_0 \) be any arbitrary point of \( X \). Since \( E(X) \subseteq T(X) \), we can choose a point \( x_1 \) in \( X \) such that \( Tx_1 = Ex_0 \). Also \( F(X) \subseteq T(X) \) we can choose a point \( x_2 \) in \( X \) such that \( Tx_2 = Fx_1 \). In general, we can choose the points \( x_{2n+1} \) and \( x_{2n+2} \) such that \( Tx_{2n+1} = Ex_{2n}, Tx_{2n+2} = Fx_{2n+1} \) for \( n = 0, 1, 2, \ldots \).

Now consider,

\[
d(Tx_{2n+1}, Tx_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})
\]

\[
(\alpha \frac{d(Tx_{2n}, Ex_{2n}) \cdot d(Tx_{2n+1}, Fx_{2n+1})}{d(Tx_{2n}, Fx_{2n+1}) + d(Tx_{2n+1}, Ex_{2n}) + d(Tx_{2n}, Tx_{2n+1})}) + \beta d(Tx_{2n}, Tx_{2n+1})
\]
\[
\alpha \frac{d(T_{2n}, T_{2n+1})}{d(T_{2n}, T_{2n+2}) + d(T_{2n+1}, T_{2n+2})} + \beta d(T_{2n}, T_{2n+1})
\]

\[
\leq \alpha \frac{d(T_{2n}, T_{2n+1})}{d(T_{2n}, T_{2n+2}) + d(T_{2n+1}, T_{2n+2})} + \beta d(T_{2n}, T_{2n+1})
\]

\[
(\alpha + \beta) d(T_{2n}, T_{2n+1})
\]

so \(d(T_{2n+1}, T_{2n+2}) \leq c d(T_{2n}, T_{2n+1})\) where \(c = (\alpha + \beta)\).

Since \((\alpha + \beta) < 1\), it follows \(c < 1\). Similarly

\[
d(T_{2n+1}, T_{2n+2}) \leq c d(T_{2n}, T_{2n+1})
\]

\[
\leq c^2 d(T_{2n-1}, T_{2n})
\]

By routine calculation the following inequalities hold for \(k > n\),

\[
d(T_{n}, T_{n+k}) \leq \sum_{i=1}^{k} d(T_{n+i-1}, T_{n+i})
\]

\[
\leq \sum_{i=1}^{k} c^{n+i-1} d(T_{0}, T_{1})
\]

\[
\leq \frac{c^n}{(1-c)}
\]

(as \(c < 1\)). Hence \(\{T_n\}\) is a Cauchy sequence. Since \(X\) is complete sequence \(\{T_n\}\) will converges to a point \(u\) in \(X\). Since \(\{E_{2n}\}\) and \(\{F_{2n+1}\}\) are subsequences of \(\{T_n\}\), they will also converges to same point \(u\). Since \(E, F\) and \(T\) are continuous, we have

\[(2,3,6) E(T_{2n}) \longrightarrow Eu, F(T_{2n+1}) \longrightarrow Fu.\]
Since $T$ commutes with $E$ and $F$, we have
$$E(Tx_{2n}) = T(Ex_{2n}); \quad F(x_{2n+1}) = T(Fx_{2n+1}) \quad \text{for all } n = 0, 1, \ldots,$$
and letting $n \to \infty$, we have
\begin{align*}
(2.3.7) \quad & E(u) = Tu = Fu \quad \text{and} \\
(2.3.8) \quad & T(Tu) = T(Eu) = E(Tu) = E(Fu) = F(Tu) = F(Fu).
\end{align*}
By (2.3.7), (2.3.7) and (2.3.8) if $Eu \neq F(Eu)$, then
\begin{align*}
d(Eu, F(Eu)) & \leq \frac{\alpha d(Tu, Eu) d(Tu, F(Tu))}{d(Tu, F(Eu)) + d(Tu, Eu) + d(Tu, F(Tu))} \\
& \leq \beta d(Eu, F(Eu))
\end{align*}
a contradiction. Hence
\begin{align*}
(2.3.9) \quad & Eu = F(Eu).
\end{align*}
By (2.3.7), (2.3.8) and (2.3.9) we have
$$Eu = F(Eu) = T(Eu) = E(Eu)$$
which implies $Eu$ is the common fixed point of $E$, $F$ and $T$.

Let $u$ and $v$ ($u \neq v$) be two common fixed points of $E$, $F$ and $T$ in $X$ such that
$$Eu = Fu = Tu = u \quad \text{and} \quad Ev = Fv = Tv = v,$$
then
\begin{align*}
d(u, v) & = d(Eu, Fv) \\
& \leq \frac{\alpha d(Tu, Eu) d(Tv, Fv)}{d(Tu, Fv) + d(Tv, Eu) + d(Tu, Tv)} + \beta d(Tu, Tv) \\
& \leq \beta d(u, v) < d(u, v)
\end{align*}
a contradiction. This implies the uniqueness of common fixed point of $E$, $F$ and $T$. 
Example 6. Let $X$ be the subset of reals with the usual metric $d$. Let $E$, $F$ and $T$ be the mappings of $X$ into itself and defined as follows:

$E(x) = x/3 + 1/6$, $F(x) = x/2 + 1/8$, $T(x) = 1/2 - x$ and $x \in X = [0, 1/2]$. Clearly $E$, $F$ and $T$ are continuous at $x = 0$, $T$ continuous with each of the maps $E$ and $F$. Also

$$E(x) = [1/6, 1/3] \subset T(x) = X$$
and

$$F(x) = [1/8, 3/8] \subset T(x) = X.$$ 

Taking $x = 0$, $y = 1/2$, $\alpha = 1/3$ and $\beta = 1/2$ we find $E$, $F$ and $T$ satisfy the condition (2.3.5) of theorem 8. Also we find that

$$1/4 = E(1/4) = F(1/4) = T(1/4).$$

Hence it follows that $1/4$ is the common fixed point of $E$, $F$ and $T$. Also it may be found that $x = 1/4$ is a unique common fixed point of $E$, $F$ and $T$. Therefore all the conditions of theorem 8 are satisfied.

Further, we extend the above theorem 8 and prove the following theorem:

Theorem 9. Let $E$, $F$ and $T$ be the self maps of a complete metric space $(X, d)$ satisfying the condition (2.3.4) of theorem 8 and

$$(2.3.10) \, d(E^r x, F^s y) \leq \frac{\alpha d(Tx, E^r x) + d(Ty, F^s y)}{d(Tx, F^s y) + d(Ty, E^r x) + d(Tx, Ty)} + \beta d(Tx, Ty)$$

for all $x, y$ in $X$, $Tx \neq Ty$, $\alpha, \beta \geq 0$ and $(\alpha + \beta) < 1$.

If some positive integers $r$ and $s$, $F^r x, F^s y$ and $T$ are
continuous, then $E$, $F$ and $T$ have a unique common fixed point.

Proof. It follows from condition (2.3.4) of theorem 8 that $E^r T = T E^r$, $F^s T = T F^s$, $E^r(X) \subset E(x) \subset T(X)$ and $F^s(X) \subset F(x) \subset T(X)$. Therefore by theorem 8, there is a unique point $u$ in $X$ such that

\[(2.3.11) \quad u = T u = E^r u = F^s u \quad i.e. \quad u \text{ is a unique common fixed point of } T, \quad E^r \text{ and } F^s.\]

Also

\[(2.3.12) \quad T(Eu) = E(Tu) = Eu = E(E^r u) = E^r(Eu) \quad \text{and} \]

\[(2.3.13) \quad T(Fu) = F(Tu) = Fu = F(F^s u) = F^s(Fu).\]

Hence it follows that $Eu$ is the common fixed point of $T$, $E^r$ and also $Fu$ is the common fixed point of $T$ and $F^s$. The uniqueness of $u$, from (2.3.11), (2.3.12) and (2.3.13) implies

\[u = Eu = Fu = Tu.\]

Remark 6. We put $T = I$ (identity map) and $E = F$ in theorem 8, we get the theorem $A_3$ of Jaggi and Dass [38].

Remark 7. On taking $T = I$ (identity map) in theorem 9, we get the result of Paliwal [53].

Remark 8. If we put $\omega = 0, r = s = 1, E = F$ and $T = I$ (identity map) in theorem 9, we get the result of Edelstein [22].

In 1986, Bagwat and Singh [4] have extended the result of Das and Gupta [17] for a pair of mappings and established the following theorem:
Theorem A11. Let $T_1$ and $T_2$ be two continuous self maps of a metric space $(X, d)$ such that

$$d(T_1 x, T_2 y) \leq \frac{d(x, T_1 x) d(x, T_2 y) + d(y, T_2 y) d(y, T_1 x)}{d(x, T_2 y) + d(y, T_1 x)}$$

for all $x, y$ in $X$. If for some $x_0 \in X$, the sequence $\{x_n\}$ of elements $x_n$, where $x_{2n+1} = T_1 x_{2n}$, $x_{2n+2} = T_1 x_{2n+1}$, --- has a convergent subsequence $\{x_{n_k}\}$ converging a point $x_0 \in X$, then $x_0$ is a unique fixed point of $T_1$ and $T_2$.

Before presenting our main theorem we define the following:

Definition 1. Let $S$, $T$ and $P$ be three self-maps of a metric space $(X, d)$, then we say that $\{S, T\}$ is a weakly commuting pair of mappings with respect to mapping $P$ if

(2.3.15) $d(PSPx, TPx) \leq d(SPPx, TPx)$ and

(2.3.16) $d(SPx, PTx) \leq d(SPx, TPPx)$

for all $x$ in $X$.

Example 7. Let $X = \{0, 1, 1/2, 1/2^2, \ldots \}$ with the usual metric. Define $P, S, T : X \to X$ such as

$P(o) = S(o) = T(o) = 0,$

$$P\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2^{n+9}} & \text{if } n \text{ is even} \\ \frac{1}{2^{n+10}} & \text{if } n \text{ is odd} \end{cases}$$

$$S\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2^{n+8}} & \text{if } n \text{ is even} \\ \frac{1}{2^{n+6}} & \text{if } n \text{ is odd} \end{cases}$$

To appeared in Mathematics Education (1990).
then it is easy to see that condition \((2.3.15)\) and \((2.3.16)\) satisfy for every \(x \in X\), i.e. \(\{S, T\}\) is a weakly commuting a pair of mappings with respect to mapping \(P\), but neither \(PS = SP\) nor \(PT = TP\).

**Remark 9.** If \(PS = SP\) and \(TP = PT\) then conditions \((2.3.15)\) and \((2.3.16)\) automatically satisfy.

We generalize the above theorem \(A_{11}\) with the use of weakly commuting pair of mappings with respect to certain mapping:

**Theorem 10.** Let \((X, d)\) be a complete metric space. Let \(P, S\) and \(T\) be continuous self-mappings as \(P, S, T : X \to X\) satisfying the following conditions:

\[(2.3.17)\] \(\{S, T\}\) is a weakly commuting pair of mappings with respect to mapping \(P\).

\[(2.3.18)\] \(d(SPx, TPy) \leq \frac{d(x, SPx) d(x, TPy) + d(y, TPy) d(y, SPx)}{d(x, TPy) + d(y, SPx)}\)

for all \(x, y \in X\). If for some \(x_0 \in X\), the sequence \(\{x_n\}\) has a convergent subsequence \(\{x_{n_k}\}\) convergent to a point \(z \in X\), then \(z\) is a unique common fixed point of \(S, P\) and \(T\).

**Proof.** Let \(x_0 \in X\) such that

\[x_{2n+1} = SPx_{2n}, x_{2n} = TPx_{2n-1}, x_{2n+2} = TPx_{2n+1}\]

We have,

\[d(x_{2n+1}, x_{2n+2}) = d(SPx_{2n}, TPx_{2n+1})\]
\[
\begin{align*}
\frac{d(x_{2n}, SPx_{2n}) d(x_{2n}, TPx_{2n+1}) + d(x_{2n+1}, TPx_{2n+1}) d(x_{2n+1}, SPx_{2n})}{d(x_{2n}, TPx_{2n+1}) + d(x_{2n+1}, SPx_{2n})} \\
\frac{d(x_{2n}, x_{2n+1}) d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}
\end{align*}
\]

or \( d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) \).

Continuing in this way, we obtain

\[
\begin{align*}
&d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) \\
&\quad \leq d(x_0, x_1). \quad \text{So we have a monotonic sequence}
\end{align*}
\]

of positive real numbers. Let this monotonic sequence converges to a real number \( l \). Since \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \) in \( X \) which converges to some \( z \in X \), therefore

\[
\lim_{n \to \infty} x_{2n_k} = z.
\]

Then we have to show that \( z \) is a unique common fixed point of \( S, P \) and \( T \). If possible we assume that \( z \neq SPz \).

Now \( d(z, SPz) = d(\lim_{k} x_{2n_k}, SP \lim_{k} x_{2n_k}) \)

\[
= \lim_{k} d(x_{2n_k}, SPx_{2n_k})
= \lim_{k} d(x_{2n_k}, x_{2n_k+1})
= \lim_{k} d(x_{2n_k+1}, x_{2n_k+2})
= \lim_{k} d(SPx_{2n_k}, TPSPx_{2n_k})
= d(\lim_{k} SPx_{2n_k}, TP \lim_{k} SPx_{2n_k})
\]

\[ (2.3.19) \]

But \( d(SPz, TPSPz) \leq \frac{d(z, SPz) d(z, TPSPz) + d(SPz, TPSPz) d(SPz, SPz)}{d(z, TPSPz) + d(SPz, SPz)} \)
From (2.3.19) and (2.3.20), we have:
\[ d(z, SPz) = d(SPz, TPSPz) \leq d(z, SPz) \]
which is a contradiction. Therefore \( z = SPz \), i.e. \( z \) is a fixed point of \( SP \).

In the same way let \( z \neq TPz \), we get
\[ d(z, TPz) = d(SPz, TPSPz) \leq d(z, TPz) \]
a contradiction. Hence \( z \) is a fixed point of \( TP \) also.

Thus
\[ (2.3.21) \quad SPz = z = TPz. \]

Now we shall show that \( z \) is the common fixed point of \( S, P \) and \( T \).

Using (2.3.15), (2.3.18) and (2.3.21) we have
\[ d(Pz, z) = d(PSPz, TPz) \leq d(SPPz, TPz) \]
\[ \leq d(Pz, Pz) + d(z, z) d(z, Pz) \]
\[ \leq 0 \]
therefore \( d(Pz, z) \leq 0 \) implies \( Pz = z \). Hence by (2.3.21) we have
\[ Sz = z = Tz. \]

Similarly using conditions (2.3.16), (2.3.21) and (2.3.18) we have
\[ d(Pz, z) = d(PSPz, TPz) \]
\[ d(SPz, TPz) \]
\[ \leq d(Pz, Pz) d(Pz, z) + d(z, z) d(z, Pz) \]
\[ \leq 0 \]

therefore \( d(Pz, z) \leq 0 \) implies \( Pz = z \). Hence by (2.3.21) we have

\[ Sz = z = Tz. \]

Similarly using conditions (2.3.16), (2.3.21) and (2.3.18), we have

\[ Pz = z = Sz = Tz. \]

Therefore, \( z \) is the common fixed point of \( S, T \) and \( P \).

For the uniqueness of fixed point, let \( z \) and \( z' \) be the two common fixed points of \( S, P \) and \( T \). Let \( z \neq z' \) then

\[ d(z, z') = d(SPz, TPz') \]
\[ \leq d(z, SPz) d(z, TPz') + d(z', TPz') d(z', SPz) \]
\[ \leq 0 \]

a contradiction. Therefore \( z \) is a unique common fixed point of \( S, P \) and \( T \).

Remark 10. Taking \( P = I \) (identity map), our definition 1 automatically satisfies and our theorem reduces to that of Bhagwat and Singh [4].

In 1986, Pathak [56] generalized the well known results of Banach [2], Kannan [40] and Fisher and Khan [27] and proved the following theorem:
Theorem A.12: Let $S$ and $T$ be two self-maps of a complete metric space $(X,d)$ and satisfy the inequality

$\quad (2.3.22) \quad d(Sx,Ty) \leq q \frac{d(x,Sx)d(x,Ty)+d(y,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} \left( d(x,y) \right)^2$

for all $x,y$ in $X$ with $0 \leq q < 1$ and $d(x,Ty)+d(y,Sx)+d(x,y) \neq 0$, where $\alpha, \beta, \gamma \geq 0$ (not all zero), then $S$ and $T$ have a common fixed point. Further, if $d(x,Ty)+d(y,Sx)+d(x,y) = 0$, then $S$ and $T$ have a unique common fixed point.

We generalize the above theorem for three mappings. Before presenting our theorem, we state the following definition:

Definition 2. Let $S$ and $T$ be the mappings of $X$ into itself. The mappings $S$ and $T$ are weakly commuting if

$\quad (2.3.23) \quad d(STx,TSx) \leq d(Sx,Tx)$

for all $x \in X$. It is obvious that two commuting mappings are also weakly commuting, but two weakly commuting do not necessarily commute as shown by following example:

Example 8. Let $X=[0,1]$ with the euclidean metric. Define $T_x = \frac{x}{x+4}$, $S_x = \frac{x}{2}$ for any $x \in X$, we have

$\quad d(STx,TSx) = \frac{x}{8+2x} - \frac{x}{8+x} = \frac{x^2}{(8+2x)(8+x)}$

$\leq \frac{x^2+2x}{(8+2x)} = \frac{x}{2} - \frac{x}{4+x} = d(Sx,Tx)$.

Thus $S$ and $T$ satisfy (2.3.23) but do not commute for any $x \neq 0$.

We prove the following theorem:
Theorem 11. Let $S$, $T$ and $I$ be three mappings of a complete metric space $(X, d)$ such that for all $x, y$ in $X$,
\[
\alpha d(Ix, Sx) d(Ix, Ty) + \beta d(Iy, Sx) d(Iy, Ty) < \xi d(Ix, Ty) + \eta d(Ix, Iy)
\]
\[
(2.3.24)d(Sx, Ty) \leq q \frac{\alpha d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy)}{\alpha d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy)}
\]
if $\alpha d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy) \neq 0$, where $0 \leq q < 1$ and $\alpha, \beta, \gamma \geq 0$ (not all zero). If the range of $I$ contains the range of $S$ and $T$, if either $I$ is continuous and weakly commuting with either $S$ or $T$, or if $S$ is continuous and weakly commuting with $I$, or if $T$ is continuous and weakly commuting with $I$, then $S$, $T$ and $I$ have a common fixed point. Further, if $\alpha d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy) = 0$, then $S, T$ and $I$ have a unique common fixed point.

Proof. Let $x_0$ be an arbitrary point in $X$. Since the range of $I$ contains the range of $S$, let $x_1$ be a point in $X$ such that $Sx_0 = Tx_1$. Since the range of $I$ contains the range of $T$, we can choose a point $x_2$ such that $Tx_1 = Ix_2$. In general, having chosen the point $x_{2n}$ such that $Sx_{2n} = Ix_{2n+1}$, choose a point $x_{2n+2}$ such that $Tx_{2n+1} = Ix_{2n+2}$ for $n = 1, 2, \ldots$. For simplicity, we put
\[
d_{2n-1} = d(Tx_{2n-1}, Sx_{2n}) \quad \text{and} \quad d_{2n} = d(Sx_{2n}, Tx_{2n+1})
\]
for $n = 1, 2, \ldots$. Now we distinguish three cases:

(i) Let $d_{2n-1} \neq 0$ and $d_{2n} \neq 0$ for $n = 1, 2, \ldots$. Using the inequality (2.3.24), we have
\[
d_{2n} = d(Sx_{2n}, Tx_{2n+1})
\]
\[
\leq \frac{\alpha d_{2n-1} d(Tx_{2n-1}, Tx_{2n+1}) + \beta d(Sx_{2n}, Sx_{2n}) d_{2n} + \gamma (d_{2n-1})^2}{\alpha d(Tx_{2n-1}, Tx_{2n+1}) + \beta d(Sx_{2n}, Sx_{2n}) + \gamma d_{2n-1}}
\]
\[
\leq q d_{2n-1}
\]
which implies that
\[ d(Sx_{2n}, Tx_{2n+1}) = d_{2n} \leq q d_{2n-1} = q d(Tx_{2n-1}, Sx_{2n}) \]
for \( n = 1, 2, \ldots \), where \( q < 1 \). Similarly, it is proved that
\[ d(Tx_{2n-1}, Sx_{2n}) = d_{2n-1} \leq q d_{2n-2} = q d(Sx_{2n-2}, Tx_{2n-1}) \]
for \( n = 1, 2, \ldots \).

It follows that the sequence
\[
(2.3.25) \quad \{ Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots \}
\]
is a Cauchy sequence in the complete metric space \( X \) and so has a limit \( w \) in \( X \). Hence the sequences \( \{ Sx_{2n} \} = \{ Tx_{2n+1} \} \) and \( \{ Tx_{2n-1} \} = \{ Sx_{2n} \} \) converge to the point \( w \), because they are subsequences of the sequence \( (2.3.25) \).

Suppose first of all that \( I \) is continuous, then
the sequences \( \{ I^2x_{2n} \} \) and \( \{ ISx_{2n} \} \) converge to the point \( Iw \).
If \( I \) weakly commutes with \( S \), we have
\[
d(SIx_{2n}, Iw) \leq d(SIx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iw)
\leq d(Sx_{2n},Ix_{2n}) + d(ISx_{2n}, Iw),
\]
which implies, on letting \( n \) tend to infinity, that the sequence
\( \{ SIx_{2n} \} \) also converges to \( Iw \). We now claim that \( Tw = Iw \).

Suppose not, then we have \( d(Iw, Tw) > 0 \) and using inequality
\( (2.3.24) \), we obtain:
\[
d(SIx_{2n}, Tw) \leq q \frac{d(I^2x_{2n}, Sx_{2n}) d(I^2x_{2n}, Tw) + \beta d(Iw, SIx_{2n})}{\alpha d(I^2x_{2n}, Tw) + \beta d(Iw, SIx_{2n}) + \gamma d(I^2x_{2n}, Iw)}.
\]
Letting \( n \) tend to infinity, we deduce that
\( d(Iw, Tw) = 0 \), a contradiction.

Now suppose that \( Sw \neq Tw = Iw \), then
A contradiction. Thus $Iw = Sw = Tw$.

A similar conclusion is achieved if $I$ weakly commutes with $T$. Let us now suppose that $S$ is continuous instead of $I$, then the sequence $\{S^{2x_{2n}}\}$ converges to the point $Sw$. Since $S$ weakly commutes with $I$, we have that the sequence $\{ISx_{2n}\}$ also converges to $Sw$. Since the range of $I$ contains the range of $S$, there exists a point $w'$ such that $Iw' = Sw$. Then if $Tw' \neq Sw = Iw'$ we have

\[
d(S^{2x_{2n}}, Tw') \leq \frac{\alpha d(Iw', Sw) + \beta d(Iw', Tw') + \gamma [d(Iw', Iw')]^2}{\alpha d(Iw', Tw') + \beta d(Iw', Sw) + \gamma d(Iw', Iw)}
\]

on letting $n \to \infty$, it follows that $d(Sw, Tw') \leq 0$, a contradiction. Thus $Sw = Tw' = Iw'$.

Now suppose that $Sw' \neq Tw' = Iw'$, then

\[
d(Iw', Sw')d(Iw', Tw') + \beta d(Iw', Sw')d(Iw', Tw')
\]

a contradiction and so

$Iw' = Sw' = Tw'$.

A similar conclusion is obtained if one assumes that $T$ is continuous and weakly commuting with $I$. 
(ii) Let $d_{2n-1} = 0$ for some $n$, then

$$I_{2n} = T_{2n-1} = S_{2n}.$$ 

We claim $I_{2n} = T_{2n}$, since otherwise if $d(I_{2n}, T_{2n}) > 0$, inequality (2.3.24) implies

$$0 < d(I_{2n}, T_{2n}) = d(S_{2n}, T_{2n})$$

which is a contradiction. Thus $I_{2n} = S_{2n} = T_{2n}$.

(iii) Let $d_{2n} = 0$ for some $n$, then $I_{2n+1} = S_{2n} = T_{2n+1}$.

and reasoning as in (ii), $I_{2n+1} = S_{2n+1} = T_{2n+1}$.

Therefore, in all cases, there exists a point $w$ such that

$$Iw = Sw = Tw.$$ 

We claim that $w$ is a fixed point of $I$, i.e., $w = Tw$. For, if possible assume that $w \neq Tw$ then we have

$$d(w, Tw) \leq d(w, S_{2n}) + d(S_{2n}, Tw)$$

$$\leq d(w, S_{2n}) + \frac{\alpha d(I_{2n}, S_{2n}) d(I_{2n}, Sw) + \beta d(I_{2n}, S_{2n}) d(Iw, Tw) + \gamma [d(I_{2n}, Sw)]^2}{\alpha d(I_{2n}, Sw) + \beta d(Iw, S_{2n}) + \gamma d(I_{2n}, Iw)}$$

Letting $n \to \infty$, we have $d(w, Tw) \leq 0$; a contradiction. Hence $w = Tw$ i.e., $w$ is a fixed point of $T$.

We have therefore proved that $W$ is a common fixed point of $I$, $S$, and $T$. 

To show the uniqueness of \( w \), we consider that
\[ \alpha d(Ix,Ty) + \beta d(Iy,Sx) + \gamma d(Ix,Iy) = 0. \]
Let \( S, T \) and \( I \) have second common fixed point \( z(z \neq w) \) in \( X \), then
\[ \alpha d(Iw,Tz) + \beta d(Iz,Sw) + \gamma d(Iw,Iz) = 0 \text{ implies} \]
\[ d(Iw,Tz) = d(Iz,Sw) = d(Iw,Iz) = 0 \] and so \( w = Tw = Iw = Sw \).
Thus \( S, T \) and \( I \) have a unique common fixed point.

Example 9. Let \( X = \{1, 2, 3, 4\} \) be a finite set with \( d \) given by
\[ d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1, \]
\[ d(1,2) = d(3,4) = 2. \]
Define \( I, S \) and \( T \) on \( X \) by,
\[ T(1) = T(2) = T(3) = T(4) = 2 \]
\[ I(1) = 1, I(2) = 2, I(3) = 4, I(4) = 3, \]
\[ S(1) = S(2) = S(4) = 2, S(3) = 3. \]
Since \( SI(1) = S(1) = 2 = I(2) = IS(1), \)
\[ SI(2) = S(2) = 2 = I(2) = IS(2) \text{ and} \]
\[ d(SI(3),IS(3)) = d(S(4),I(3)) = d(2,4) = 1 < 2 = d(3,4) = d(S(4),I(3)) \text{ and} \]
\[ d(SI(4),IS(4)) = d(S(3),I(2)) = d(3,2) = 1 = d(2,3) = d(S(4),I(4)), \]
we have that \( I \) weakly commutes with \( S \). Clearly \( I \) (or \( S \)) is
continuous and \( S(X) = \{2, 3\} \subset X = I(X) \) and \( T(X) = 2 \subset X = I(X) \). Further, an easy routine calculation shows that
inequality \((2, 3, 24)\) holds, for \( q < 1 \). Therefore all the
conditions of theorem 11 are satisfied and \( 2 \) is a unique
common fixed point of \( I, S \) and \( T \).

Remark 11. Assuming \( I \) is identity on \( X \), we obtain
theorem 1 of Pathak [56].
Corollary 1. Let $S$ and $T$ be mappings of a complete metric space $(X,d)$ into itself such that for all $x,y$ in $X$,
\begin{equation}
(2.3.26) \quad d(Sx, Ty) \leq \frac{q \alpha d(x, Sx) d(x, Ty) + \beta d(y, Sx) d(y, Ty) + \gamma d(x, y)}{\alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y)} \leq q \quad \text{if } \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y) \neq 0,
\end{equation}
where $0 \leq q < 1$ and $\alpha, \beta, \gamma \geq 0$ (not all zero), then $S$ and $T$ have a common fixed point. Further, if $\alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y) = 0$, then $S$ and $T$ have unique common fixed point.

Remark 12. If we put $S = T$ and $\alpha = \beta = 0$ in corollary 1, we obtain the celebrated Banach contraction principle [2].

Remark 13. If we put $S = T$ and on taking (i) $\beta = \gamma = 0$ (ii) $\alpha = \gamma = 0$ in (2.3.26) then taking A.M., we obtain theorem of Kannan [40].

In 1981, Sharma and Yael [34] proved the following theorem:

Theorem 5. Let $f$ be a self-map of $X$ into itself of a complete metric space $(X,d)$ satisfying the inequality,
\begin{equation}
(2.3.27) \quad d(fx, fy) \leq a_1 d(x, fx) + a_2 d(y, fy) + a_3 d(x, fy) + a_4 d(y, fx) + a_5 d(x, y)
\end{equation}
for all $x, y \in X$, where $a_i \geq 0$ and $\sum a_i < 1$ for $i=1,2,--5$. Such a condition is equivalent to the condition
\begin{equation}
(2.3.28) \quad d(fx, fy) \leq a \left[d(x, fx) + d(y, fy)\right] + b \left[d(x, fy) + d(y, fx)\right] + c d(x, y)
\end{equation}
by interchanging $x$ and $y$: where $a, b, c \geq 0$ and $(a+b+c) < 1$, then $f$ has a unique fixed point in $X$, if $f$ is asymptotically regular at some point in $X$. 
We establish a common fixed point theorem in complete metric space without considering the usual sequence of successive approximations in order to show the existence of common fixed point. Our result is generalization of Sharma and Yuel[94] with the use of weakly commuting mappings.

Theorem 12. Let $A, S$ and $T$ be three self-maps of a complete metric space $(X, d)$ satisfying

\[ d(Sx, Ty) \leq a_1 d(Ax, Sx) + a_2 d(Ay, Ty) + a_3 d(Ax, Ty) + a_4 d(Ay, Sx) + a_5 d(Ax, Ay) \]

for all $x, y$ in $X$, where $a_i = a_i(x, y), i = 1, 2, ..., 5$, are non-negative functions such that

\[ \max \left\{ \sup_{x,y \in X} (a_2 + a_3), \sup_{x,y \in X} (a_1 + a_4), \sup_{x,y \in X} (a_3 + a_4 + a_5) \right\} < 1, \]

(2.3.31) if $A$ is continuous,

(2.3.32) the map $A$ weakly commutes with $S$ and $T$ and

(2.3.33) there exists a sequence which is asymptotically $S$-regular and $T$-regular with respect to $A$, then $A, S$ and $T$ have a unique common fixed point.

Proof. Let $\{x_n\}$ be a sequence satisfying (2.3.33).

Using (2.3.29),

\[ d(Ax_n, Ax_m) \leq d(Ax_n, Sx_n) + d(Sx_n, Tx_m) + d(Tx_m, Ax_m) \]

\[ \leq d(Ax_n, Sx_n) + a_1 d(Ax_n, Sx_n) + a_2 d(Ax_n, Tx_m) + a_3 d(Ax_n, Tx_m) + a_4 d(Ax_m, Sx_n) + a_5 d(Ax_n, Ax_m) + d(Tx_m, Ax_m) \]
where $a_i = a_i(x_n, x_m)$. Therefore
\[
(1+a_3-a_4-a_5) \cdot d(Ax_n, Ax_m) \leq (1+a_1+a_4) \cdot d(Ax_n, Sx_n)
\]
\[+(1+a_2+a_3) \cdot d(Ax_m, Tx_m)
\]
which from (2.3.30) and (2.3.33), implies that $\{Ax_n\}$ is Cauchy. Since $X$ is complete, let $z = \lim Ax_n$. Being
\[d(Sx_n, z) \leq d(Sx_n, Ax_n) + d(Ax_n, z)
\]
$\{Sx_n\} \rightarrow z$. Similarly, $\{Tx_n\} \rightarrow z$. Also, using (2.3.31),
\[\{A^2x_n\} \rightarrow Az, \{ASx_n\} \rightarrow Az\text{ and }\{ATx_n\} \rightarrow Az.
\]
From (2.3.32),
\[d(SAx_n, Az) \leq d(SAx_n, ASx_n) + d(ASx_n, Az)
\]
\[\leq d(Ax_n, Sx_n) + d(ASx_n, Az)
\]
whence $\{SAx_n\} \rightarrow Az$. Similarly, $\{TAx_n\} \rightarrow Az$.
Further from (2.3.29), with $a_i = a_i(Ax_n, z)$,
\[d(Az, Tz) \leq d(Az, SAx_n) + d(SAx_n, Ty)
\]
\[\leq d(Az, SAx_n) + a_1 d(A^2x_n, SAx_n) + a_2 d(Az, Tz) + a_3 d(A^2x_n, Tz)
\]
\[+ a_4 d(Az, SAx_n) + a_5 d(A^2x_n, Az)
\]
\[\leq d(Az, SAx_n) + a_1 d(A^2x_n, SAx_n) + (a_2 + a_3) d(Az, Tz)
\]
\[+ (a_3 + a_4 + a_5) \max \{d(Az, SAx_n), d(A^2x_n, Az)\}.
\]
Taking the lim-sup, we have
\[d(Az, Tz) \leq \sup_{x, y \in X} (a_2 + a_3) d(Az, Tz).
\]
Similarly,
\[d(Az, Sz) \leq \sup_{x, y \in X} (a_1 + a_4) d(Az, Sz).
\]
Then, from (2.3.30) it follows, $Az = Sz = Tz$. 

From (2.3.29), with $a_i = a_i(x_n, Ax_n)_i$,

\[
d(Sx_n, TAx_n) \leq a_1 d(Ax_n, Sx_n) + a_2 d(A^2x_n, TAx_n) + a_3 d(Ax_n, TAx_n) \\
+ a_4 d(A^2x_n, Sx_n) + a_5 d(Ax_n, A^2x_n)
\]

\[
\leq a_1 d(Ax_n, Sx_n) + a_2 d(A^2x_n, TAx_n) + (a_3 + a_4 + a_5) \max \left\{ d(Ax_n, TAx_n), d(A^2x_n, Sx_n), d(Ax_n, A^2x_n) \right\}
\]

Taking $\lim \sup$ of both sides, yields

\[
d(z, Az) \leq \sup_{x, y \in X} (a_3 + a_4 + a_5) d(z, Az),
\]

which, from (2.3.30), implies $z = Az$, and hence $z$ is a common fixed point of $A$, $S$ and $T$.

To prove the uniqueness of $z$, suppose $z$ and $w$ are common fixed points of $A$, $S$ and $T$. From (2.3.29), with $a_i = a_i(z, w)$,

\[
d(z, w) = d(Sz, Tw) \leq a_1 d(Az, Sz) + a_2 d(Aw, Tw) + a_3 d(Az, Tw) \\
+ a_4 d(Aw, Sz) + a_5 d(Az, Aw)
\]

\[
\leq (a_3 + a_4 + a_5) d(z, w)
\]

which, from (2.3.30), implies $z = w$.

Example 10. Let $X = [0, 1]$ with Euclidean metric and $S = T$, $A : X \longrightarrow X$, define as

\[
S(x) = \frac{x}{4 + x}, \quad A(x) = \frac{x}{2}
\]

The map $S$ and $A$ weakly commute and let $\{x_n\}$ be a sequence in $X$ converging to $0$. Since
\[ d(Sx, Sy) = \frac{x - y}{4 + x} - \frac{y - x}{4 + y} = \frac{4|x - y|}{(x + 4)(y + 4)} \leq \frac{4|x - y|}{16} = \frac{x - y}{4} = \frac{1}{2} \frac{|x - y|}{2} = \frac{1}{2} d(Ax, Ay). \]

\( \{x_n\} \) is asymptotically \( S \)-regular with respect to \( A \).

For every \( x, y \in X \),

\[ d(Sx, Sy) = \frac{x - y}{4 + x} - \frac{y - x}{4 + y} = \frac{4|x - y|}{(x + 4)(y + 4)} \leq \frac{4|x - y|}{16} = \frac{x - y}{4} = \frac{1}{2} \frac{|x - y|}{2} = \frac{1}{2} d(Ax, Ay). \]

\( A \) is continuous in \( X \) and it suffices to assume \( a_5 = \frac{1}{2}, a_i = 0 \)

for \( i = 1, 2, 3, 4 \) in order to satisfy theorem 1.

Remark 14. If we put \( S = T \) and \( A \) is the identity map and \( a_i \) are the constants in theorem 12, we get the

result of Sharma and yuel[54].

Let \( R^+ \) be the set of non-negative reals, \( N \) the set of positive integers and let \( (X, d) \) be a complete metric space.

Consider a real function \( f \in F \) where \( F \) be the set of all real functions satisfying the following properties:

(i) \( f \) is upper semi-continuous,
(ii) \( f \) is non-decreasing in each co-ordinate variables,
(iii) \( f(t) < t \) for any \( t > 0 \).

A map \( S : X \rightarrow X \) is called an idempotent if \( S^2 = S \)
and an involution if \( S^2 = I \), where \( I \) denotes the identity map of \( X \).

We also use the following notion of weak \( \ast \ast \) commuting self-mappings of \( (X, d) \).
Definition 3. Two self-maps $A$ and $S$ of $(X, d)$ are called weak* commuting, if $A(X) \subseteq S(X)$ and

\[ d(A^2 S^2 x, S^2 A^2 x) \leq d(A^2 S x, S A^2 x) \leq d(A S^2 x, S^2 A x) \leq d(A S x, S A x) \leq d(S^2 x, A^2 x) \]

for all $x \in X$.

Clearly two commuting mappings also commute weak* but two weak* commuting mappings are not necessarily commuting, as it is shown in the following example.

Example 11. Let $X = [0,1]$ with Euclidean metric $d$ and let $A, B, S$ and $T$ be defined as $A(x) = \frac{x}{(x+4)}, B(x) = \frac{x}{(x+6)}$, $S(x) = \frac{x}{2}$ and $T(x) = \frac{x}{3}$ for all $x \in X$. Then $A(X) = [0,1/5] \subseteq [0,1/2] = S(X)$ and

\[ d(A^2 S^2 x, S^2 A^2 x) = \frac{x}{5x + 64} - \frac{x}{20x+64} = \frac{15 x^2}{(5x+64)(20x+64)} \]

\[ \frac{15 x^2}{(15x + 96)(10x + 32)} = \frac{x}{5x + 32} - \frac{x}{10x + 32} = d(A^2 S x, S A^2 x) \]

\[ d(A^2 S x, S A^2 x) = \frac{5x^2}{(5x + 32)(10x + 32)} = \frac{5/4 x^2}{(5/2x+16)(5x+16)} \]

\[ \frac{3x^2}{(x+16) (4x + 16)} = \frac{x}{(x + 16)} - \frac{x}{(4x + 16)} = d(A S^2 x, S^2 A x) \]
Hence we conclude that

\[ d(A^2S^2x, S^2A^2x) \leq d(A^2Sx, SA^2x) \leq d((AS^2x, S^2A^2x) \leq d(ASx, SAx) \leq d(S^2x, A^2x) \]

for all \( x \) in \( X \) but for any \( x \neq 0 \), we have

\[ ASx = \frac{x}{x + 8} > \frac{x}{2x + 8} = SAx. \]

Remark 15. If we retain only last inequality, the definition reduces to weak* commuting mappings of a pair \( \{S, A\} \) defined by Pathak[92].

Remark 16. If we consider, \( A \) and \( S \) as idempotent mappings i.e. \( A^2 = A \) and \( S^2 = S \) our definition reduces to weakly commuting pair of mappings \( \{S, A\} \) defined by Sessa[71].

Recently Chang[96], generalizing the result of Husain and Sehgal[95] and Iseki[91], established the following result for a family of f-contraction mappings:
Theorem B6. Let $S, T : X \rightarrow X$ be continuous. Then $S$ and $T$ have a common fixed point $w$ if and only if there exist two self-maps $A, B$ of $X$ and a function $f \in F$ such that

(2.3.35) $A(x) \cup B(x) \subseteq S(x) \cap T(x)$,

(2.3.36) both $A$ and $B$ commute with $S$ and $T$,

(2.3.37) $d(Ax, By) \leq f \left( \max \left\{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2} [d(Sx, By) + d(Ty, Ax)] \right\} \right)$

for any $x, y \in X$. Further, $w$ is the unique common fixed point of $A, B, S$ and $T$.

We improve the result of Chang[96] under considerably weaker condition of mappings.

Theorem 13. Let $A, B, S$ and $T$ be four self-maps of $(X, d)$ such that

(2.3.38) $A^2(x) \subseteq T^2(x)$ and $B^2(x) \subseteq S^2(x)$,

(2.3.39) $d(A^2x, B^2y) \leq f \left( \max \left\{ d(S^2x, T^2y), d(S^2x, A^2x), d(T^2y, B^2y), \frac{1}{2} [d(S^2x, B^2y) + d(T^2y, A^2x)] \right\} \right)$

for all $x, y$ in $X$, where $f$ satisfy (i), (ii) and (iii). If one of $A, B, S$ and $T$ is continuous and if $A$ and $B$ weak commute with $S$ and $T$ respectively, then $A, B, S$ and $T$ have a common fixed point $z$. Further $z$ is the unique common fixed point of $A, B, S$ and $T$.

Proof. Let $x_0$ be an arbitrary point of $X$ and $x_1, x_2$ in $X$ such that $A^2x_0 = T^2x_1$, $B^2x_1 = S^2x_2$. This can be done since (2.3.38) hold. According to Fisher[98], we can inductively define a sequence.
\[(2.3.40) \quad A^2x_0, B^2x_1, A^2x_2, B^2x_3, \ldots, A^2x_{2n}, B^2x_{2n+1}, A^2x_{2n+2}, \ldots, \text{such that} \]

\[A^2x_{2n} = T^2x_{2n+1}, B^2x_{2n+1} = S^2x_{2n+2} \]

for each integer \(n \in \mathbb{N} \cup \{0\}\). Employing the method of proof of Meade and Singh [37], it is proved that the sequence \((2.3.40)\) is a Cauchy sequence and thus it converges to a point \(z\).

Suppose that \(S\) is continuous. Since the sequence \(\{A^2x_{2n}\} = \{T^2x_{2n+1}\}\) and \(\{B^2x_{2n-1}\} = \{S^2x_{2n}\}\) converges also to \(z\), we have that the sequence \(\{SA^2x_{2n}\}\) converges to \(Sz\). \(A\) being weak commuting with \(S\) we deduce that

\[d(A^2Sx_{2n}, Sz) \leq d(A^2Sx_{2n}, SA^2x_{2n}) + d(SA^2x_{2n}, Sz) \leq d(S^2x_{2n}, A^2x_{2n}) + d(SA^2x_{2n}, Sz)\]

which implies, as \(n \to \infty\), that \(\{A^2Sx_{2n}\}\) converges to \(Sz\).

Now using \((2.3.39)\) and the fact that \(\{S^2x_{2n+1}\}\) converges to \(Sz\), we have

\[d(A^2Sx_{2n}, B^2x_{2n+1}) \leq \left(\max \left\{d(S^3x_{2n}, T^2x_{2n+1}), d(S^3x_{2n}, A^2Sx_{2n}), d(T^2x_{2n+1}, B^2x_{2n+1}), \frac{1}{2} \left[d(S^3x_{2n}, B^2x_{2n+1}) + d(T^2x_{2n+1}, A^2Sx_{2n})\right]\right\}\]

Letting \(n \to \infty\), we have

\[d(Sz, z) \leq \left(\max \left\{d(z, z), d(Sz, Sz), d(z, z), \frac{1}{2} [d(Sz + d(z, Sz))]\right\}\right) \leq \left(\max \left\{0, 0, 0, d(Sz, z)\right\}\right) \leq d(Sz, z)\]

giving therefore by \(Sz = z\) and so \(S^2z = z\).
Now,
\[d(A^2z, B^2x_{2n+1}) \leq f\left( \max \left\{ d(s^2z, t^2x_{2n+1}), d(s^2z, A^2z),
\frac{1}{2} d\left(s^2z, B^2x_{2n+1}\right) + d(t^2x_{2n+1}, A^2z) \right\} \right)\]  

Letting \( n \to \infty \), we have
\[d(A^2z, z) \leq f\left( \max \left\{ d(z, z), d(z, A^2z), d(z, z),
\frac{1}{2}d(z, A^2z) \right\} \right)\]  
\[\leq f\left( \max \left\{ 0, d(z, A^2z), 0, \frac{1}{2}d(z, A^2z) \right\} \right)\]  
\[\leq f\left( d(A^2z, z) \right) < d(A^2z, z)\]  
a contradiction and therefore \( A^2z = S^2z = z \). 

Since the range of \( T^2 \) contains the range of \( A^2 \), let \( z' \) be 
a point in \( X \) such that \( T^2z' = z \).

Then using (2.2.29) we have
\[d(z, B^2z') = d(A^2z, B^2z')\]  
\[\leq f\left( \max \left\{ d(s^2z, t^2z'), d(s^2z, A^2z), d(t^2z', B^2z'),
\frac{1}{2} d\left(s^2z, B^2z'\right) + d(t^2z', A^2z) \right\} \right)\]  
\[\leq f\left( \max \left\{ d(z, z), d(z, z), d(z, B^2z'),
\frac{1}{2} d(z, B^2z') \right\} \right)\]  
\[\leq f\left( \max \left\{ 0, 0, d(z, B^2z'), \frac{1}{2} d(z, B^2z') \right\} \right)\]  
\[\leq f\left(d(z, B^2z')\right) < d(z, B^2z')\]  
which implies \( z = B^2z' \) by property (iii). Since \( B \) is weak**
commuting with \( T \), we have
\[B^2T^2z' = T^2B^2z' \]  
which implies \( B^2z = T^2z \).
Using again (2.3.39) and (iii), one deduces

\[ B^2z = T^2z = z. \]

Since \( A \) weak** commutes with \( S \) we have,

\[ S^2Az = AS^2z \quad \text{and} \quad S^2Az = A^3z = Az. \]

Now \( d(Az,z) = d(A^3z,B^2z) \)

\[ \leq f\left( \max \left\{ d(S^2Az,T^2z), d(S^2Az,A^3z), d(T^2z,B^2z), \right. \right. \\
\left. \left. \frac{1}{2} \left[ d(S^2Az,B^2z) + d(T^2z,A^3z) \right] \right\} \right) \]

\[ \leq f\left( \max\{d(Az,z),0,0,\frac{1}{2}[d(Az,z) + d(z,Az)]\} \right) \]

\[ \leq f(d(Az,z)) < d(Az,z), \]

which implies \( Az = z \). Therefore \( Az = Sz = z \).

Using (2.3.39), (iii) and weak** commutativity of \( B,T \) one deduces, \( Tz = Bz = z \).

Therefore \( z \) is a common fixed point of \( A,B,S \) and \( T \).

Analogous proof can be given if one suppose the continuity of \( T \) instead of \( S \).

Now we suppose that continuity of \( A \). Then the sequence \( \{AS^2x_{2n}\} \) converges to \( Az \). Since \( A \) weak** commutes with \( S \) we have

\[ d(S^2Ax_{2n},Az) \leq d(S^2Ax_{2n},AS^2x_{2n}) + d(AS^2x_{2n},Az) \]

\[ \leq d(S^2x_{2n},A^2x_{2n}) + d(AS^2x_{2n},Az) \]

which implies, as \( n \to \infty \), that the sequence \( \{S^2Ax_{2n}\} \) converges to \( Az \). Using (2.3.39) and properties (i),(ii), (iii) and observing that \( \{A^3x_{2n}\} \) converges also to \( Az \).
One proves that $A_z = z$.

As above, one shows that

$T^2 z = B^2 z = z$.

Since the range of $S^2$ contains the range of $B^2$, let $z^*=z$ be a point in $X$ such that $S^2 z^* = z$.

Using again (2.3.39), we have

\[
d(A^2 z^*, z) = d(A^2 z^*, B^2 z)
\]

\[
f\left(\max \left\{d(S^2 z^*, T^2 z), d(S^2 z^*, A^2 z^*), d(T^2 z, B^2 z),
\frac{1}{2} \left[d(S^2 z^*, B^2 z) + d(T^2 z, A^2 z^*)\right]\right\}\right)
\]

\[
f\left(\max \left\{d(z, z), d(z, A^2 z^*), d(z, z), \frac{1}{2} \left[d(z, z) + d(z, A^2 z^*)\right]\right\}\right)
\]

\[
f\left(d(z, A^2 z^*)\right) < d(z, A^2 z^*)
\]

which implies $A^2 z^* = z$ by property (iii).

Since $A$ weak** commuting with $S$ we have

\[
d(A^2 S^2 z^*, S^2 A^2 z^*) \leq d(S^2 z^*, A^2 z^*) = d(z, z) = 0
\]

and therefore

\[
S^2 z = S^2 A^2 z^* = A^2 S^2 z^* = A^2 z = z.
\]

Since $A$ weak* commutes with $S$, we have

\[
A^2 S z = S A^2 z \text{ and so } A^2 S z = S^3 z = S z.
\]

Now,

\[
d(S z, z) = d(A^2 S z, B^2 z)
\]

\[
f\left(\max \left\{d(S^3 z, T^2 z), d(S^3 z, A^2 S z), d(T^2 z, B^2 z),
\frac{1}{2} \left[d(S^3 z, B^2 z) + d(T^2 z, A^2 S z)\right]\right\}\right)
\]

\[
f\left(\max \left\{d(S z, z), d(S z, S z), d(z, z), \frac{1}{2} \left[d(S z, z) + d(z, S z)\right]\right\}\right)
\]
\[
\leq f \left( \max \left\{ d(Sz, z), d(Sz, S^2z), d(Sz, A^2z) \right\} \right) \\
\leq f \left( d(Sz, z) \right) < d(Sz, z)
\]
which implies \( Sz = z \) and so
\[
Az = Sz = z.
\]

Since \( B \) weakly** commutes with \( T \), we have \( B^2Tz = T B^2z \) and and \( T^2Bz = B T^2z \) and so
\[
B^2Tz = T^2z = Tz \quad \text{and} \quad T^2Bz = B^2z = Bz.
\]

Now, \( d(z, Tz) = d(A^2z, B^2Tz) \)
\[
\leq f \left( \max \left\{ d(S^2z, T^3z), d(S^2z, A^2z), d(T^3z, B^2Tz), \right. \right.
\]
\[
\left. \left. \frac{1}{2} \left[ d(S^2z, B^2Tz) + d(T^3z, A^2z) \right] \right\} \right) \\
\leq f \left( \max \left\{ d(z, Tz), d(z, z), d(Tz, Tz), \frac{1}{2} d(z, Tz) + d(Tz, z) \right\} \right) \\
\leq f(d(z, Tz)) < d(z, Tz)
\]
which implies \( Tz = z \). Similarly we can prove that \( Bz = z \) and so \( Tz = Bz = z \), we have therefore proved that \( z \) is again a common fixed point of \( A, B, S \) and \( T \).

If the mapping \( B \) is continuous instead of \( A \), there the proof that \( z \) is again a common fixed point of \( A, B, S \) and \( T \) is similar.

Using (2.3.39) the uniqueness of \( z \) is easily proved.

Corollary 2. Let \( S, T : X \rightarrow X \) and either \( S \) or \( T \) be continuous. Then \( S \) and \( T \) have a common fixed point \( w \) if and only if there exist two self-mappings \( A, B \) of \( X \) and a function \( f \in F \) such that (i) and (iii) hold and \( A \) (resp. \( B \)) weakly commutes with \( S \) (resp. \( T \)). Further \( w \) is the unique common fixed point of \( A, B, S \) and \( T \).
Remark 11. The corollary generalize theorem of Chang[96] and note that Chang suppose the continuity of both $S$ and $T$ and the commutativity of both $A$ and $B$ with $S$ and $T$, but an assumptions in the corollary 2 are clearly more weaker than that of Chang[96].

Following example shows that theorem 13 is stronger than theorem $B_6$ of Chang[96].

Example 12. Let $X$ be the subset of $R^2$ defined by $X = \{F, G, H, I, J\}$, where

$F = (0,0), G = (1,0), H = (0,1), I = (\frac{1}{2},0), J = (-1,0)$.

Let $A, B, S, T: X \rightarrow X$ be given by

$A(F) = A(G) = A(H) = G, A(I) = A(J) = F,$
$B(F) = B(G) = B(H) = G, B(I) = B(J) = F,$
$S(F) = S(G) = S(H) = G, S(I) = S(J) = G,$
$T(F) = T(G) = G, T(H) = F, T(I) = T(J) = G.$

Further $T^2(X) = G = A^2(x)$
$s^2(x) = G = B^2(x)$.

By routine calculation one can easily verify the weak** commutativity of pairs $\{A, S\}$ and $\{B, T\}$. Then $A, B, S$ and $T$ satisfy condition (2.3.39) for all $x, y \in X$ and $B$ is the unique common fixed point of $A, B, S$ and $T$.

However, $A, B, S$ and $T$ do not satisfy condition (2.3.37). For otherwise, choosing $x = F, y = J$, we would have

$$d(A(F), B(J)) \leq f\left(\max \left\{ d(S(F), T(J)), d(S(F), A(F), d(T(J), B(J)),
\frac{1}{2}[d(S(F), B(J)) + d(T(J), A(F))]\right\}\right)$$

$\leq f\left(\max \left\{ 0, 0, 1, \frac{1}{2}\right\}\right) = f(1)$

which is a contradiction to the required condition (iii).