CHAPTER - I

INTRODUCTION

1.1. Functional analysis is an abstract branch of mathematics that originated from classical analysis. Its development started about eighty years ago. Functional analysis is a scientific discipline of fairly recent origin. It provides a powerful tool to discover solutions to problems occurring in pure, applied and social sciences, for instance physics, engineering, medicine, agro-industries, ecology, economics and bio-economics. Functional analysis has developed as a consequence of the attempts to generalize the results of classical analysis, algebra and geometry. The fixed point theory is one of the important part of functional analysis.

A topological space $X$, is said to be fixed point space, if every continuous mapping $T : X \rightarrow X$ has a point in the sense that $Tx_0 = x_0$ for some $x_0 \in X$ and also the point $x_0$ is said to be a fixed point of $T$ in $X$. Under certain conditions it may happen that $T$ admits one or more fixed points.

Fixed point theorem play a prominent role in pure and applied mathematics. A fixed point theorem has varied application to differential equations, integral equations and more generally to operator equations. Recently, fixed point methods related to economics and mathematical programming have been applied with great
success and it has emerged out as one of the most fundamental necessary tools in different problems of mathematical economics, theory of games, as well as in control system theory and neutron transport theory. For these applications we refer to Kolmogorov and Fomin [47], Chen and Shin [14], Smart [82], Szebehely [83] and Swaminathan [84].

In 1895, the first result on fixed point was motivated by Poincare [60] while studying the vector distribution as a map of the surface to itself by translating a point as a vector based at that point. He found the isolated singularities of such distributions to which an index was assigned. These singularities are the fixed points.

In 1910, Brouwer's proved a theorem which asserts that every continuous map $T$ of the closed unit ball $B = \{ (x(1), \ldots, x(n)) \in \mathbb{R}^n : x(1)^2 + \ldots + x(n)^2 \leq 1 \}$ in $\mathbb{R}^n$ (Euclidean $n$ dimensional space) to itself has a fixed point. The Brouwer's fixed point theorem remains true, if we replace $B \subseteq \mathbb{R}^n$ by any topological space $X$ homeomorphic with $B$. Since any non-empty, closed, bounded and convex subset $S$ of $\mathbb{R}^m$ is homeomorphic to the closed unit ball $B$ in $\mathbb{R}^m$ for some $m \leq n$, it follows that every continuous map of $S$ to itself has a fixed point.

In 1922, a Polish mathematician Banach [2] proved an important result on fixed point that is known as
Banach contraction principle which states as follows:

If $T$ is a mapping of a complete metric space $(X, d)$ into itself and satisfies:

\[(1.1.1) \quad d(Tx, Ty) \leq K \cdot d(x, y)\]

for all $x, y$ in $X$ and for some $0 \leq K < 1$, then $T$ has a unique fixed point.

A mapping satisfying (1.1.1) is known as contraction mapping. A contraction mapping is always continuous, however a continuous mapping is not necessarily a contraction.

Due to pretty statement, easy proofs and important applications, Banach contraction principal has given much inspiration to a number of mathematicians and research workers over the field of fixed point spaces. Banach contraction principle is the foundation stone over which the whole bulk of results on fixed points rests.

In 1965, Chu and Diaz [15] found out that even if $T$ is not a contraction it is possible that $T^n$ the $n^{th}$ iterate of $T$ is a contraction for some positive fixed integer $n$. Further in 1969, Sehgal [80] proved the following result.

Let $T : X \to X$ be a continuous mapping of a complete metric space $(X, d)$. If for each $x$ in $X$ there exists a positive integer $n(x)$, such that for all $y \in X$,

\[(1.1.2) \quad d(T^n(x), T^n(y)) \leq K \cdot d(x, y)\]
for some constant $K$, $0 \leq K < 1$, then $I$ has a unique fixed point $u \in X$ and $r^n(x_0) \rightarrow u$ for each $x_0 \in X$.

Sehgal’s result was further generalized by Guseman [30], Ciric [11], Khasanchi [48], Sharma [77], Iseki [36] and Rhoades [66], Rakotch [69], Browder [5], Boyd and Wang [6] further generalized the Banach contraction principle replacing the Lipschitz constant $K$ by some real valued functions where values are less than unity.

In 1968, Kannan [40] considered a self map $T$ satisfying the condition

\[(1.1.3) \quad d(Tx, Ty) \leq a \left[ d(x, Tx) + d(y, Ty) \right] \]

for all $x, y \in X$, $0 \leq a < \frac{1}{2}$ and obtained a unique fixed point.

It is obvious that mapping satisfying the condition (1.1.3) need not be contraction to have a unique fixed point. Reich [60] unified the theorems of Banach [2] and Kannan [40] and obtained a mapping $T$ of a complete metric space $X$ into itself satisfying the following condition,

\[(1.1.4) \quad d(Tx, Ty) \leq a d(x, Tx) + b d(y, Ty) + c d(x, y) \]

where $0 \leq a + b + c < 1$ for all $x, y \in X$, then $T$ has a unique fixed point in $X$.

1.2 Common fixed point for a pair of mappings in metric space.

A point $z \in X$ is said to be a common fixed point of a pair of mappings $S$ and $T$ on $X$, if $S(z) = z = T(z)$.
In the beginning, a number of mathematicians conjectured that two continuous mappings on \([0,1]\) have a common fixed point. In 1969, Boyce [7] and Hunke [31] showed that their conjecture is false. Many workers tried to seek for the sufficient condition on a pair of mappings which guarantees the existence of a common fixed point and its uniqueness. The problem of finding a sufficient condition for the existence of a unique common fixed point for a pair of mappings each defined on a complete metric space was first taken by Kannan [40] in 1968. A great deal of work on common fixed point has been done by Kannan [40], Yeh [89], Iseki [36], Rhoades [67], Ciric [12] and many others.

In 1976, Jungck [39] obtained a common fixed point theorem by generalizing the celebrated Banach contraction principle. We notice that in his generalization he took a continuous mapping in place of the identity mapping. Jungck's result reads as follows:

Theorem \(A_1\). Let \(S\) be a continuous self map of a complete metric space \((X,d)\). If there exists a map \(T: X \rightarrow X\) and a constant \(0 \leq \alpha < 1\) such that

\[
\begin{align*}
(a) & \quad S \text{ and } T \text{ commute}, \\
(b) & \quad T(X) \subset S(X), \\
(c) & \quad d(Tx,Ty) \leq \alpha d(Sx,Sy)
\end{align*}
\]

for all \(x,y\) in \(X\), then \(S\) and \(T\) have a unique common fixed point.
In recent years Jungck's result has been generalized and extended in various ways by many workers. In 1978, Fisher and Khan [27] proved the following theorem:

**Theorem A.2.** If $S$ and $T$ are the self mappings of a complete metric space $(X,d)$ and satisfying the inequality,

$$d(Sx, Ty) \leq \frac{b \ d(x, Sx) \ d(x, Ty) + c \ d(y, Sx) \ d(y, Ty)}{d(x, Ty) + d(y, Sx)}$$

for all $x, y$ in $X$ with $d(x, Ty) + d(y, Sx) \neq 0$, where $b, c \geq 0$ and $(b+c) \leq 1$, then $S$ and $T$ have common fixed point. Further, if $d(x, Ty) + d(y, Sx) = 0$ implies $d(Sx, Ty) = 0$, then $S$ and $T$ have a unique common fixed point.

In Chapter II we first establish a common fixed point theorem for a pair of mappings in complete metric space. We have used this result for solving non-linear equations and some geometrical problems.

We prove the following theorem:

**Theorem 1.** Let $S$ and $T$ be two self mappings of a complete metric space $(X,d)$ satisfying the inequality,

$$d(Sx, Ty) \leq \frac{\alpha_1 d(x, Sx) d(x, Ty) + \alpha_2 d(y, Sx) d(y, Ty)}{d(x, Ty) + d(y, Sx)}$$

$$+ \frac{\beta_1 d(x, Sx) d(y, Sx) + \beta_2 d(x, Ty) d(y, Ty)}{d(x, Sx) + d(x, Ty)}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are non-negative constants such that $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leq 1$. Then $S$ and $T$ have a unique common fixed point.
for all \( x, y \) in \( X \) with \( d(x,Ty) + d(y,Sx) \neq 0 \) and \\
d\((x,Sx) + d(x,Ty) \neq 0 \), where \( \alpha_1, \beta_1 \geq 0 \) (\( i = 1,2 \)) and \\
\((\alpha_1 + \alpha_2 + \beta_1 + \beta_2) < 1 \), then \( S \) and \( T \) have a common fixed. Further, if \( d(x, Ty) + d(y, Sx) = 0 \) or \\
d\((x,Sx) + d(x,Ty) = 0 \), then \( S \) and \( T \) have a unique common fixed point.

It may be remarked the theorem \( A_2 \) is particular case of theorem 1.

Further, we extend the theorem 1 for a sequence of mappings in complete metric space.

**Theorem 2.** Let \( \mathbb{I}_+ \) be the set of positive integers. Further, let \( T_0 \) and \( \{ T_n : n \in \mathbb{I}_+ \} \) be mappings of a non-empty complete metric space \( X \) into itself satisfying the inequality,

\[
(1.2.4) d(T_0x, T_ny) \leq \frac{\alpha_1 d(x, T_0x)d(x, T_ny) + \alpha_2 d(y, T_0x)d(y, T_ny)}{d(x, T_ny) + d(y, T_0x)} + \frac{\beta_1 d(x, T_0x)d(y, T_0x) + \beta_2 d(x, T_ny)d(y, T_ny)}{d(x, T_0x) + d(x, T_ny)}
\]

for all \( x, y \) in \( X \) and for each \( n = 1,2, \ldots \), for which \( d(x, T_ny) + d(y, T_0x) \neq 0 \) and \( d(x, T_0x) + d(x, T_ny) \neq 0 \), where \( \alpha_1, \beta_1 \geq 0 \) (\( i = 1,2 \)) and \((\alpha_1 + \alpha_2 + \beta_1 + \beta_2) < 1 \); then there exists a point \( z \in X \) such that \( T_nz = z \), \( n = 0,1,2 \ldots \), and for an arbitrary point \( x_0 \in X \), the sequence

\[
x_1 = T_0x_0, \ x_2 = T_1x_1, \ x_3 = T_0x_2, \ldots \ x_{2n-1} = T_0x_{2n-2}, \ x_{2n} = T_nx_{2n-1}, \ x_{2n+1} = T_0x_{2n}, \ldots \ (n = 1,2, \ldots)
\]

converges to \( z \). Further, if \( d(x, T_ny) + d(y, T_0x) = 0 \) or
For solving non-linear functional equations, we prove the following theorem:

Theorem 3. Let \( \{f_n\} \) and \( \{g_n\} \) be the sequences of elements in complete metric space \((X, d)\). And let \( v_n \) be the unique solution of the equations \( u - Su = f_n \) and \( u - Tu = g_n \), where \( S \) and \( T \) are the mappings of \( X \) into itself satisfying the condition (1.2.3) of theorem 1. If \( f_n, g_n \rightarrow 0 \) as \( n \rightarrow \infty \), then sequence \( \{v_n\} \) converges to the common solution of the equations \( u = Su \) and \( u = Tu \).

In the sequence, we prove the following geometrical problem:

Theorem 4. Let \( X = \mathbb{R} \), be the usual metric space of real numbers and \([a, b] \subset \mathbb{R}; S, T : [a, b] \rightarrow [a, b] \) be two differentiable functions such that

\[
|S'(x)| \leq \alpha_1 < 1, \quad |T'(x)| \leq \alpha_2 < 1, \quad \beta_1 = \frac{\alpha_1 - \alpha_2}{\alpha_1} < 1,
\]

\[
\beta_2 = \frac{\alpha_2}{\alpha_2} < 1 \quad \text{with} \quad (\alpha_1 + \beta_2 + \beta_1) < 1.
\]

Further, let the functions \( S \) and \( T \) be such that

\[
(1.2.5) \quad |S(x) - T(y)| = \frac{1}{|x-y|} \left[ |S(x)-S(y)||x-S(x)||x-T(y)| + |T(x)-T(y)| \right]
\]

\[
|T(x)-T(y)-S(x)+S(y)| \left[ \frac{|x-S(x)||y-Sx|}{|S(x)-S(y)|} + \frac{|x-T(y)||y-T(y)|}{|T(x)-T(y)|} \right]
\]

\[
+ \frac{|x-S(x)| + |x-T(y)|}{|y-S(x)| + |y-T(y)|}
\]

for all \( x, y \in [a, b] \) with \( |x-y| \neq 0, |S(x)-S(y)| \neq 0 \),
\[ T(x) - T(y) \neq 0, \quad x - T(y) + y - S(x) \neq 0 \quad \text{and} \quad x - S(x) + x - T(y) \neq 0, \]
then there exists a unique solution of the equations
\[ x = Sx = Tx. \]

In 1984, Wang et al. [87] proved the following fixed point theorems for expansion mappings:

Theorem A3. If there is a constant \( a > 1 \) such that
\[ (1.2.6) \quad d(fx, fy) \geq \min \left\{ d(x, fx), d(fy, y), d(x, y) \right\} \]
for any \( x, y \) in \( X \) and \( f \) is on to continuous, then \( f \) has a fixed point.

Theorem A4. If there exists a real constant \( a \) with \( a > 1 \) such that
\[ (1.2.7) \quad d(fx, fy) \geq a \cdot d(x, y) \]
for any \( x, y \) in \( X \) and \( f \) is onto, then \( f \) has a unique fixed point.

In 1985, Rhoades [63] generalized the above theorems for a pair of mappings. He proved the following theorems:

Theorem A5. Let \( f, g \) be surjective self-maps of a complete metric space \((X, d)\). Suppose there exists a constant \( a > 1 \) such that
\[ (1.2.8) \quad d(fx, gy) \geq a \cdot d(x, y) \]
for each \( x, y \) in \( X \), then \( f \) and \( g \) have a common fixed point.

Theorem A6. Let \( f, g \) be surjective continuous self-maps of a complete metric space \( X \). If there exists a real number \( a > 1 \) such that
\[ (1.2.9) \quad d(fx, gy) \geq a \min \left\{ d(x, fx), d(y, gy), d(x, y) \right\} \]
for each \( x, y \in X \), then \( f \) or \( g \) has a fixed point, or
\( f \) and \( g \) have a common fixed point.

As an extension of theorem 5, we prove the following theorem:

**Theorem 5.** Let \( f \) and \( g \) be surjective continuous self maps of a complete metric space \( (X, d) \). Suppose there exists a constant \( a > 1 \) such that
\[ (1.2.10) \quad [d(fx, gy)]^2 \geq a [d(x, fx) d(y, gy)] \]
for each \( x, y \in X \), then \( f \) and \( g \) have a unique common fixed point.

Our next theorem is a generalization of theorem 5 for sequence of mappings.

**Theorem 6.** Let \( \{f_n\} \) \((n = 1, 2, \ldots)\) be a sequence of surjective continuous self-maps of a complete metric space \( X \). If there exists a real number \( a > 1 \) such that for all \( n \in \mathbb{N} \),
\[ (1.2.11) \quad d(f_0x, f_ny) \geq a \min \left\{ d(x, y), d(x, f_0x), d(y, f_ny) \right\} \]
for each \( x, y \in X \), then there exists a common fixed point of \( f_n \) in \( X \).

A metric space \( X \) with two metrics \( d \) and \( d_1 \) have been considered by Maia \([50]\) in 1968, he proved the following theorem:

**Theorem A7.** Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \) and \( T \) a self-map of \( X \). Let \( X \) satisfying the
following conditions:

(1.2.12) \( d_1(x, y) \leq d(x, y) \) for all \( x, y \in X \)

(1.2.13) \( X \) is complete with respect to \( d_1 \).

(1.2.14) \( T \) is continuous with respect to \( d_1 \).

(1.2.15) \( d(Tx, Ty) \leq \alpha d(x, y) \)
for every \( x, y \in X \) and \( \alpha \in [0, 1) \), then \( T \) has a
unique fixed point.

In 1986, an extension of the above theorem \( A_7 \)
has been carried out by Dhange and Dhobale [20]. A
brief statement of the theorem is given below:

Theorem \( A_8 \). Let \( X \) be a metric space with two
metrics \( d \) and \( d_1 \). Let \( T \) be a self-map of \( X \) and \( X \) satis­
fying the following conditions:

(1.2.16) \( d_1(x, y) \leq a [d(x, Tx) + d(y, Ty)] \), \( a \geq 1 \) and
for every \( x, y \in X \),

(1.2.17) \( X \) is \( T \) orbitally complete with respect to \( d_1 \),

(1.2.18) there exists a real number \( \beta \) such that

\[
\min \left\{ d(Tx, Ty), d(x, Tx), d(y, Tx) \right\} + \beta \left\{ d(x, Ty), d(y, Tx) \right\}
\leq p d(x, Tx) + q d(x, y)
\]

for every \( x, y \in X \), where \( p \) and \( q \) are non-negative
constants such that \( p + q = \beta \in [0, 1) \), then \( T \) has a fixed
point.

We establish a fixed point theorem in complete
metric space, which satisfies a new condition different
from Dhange and Dhoble [20].
Theorem 7. Let $X$ be a metric space with two metrics $d$ and $d_1$. Let $T$ be a self-map of $X$ and $X$ satisfies the following conditions:

(1.2.19) $d_1(x, y) \leq a [d(x, Tx) + d(y, Tx)]$, $a \geq 1$ and every $x, y \in X$.

(1.2.20) $X$ is $T$-orbitally complete with respect to $d_1$.

(1.2.21) $\min \left\{ d(x, Tx) d(x, Ty) d(Tx, Ty), \frac{1}{2} [d(x, y)]^2 d(x, Ty), \frac{1}{2} d(x, Tx) d(y, Ty) \left[ d(x, Ty) + d(Tx, y) \right], \min \left\{ d(x, Tx) d(x, y) d(Tx, Ty), \frac{1}{2} d(x, Ty) d(y, Tx) \left[ d(x, Ty) + d(Tx, y) \right] \right\} \right\} \leq \frac{1}{4} q d(x, Tx) d(x, y) \left[ d(x, Ty) + d(Tx, y) \right]

for all $x, y \in X$, $q \in [0, 1)$, then $T$ has a fixed point.

1.3. Common fixed point of three mappings in metric space

If $A$, $S$ and $T$ are the three self-mappings of a metric space $(X, d)$ and for any $x \in X$, if

(1.3.1) $Ax = Sx = Tx = x$, then $x$ is said to be the common fixed point of $A$, $S$ and $T$.

In 1979, Yeh [88] extended the result of Jungck [39] on common fixed point for continuous and commuting self-mappings of a complete metric space and obtained a unique common fixed point of three continuous self-mappings of a complete metric space. In 1980, Singh and Singh [75] obtained common fixed point of three mappings under less stringent conditions than Yeh [88].

In 1980, Jaggi and Dass [38] extended the result of Banach [2] and proved the following theorem:
Theorem B.1. Let \( f \) be a self-map defined on a metric space \((X,d)\) satisfying following conditions:

(1.3.1) for some \( \alpha, \beta \in [0,1) \) with \((\alpha + \beta) < 1\),

\[
d(fx, fy) \leq \frac{\alpha d(x,fx) d(y,fy)}{d(x,fx)+d(y,fx)+d(x,y)} + \beta d(x,y)
\]

for all \( x, y \in X, x \neq y \).

(1.3.2) there exists \( x_0 \in X \),

\[
\{ f^n x_0 \} \supset \{ f^n k x_0 \} \text{ with } \lim_{n \to \infty} f^n x_0 \in X,
\]

then \( f \) has a unique fixed point \( u = \lim_{k \to \infty} f^n x_0 \).

In 1988, Paliwal [53] extended the above result and proved the following theorem:

Theorem B.2. Let \( T_1 \) and \( T_2 \) be two continuous self-maps of a metric space \((X,d)\) such that

(1.3.3) \( d(T_1^{r} x, T_2^{s} y) \leq \frac{\alpha d(x,T_1^{r} x) d(y,T_2^{s} y)}{d(x,T_2^{s} y)+d(y,T_1^{r} x)+d(x,y)} + \beta d(x,y)\)

for all \( x, y \in X, x \neq y \), where \( r > 0, s > 0 \) are integers and \( \alpha, \beta \) are non-negative real numbers such that \((\alpha + \beta) < 1\).

If for some \( x_0 \in X \), the sequence \( \{ x_n \} \) consisting of points

\[
x_{2n+1} = T_1^{r} x_{2n}, x_{2n+2} = T_2^{s} x_{2n+1},
\]

has a subsequence \( \{ x_{n_k} \} \) converges to a point \( u \), then \( T_1 \) and \( T_2 \) have a unique common fixed point.

We shall prove some unique common fixed point theorems for three mappings on complete metric space which generalize the results of Jaggi and Dass [38], Paliwal [53] and Edelstein [22]. In fact, we prove the following theorem:
Theorem 8. Let \( E, F \) and \( T \) be the three continuous self maps of a complete metric space \((X, d)\) and satisfying the following conditions:

\[
\begin{align*}
(1.3.4) & \quad ET = TE, FT = TF, E(X) \subseteq T(X) \text{ and } F(X) \subseteq T(X) \\
(1.3.5) & \quad d(Ex, Fy) \leq \frac{\alpha d(Tx, Ex) d(Ty, Fy)}{d(Tx, Fy) + d(Ty, Ex) + d(Tx, Ty)} + \beta d(Tx, Ty)
\end{align*}
\]

for all \( x, y \) in \( X \), \( Tx \neq Ty \), \( \alpha, \beta \geq 0 \) and \( (\alpha + \beta) < 1 \), then \( E, F \) and \( T \) have a unique common fixed point.

Further, we extend the above theorem 8 and prove the following theorem:

Theorem 9. Let \( E, F \) and \( T \) be the self mappings of a complete metric space \((X, d)\) satisfying the condition \((1.3.4)\) of theorem 8 and

\[
\begin{align*}
(1.3.6) & \quad d(E^r x, F^s y) \leq \frac{\alpha ^r d(Tx, E^r x) d(Ty, F^s y)}{d(Tx, F^s y) + d(Ty, E^r x) + d(Tx, Ty)} + \beta d(Tx, Ty)
\end{align*}
\]

for all \( x, y \) in \( X \), \( Tx \neq Ty \), \( \alpha, \beta \geq 0 \) and \( (\alpha + \beta) < 1 \).

If some positive integers \( r \) and \( s \), \( F^r x, F^s y \) and \( T \) are continuous, then \( E, F \) and \( T \) have a unique common fixed point.

In 1986, Pathak [56] generalized the known results of Banach [2], Kannan [40] and Fisher and Khan [27] and proved the following theorem:

Theorem 1.3. Let \( S \) and \( T \) be two self maps of a complete metric space \((X, d)\) and satisfy the inequality,

\[
(1.3.7) \quad d(Sx, Ty) \leq q \frac{\alpha d(x, Sx) d(x, Ty) + \beta d(y, Sx) d(y, Ty) + \gamma d(x, y)^2}{\alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y)} + \rho d(x, y)
\]

for all \( x, y \) in \( X \) with \( 0 \leq q < 1 \) and \( \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y) \neq 0 \), where \( \alpha, \beta, \gamma, \rho \geq 0 \) (not all zero), then
S and T have a common fixed point. Further, if
\[ \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y) = 0, \]
then S and T have a unique common fixed point.

In 1982 Sessa introduced the new concept of 'Weekly commuting pair' of mappings in metric space which reads as follows:

Let A and S be two self mappings on X, then A, S is said to be weakly commuting pair if
\[ d(ASx, SAx) \leq d(Ax, Sx) \]
for all \( x \in X \).

Clearly, a commuting pair is weakly commuting but the converse is not true.

We generalize the theorem B for three mappings using this new concept weakly commuting and prove the following theorem:

Theorem 10. Let S, T and I be three mappings of a complete metric space \((X, d)\) such that for all \( x, y \) in \( X \),
\[
(1.3.9) \quad d(Sx, Ty) \leq q \frac{\alpha d(Ix, Sx) d(Ix, Ty) + \beta d(Iy, Sx) d(Iy, Ty) + \gamma d(Ix, Iy)^2}{d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy)}
\]
if \( \alpha d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy) \neq 0 \), where \( 0 \leq q < 1 \) and \( \alpha, \beta, \gamma \geq 0 \) (not all zero). If the range of I contains the range of S and T, if either I is continuous and weakly commuting with either S or T, or if S is continuous and weakly commuting with I, or if T is continuous and weakly commuting with I, then S, T and I have a common fixed point. Further, if \( \alpha d(Ix, Ty) + \beta d(Iy, Sx) + \gamma d(Ix, Iy) = 0 \), then S, T and I have a unique common fixed point.
In 1986, Bhagwat and Singh [4] extended the result of Desa and Gupta [17] for a pair of mappings and proved the following theorem:

Theorem B. Let $T_1$ and $T_2$ be two continuous self mappings of a metric space $(X,d)$ such that

\[
(1.3.10) d(T_1x, T_2y) \leq \frac{d(x, T_1x) d(x, T_2y) + d(y, T_2y) d(y, T_1x)}{d(x, T_2y) + d(y, T_1x)}
\]

for all $x, y$ in $X$. If for some $x_0 \in X$, the sequence $\{x_n\}$ of elements $x_n$ where $x_{2n+1} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1}$, \ldots has a convergent subsequence $\{x_{n_k}\}$ converging to a point $x \in X$, then $x$ is a unique fixed point of $T_1$ and $T_2$.

We generalize the above theorem for three mappings and prove the following theorem:

Theorem 11. Let $(X,d)$ be a complete metric space. Let $S$, $P$ and $T$ be continuous self mappings as $P, S, T : X \to X$ satisfying the following conditions:

\[(1.3.11) \{(S, T)\} \text{ is a weakly commuting pair of mappings with respect to mapping } P,\]

\[(1.3.12) (SPx, TPy) \leq \frac{d(x, SPx) d(x, TPy) + d(y, TPy) d(y, SPx)}{d(x, TPy) + d(y, SPx)}\]

for all $x, y$ in $X$. If for some $x_0 \in X$, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to a point $z \in X$, then $z$ is a unique common fixed point of $S$, $P$ and $T$. 
Recently, Browder and Petryshyn introduced the notion of asymptotic regularity for a Banach space. Its equivalent form in metric space is given as follows:

Let $A$ and $S$ be two self maps of $X$ and $\{x_n\}$ a sequence in $X$. Then $\{x_n\}$ is said to be asymptotically $S$-regular with respect to $A$ in $d(Ax_n, Sx_n) \to 0$ as $n \to \infty$.

The notion of asymptotic regularity is very useful in proving the existence of fixed points of mappings.

In 1981, Sharma and Yuel proved the following theorem:

Theorem B.5. Let $f$ be a self-map of $X$ into itself of a complete metric space $(X, d)$ satisfying the inequality

\[(1.3.13) \quad d(fx, fy) \leq a_1 d(x, fx) + a_2 d(y, fy) + a_3 d(x, fy) + a_4 d(y, fx) + a_5 d(x, y)\]

for all $x, y \in X$, where $a_i \geq 0$ and $\sum a_i < 1$ for $i=1, 2, \ldots, 5$. Such a condition is equivalent to the condition

\[(1.3.14) \quad d(fx, fy) \leq a \left[d(x, fx) + d(y, fy)\right] + b \left[d(x, fy) + d(y, fx)\right] + c d(x, y)\]

by interchanging $x$ and $y$; where $a, b, c \geq 0$ and $(a + b + c) < 1$, then $f$ has a unique fixed point in $X$, if $f$ is asymptotically regular at some point in $X$.

We generalize the above theorem with the use of weakly commuting mappings. We prove the following theorem:
Theorem 12. Let $A$, $S$ and $T$ be three self-maps of a complete metric space $(X,d)$ satisfying

\begin{equation}
\tag{1.3.15}
d(Sx,Ty) \leq a_1 d(Ax,Sx) + a_2 d(Ay,Ty) + a_3 d(Ax,Ty) + a_4 d(Ay,Sx) + a_5 d(Ax,Ay),
\end{equation}

for all $x,y$ in $X$ where $a_i = a_i(x,y)$, $i = 1, 2, \ldots, 5$, are non-negative functions such that

\begin{equation}
\tag{1.3.16}
\max \left\{ \sup_{x,y \in X} (a_2 + a_3), \sup_{x,y \in X} (a_1 + a_4), \sup_{x,y \in X} (a_3 + a_4 + a_5) \right\} < 1,
\end{equation}

\begin{equation}
\tag{1.3.17}
\text{if } A \text{ is continuous,}
\end{equation}

\begin{equation}
\tag{1.3.18}
\text{the map } A \text{ weakly commutes with } S \text{ and } T,
\end{equation}

\begin{equation}
\tag{1.3.19}
\text{there exists a sequence which is asymptotically } S\text{-regular and } T\text{-regular with respect to } A, \text{ then } A, S \text{ and } T \text{ have a unique common fixed point.}
\end{equation}

Let $R^+$ be the set of non-negative reals, $N$ the set of positive integers and let $(X,d)$ be a complete metric space. Consider a real function $f \in F$ where $F$ be the set of all real functions satisfying the following properties:

\begin{itemize}
    \item [(i)] $f$ is upper semi continuous,
    \item [(ii)] $f$ is non-decreasing in each co-ordinate variables,
    \item [(iii)] $f(t) < t$ for any $t > 0$.
\end{itemize}

In 1984, Chang [36] generalizing the result of Husain and Sehgal [35] and Iseki [31], and established the following result for a family of $f$-contraction mappings:
Theorem B. Let \( S, T : X \rightarrow X \) be continuous.

Then \( S \) and \( T \) have a common fixed point \( w \) if and only if there exist two self-maps \( A, B \) of \( X \) and a function \( f \in F \) such that

\[
(1.3.20) \quad A(X) \cup B(X) \subseteq S(X) \cap T(X),
\]

\[
(1.3.21) \quad \text{both } A \text{ and } B \text{ commute with } S \text{ and } T,
\]

\[
(1.3.22) \quad d(Ax, By) \leq f\left(\max \left\{ d(Sx, Ty), d(Sx, Ax), d(Ty, By)\right\}, \frac{1}{2} \left[ d(Sx, By) + d(Ty, Ax)\right]\right)
\]

for any \( x, y \in X \). Further, \( w \) is the unique common fixed point of \( A, B, S \) and \( T \).

In the last theorem of this chapter we improve the above result under considerable weaker condition of mappings. We prove the following theorem:

Theorem 13. Let \( A, B, S \) and \( T \) be four self-maps of \((X, d)\) such that

\[
(1.3.23) \quad A^2(X) \subseteq T^2(X) \text{ and } B^2(X) \subseteq S^2(X),
\]

\[
(1.3.24) \quad d(A^2 x, B^2 y) \leq f \left( \max \left\{ d(S^2 x, T^2 y), d(S^2 x, A^2 x), d(T^2 y, B^2 y), \frac{1}{2} \left[ d(S^2 x, B^2 y) + d(T^2 x; A^2 x)\right]\right\}\right)
\]

for all \( x, y \in X \), where \( f \) satisfy (i), (ii) and (iii).

If one of \( A, B, S \) and \( T \) is continuous and if \( A \) and \( B \) weak** we commute with \( S \) and \( T \) respectively, then \( A, B, S \) and \( T \) have a common fixed point \( z \). Further \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

1.4 Common fixed point theorems in 2-metric space

In 1963 Gähler [28] introduced the concept of 2-metric space giving the following definition:
**Definition**: A 2-metric space is a space $X$ in which, for each triple of points $a, b, c$ there exists a non-negative real valued function $d(a, b, c)$ such that (1.4.1) to each pair of points $a, b$ with $a \neq b$ in $X$, there exists a point $c \in X$ such that $d(a, b, c) \neq 0$,

(1.4.2) $d(a, b, c) = 0$, when at least two of the points are equal,
(1.4.3) $d(a, b, c) = d(b, c, a) = d(a, c, b),
(1.4.4) d(a, b, c) \leq d(a, b, w) + d(a, w, c) + d(w, b, c)$

for all $a, b, c$ and $w \in X$.

The non-negative real valued function $d$ is called 2-metric on $X$. The 2-metric $d$ is continuous function of each of its three variable but it is not necessarily continuous functions of two variables.

And extensive bibliography regarding the related works and fixed point theorems in 2-metric space appears in the paper of Singh, Tiwari and Gupta [72].

In 1988, Paliwal established the following fixed point theorem:

**Theorem C.** If $T$ is a mapping of a complete 2-metric space $X$ into itself, satisfying the condition

\[(1.4.5) d(Tx, Ty, a) \leq b_1 \left\{ \frac{d(x, Tx, a) \cdot d(y, Ty, a)}{d(x, y, a)} \right\} + b_2 \left\{ d(x, Tx, a) + d(y, Ty, a) \right\} + b_3 \left\{ d(x, Ty, a) + d(y, Tx, a) \right\} + b_4 d(x, y, a) \]
In chapter III, we generalize above theorem for a pair of mappings. We first prove the following theorem:

**Theorem 14.** Let \( T_1 \) and \( T_2 \) be an orbitally continuous self mappings from \( 2 \)-metric space \( X \) in to itself if \( T_1 \) and \( T_2 \) satisfy

\[
(1.4.6) \quad d(T_1 x, T_2 y, a) \leq b_1 \left( \frac{d(x, T_1 x, a) + d(y, T_2 y, a)}{d(x, y, a)} \right)
\]

\[
+ b_2 \left[ d(x, T_1 x, a) + d(y, T_2 y, a) \right] + b_3 \left[ d(x, T_2 y, a) + d(y, T_1 x, a) \right]
\]

\[
+ b_4 d(x, y, a)
\]

for all \( x, y \) and \( a \) in \( X \), where \( 0 \leq \frac{(b_2 + b_3 + b_4)}{(1 - b_1 - b_2 - b_3)} \leq 1 \), \( (b_2 + b_3) < 1 \), \( (2b_3 + b_4) < 1 \), \( b_3 \geq 0 \), then \( T_1 \) and \( T_2 \) have a unique common fixed point.

Since sequence of mappings the more general so we extend the theorem 14 and prove the following theorem:

**Theorem 15.** Let \( X \) be a complete \( 2 \)-metric space, \( \{T_n\} \), \( n = 1, 2, \ldots \), a sequence of mappings of \( X \) into itself such that for all \( x, y \) and \( a \) in \( X \),

\[
(1.4.7) \quad d(T_1 x, T_1 y, a) \leq \left\{ \frac{b_1 d(x, T_1 x, a) + d(y, T_1 y, a)}{d(x, y, a)} \right\}
\]

\[
+ b_2 \left[ d(x, T_1 x, a) + d(y, T_1 y, a) \right] + b_3 \left[ d(x, T_1 y, a) + d(y, T_1 x, a) \right]
\]

\[
+ b_4 d(x, y, a).
\]
\[ b_1, b_2, b_3 \text{ and } b_4 \text{ are as in theorem 12, then sequence } \{T_n\} \text{ has a common fixed point.} \]

Park and Rhoades [61], Popa and Khan et al [62] have studied some fixed point theorems for expansion mappings, extending these results Wang et al [87] and proved the following fixed point theorems in complete metric space.

**Theorem C_2.** If there exists a constant \( q > 1 \) such that

\[
1.4.8 \quad d(fx, fy) \geq q d(x, y)
\]

for all \( x, y \) in complete metric space \((X, d)\) and \( f \) is onto mapping, then \( f \) has a unique fixed point.

**Theorem C_3.** If there exists a real constant with \( q > 1 \) such that

\[
1.4.9 \quad d(fx, fy) \geq q \min \left\{ d(x, fx), d(fy, y), d(x, y) \right\}
\]

for all \( x, y \) in complete metric space \((X, d)\) and \( f \) is onto continuous mapping, then \( f \) has a fixed point.

We establish two theorems on common fixed point for expansion mappings in 2-metric space, which extend the results of Rhoades [63] and also generalize the results of Wang et al [87]. The condition of continuity has also been replaced by a weaker orbital continuity. We prove the following theorems:

**Theorem 16.** Let \( S \) and \( T \) be two surjective self mappings of a complete 2-metric space \( X \). Suppose there exists a constant \( q > 1 \) such that
(1.4.10) \( d(Sx, Ty, a) \geq q \ d(x, y, a) \)

for each \( x, y \) and \( a \) in \( X \), then \( S \) and \( T \) have a unique
common fixed point.

Theorem 17. Let \( S \) and \( T \) be two surjective orbitally continuous mappings of a complete 2-metric
space \( X \). If there exists a real number \( q > 1 \) such that
(1.4.11) \( d(Sx, Ty, a) \geq q \ \min \left\{ d(x, Sx, a), d(y, Ty, a), d(x, y, a) \right\} \)
for each \( x, y, a \in X \), then \( S \) or \( T \) has a fixed point or \( S \)
and \( T \) have a common fixed point.

In the last, we have presented a fixed point theorem for continuous nearly densifying mappings, which
is straight forward extension of the corresponding result
of Chattopadhyay [16]. We prove the following theorem:

Theorem 18. Let \( (X, d) \) be a complete 2-metric
space, with property (S) and \( d \) continuous. Let \( T_1 \) and \( T_2 \)
be commuting continuous and nearly densifying maps on \( X \).
Suppose that for some \( x_0 \) in \( X \), \( \mathcal{X}_i (x_0) = \left\{ T_1^i T_2^j x_0 : i, j \text{ are non negative integers} \right\} \) is bounded and \( X \setminus \mathcal{X}_i (x_0) \neq \emptyset \).

Further assume that
(1.4.12) \[ d(T_2^i x, T_2^j y, a) \leq \max \left\{ d(T_1^i x, T_1^j y, a), d(T_1^i x, T_2^i x, a), \right. \]
\[ d(T_1^i y, T_2^j y, a), \frac{1}{2} \left[ d(T_1^i x, T_2^i y, a) + d(T_1^i y, T_2^i x, a) \right] \]
for distinct \( T_1^i x, T_1^i y, a, \) distinct \( T_2^i x, T_2^i y, a \) in \( X \), then \( T_1 \)
and \( T_2 \) have a unique common fixed point.

1.5 Fixed point theorems in L-space

In 1975, Kasahara [49] introduced L-space in
fixed point theory. It is observed that in many fixed
point theorems, the metric properties, in particular the
axiom of triangle inequality, are not essential in their
proofs.

Let \( N \) denote the set of all non-negative integers.

A pair \((X, \rightarrow)\) of a set \( X \) and a subset \( \rightarrow \) of the
set \( X^N \times X \) is called an \( L \)-space if the following two condi-
tions are satisfied:

\[ (1.5.1) \text{ if } x_n = x \in X \text{ for all } n \in N, \text{ then } (\{x_n\}_{n \in N}, x) \in \rightarrow, \]
\[ (1.5.2) \text{ if } (\{x_n\}_{n \in N}, x) \in \rightarrow, \text{ then } (\{x_{n_i}\}_{i \in N}, x) \in \rightarrow \]

for every subsequence \( \{x_{n_i}\}_{i \in N} \) of \( \{x_n\}_{n \in N} \).

In what follows, we shall write \( \{x_n\}_{n \in N} \rightarrow x \) or
\( x_n \rightarrow x \) instead of \( (\{x_n\}_{n \in N}, x) \in \rightarrow \) and read
\( \{x_n\}_{n \in N} \) converges to \( x \).

Kasahara suggests that the notion of metric may
not be essential in Banach contraction theorem with this
view, we prove the following theorems in Chapter IV.

Theorem 19. Let \((X, \rightarrow)\) be a separated \( L \)-space
which is \( d \)-complete for a non-negative real valued function
\( d \) on \( X \times X \) with \( d(x, x) = 0 \) for each \( x \) in \( X \). Let \( S \) and \( T \)
be continuous self mappings of \( X \) satisfying the following
conditions:

\[ (1.5.3) \ ST = TS, \ S(X) \subseteq T(X) \]
\[ (1.5.4) \ d(Sx, Tx) + d(Sy, Ty) \leq \alpha d(Tx, Ty) + \beta d(Tx, Ty) \]
for all \(x,y\) in \(X\), where \(1 \leq \alpha \leq 2\) and \(0 \leq \beta\), then \(S\) and \(T\) have a common fixed point. Further, if \(0 \leq \alpha \leq 1\) and \(0 \leq \beta\), then \(S\) and \(T\) have a unique common fixed point.

**Theorem 20.** Let \((X, \rightarrow)\) be a separated \(L\)-space which is \(d\)-complete for a non-negative real valued function \(d\) on \(X \times X\) with \(d(x,x) = 0\) for each \(x\) in \(X\). Let \(S\) and \(T\) be continuous self mappings of \(X\) satisfying the following conditions:

\[(1.5.3)\] of theorem 19 and
\[(1.5.5)\]  
\[
\alpha_1 d(Sx,Sy) + \alpha_2 d(Sx,Tx) + \alpha_3 d(Sy,Ty) \leq \min \left\{ d(Sx,Ty), d(Sy,Tx) \right\} \leq \beta d(Tx,Ty)
\]

for all \(x,y\) in \(X\), where \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\) such that

\[(\beta - \alpha_3) \geq 0, \ \alpha_1 + \alpha_2 + \alpha_3 > \beta\], then \(S\) and \(T\) have a common fixed point. Further, if \((\alpha_1 - 1) > \beta\), then \(S\) and \(T\) have a unique common fixed point.

In 1979, Yeh [90] proved the following fixed point theorem:

**Theorem D.** Let \((X, \rightarrow)\) be a separated \(L\)-space which is \(d\)-complete for a non-negative real valued function \(d\) on \(X \times X\) with \(d(x,x) = 0\) for each \(x\) in \(X\). Let \(E\) and \(T\) be continuous self mappings of \(X\) satisfying the following conditions:

\[(1.5.6)\]  
\[E T = T E, \ E(X) \subseteq T(X)\]
\[(1.5.7)\]  
\[
\left\{ d(Ex, Ey) \right\}^2 \leq a d(Tx, Ex)d(Ty, Ey) + b d(Tx, Ey)d(Ex, T)\]

for all \(x,y\) in \(X\) where \(0 \leq a < 1\) and \(0 \leq b\), then \(E\) and \(T\) have a common fixed point. Further, if \(0 \leq a, b < 1\), then \(E\) and \(T\) have a unique common fixed point.
We generalize the above theorem for three mappings and prove the following theorem:

**Theorem 21.** Let $(X, \rightarrow)$ be a separated $L$-space which is $d$-complete for a non-negative real valued function $d$ on $X \times X$ with $d(x, x) = 0$ for each $x$ in $X$. Let $E, F$ and $T$ be three continuous mappings of $X$ satisfying the following conditions:

\begin{align*}
1.5.8 & \quad E \circ T = T \circ E, \quad F \circ T = T \circ F, \quad E(X) \subseteq T(X) \quad \text{and} \quad F(X) \subseteq T(X) \\
1.5.9 & \quad \left\{ d(Ex, Fy) \right\}^2 \leq ad(Tx, Ex)d(Ty, Fy) + bd(Tx, Fy)d(Ex, Ty)
\end{align*}

for all $x, y$ in $X$, where $0 \leq a < 1$ and $0 \leq b$, then $E$, $F$ and $T$ have a common fixed point. Further, if $0 \leq a, b < 1$, then $E$, $F$ and $T$ have a unique common fixed point.

Further, we prove the following theorems:

**Theorem 22.** Let $(X, \rightarrow)$ be a separated $L$-space which is $d$-complete for a non-negative real valued function $d$ on $X \times X$ with $d(x, x) = 0$ for each $x$ in $X$. Let $E, F$ and $T$ be three continuous self mappings of $X$ satisfying the condition (1.5.8) of theorem 21 and

\begin{align*}
1.5.10 & \quad d(Ex, Fy) \leq \frac{ad(Tx, Ex)d(Ty, Fy)}{d(Tx, Ty)} + b \cdot d(Tx, Ty)
\end{align*}

for all $x, y$ in $X$, where $a, b \geq 0$ and $(a + b) < 1$ with $Tx \neq Ty$, then $E$, $F$ and $T$ have a common fixed point. Further, if $a, b \geq 0$ and $(a + b) < 1$, then $E$, $F$ and $T$ have a unique common fixed point.

**Theorem 23.** Let $(X, \rightarrow)$ be a separated $L$-space which is $d$-complete for a non-negative real valued function $d$ on $X \times X$ with $d(x, x) = 0$ for each $x$ in $X$. Let $E, F$ and $T$
be three continuous self mappings of $X$ satisfying the following conditions:

$$(1.5.11) \quad \text{EFT} = \text{TEF}, \text{PET} = \text{TEF}, \text{EF}(X) \subset T(X) \text{ and } FE(X) \subset T(X);$$

$$(1.5.12) \quad \text{d}(\text{EF}x, \text{FE}y) \leq \frac{\text{ad}(\text{Tx, EFX})\text{d}(\text{Ty, FEy})}{\text{d}(\text{Tx, Ty})} - \text{bd}(\text{Tx, Ty})$$

for all $x, y \in X$ with $\text{Tx} \neq \text{Ty}$, $a, b \geq 0$, $a + b > 1$.

Further, if $T, EF$ and $FE$ are continuous, then $E, F$ and $T$ have a unique common fixed point in $X$.

1.6 Fixed point theorem in normed linear space

A set $N$ is called normed linear space, if:

$$(1.6.1) \quad N \text{ is a linear system with real (complex) numbers as ring multipliers;}$$

$$(1.6.2) \quad \text{to every element } x \text{ of the linear system } N \text{ is assigned a unique real number, called the norm of this element and denoted by } \|x\|, \text{ satisfying the following axioms}:$$

$$(1.6.3) \quad \|x\| \geq 0 \text{ and } \|x\| = 0, \text{ iff } x = 0;$$

$$(1.6.4) \quad \|x + y\| \leq \|x\| + \|y\| \text{ (triangle inequality)}$$

$$(1.6.5) \quad \|\lambda x\| = |\lambda| \|x\| \text{ (homogeneity of the norm)}.$$

In a normed linear space, a metric (distance) can be introduced by $d(x, y) = \|x - y\|$.

In 1983, Naimpally and Singh [52] extended a fixed point theorem of Rhoades [66] for a map $T$, with using $I$-scheme. They proved the following theorem:

Theorem $D_1$. Let $N$ be a normed linear space and $X$ be a closed convex subset of $N$. Let $T: X \rightarrow X$ be a
mapping satisfying

\[(1.6.6) \|Tx - Ty\| \leq k \max \left\{ \|x - y\|, \|x - Tx\| + \|y - Ty\|, \frac{\|x - Ty\| + \|y - Tx\|}{2} \right\},\]

\([x_n]\) the sequence of the I-scheme associated with T and such that \([\alpha_n]\) is bounded away from zero. If \([x_n]\) converges to \(p\), then \(p\) is a fixed point of \(T\).

In Chapter V, we first establish a fixed point theorem for a pair of mappings in normed linear space which extends the above theorem of Naimpally and Singh [52] with the use of the G-iterative process. We prove the following theorem:

**Theorem 24.** Let \(X\) be a closed convex bounded subset of a normed linear space \(N\) and let \(T_1\) and \(T_2\) be two self mappings of \(X\) satisfying

\[(1.5.7) \|T_1x - T_2y\| \leq q \max \left\{ c \|x - y\|, \frac{\|x - T_1x\| + \|y - T_2y\|}{2} \right\},\]

for all \(x, y\) in \(X\) where \(c \geq 0, 0 < q < 1\). Let the sequence \([x_n]\) be defined in accordance with the G-iterates associated with \(T_1\) and \(T_2\) as given below:

For \(x_0 \in X\), set

\[(1.6.8) x_{2n+1} = (\mu_{2n} - \lambda_{2n})x_{2n} + \lambda_{2n}T_1x_{2n} + (1 - \mu_{2n})T_2x_{2n-1}\]

\[(1.6.9) x_{2n+2} = (\mu_{2n+1} - \lambda_{2n+1})x_{2n+1} + \lambda_{2n+1}T_1x_{2n} + (1 - \mu_{2n+1})T_2x_{2n-1}\]

for \(n = 0, 1, 2, \ldots\), where \(\{\lambda_n\}\) and \(\{\mu_n\}\) satisfy

(i) \(\lambda_0 = \mu_0 = 1\) (ii) \(0 < \lambda_n < 1, 0 \leq \mu_n \leq 1\) such that \(\mu_n \geq \lambda_n\),

\(n > 0\) (iii) \(\lim_{n \to \infty} \lambda_n = h > 0\) (iv) \(\lim_{n \to \infty} \mu_n = 1\).
If \( \{x_n\} \) converges to \( z \) in \( X \), then \( z \) is a common fixed point of \( T_1 \) and \( T_2 \). Further, if \( \max \{q, 2q\} < 1 \), then mappings \( T_1 \) and \( T_2 \) have a unique common fixed point.

In 1984, Khan and Imdad \[45\] have extended the result of Goebel and Zlotkiewicz \[28\] on common fixed point for two maps and proved the following theorem:

Theorem D\(_2\). Let \( X \) be a closed and convex subset of \( N \). Let \( E : X \rightarrow X \) and \( F : X \rightarrow X \) satisfy the following conditions:

\[
(1.6.10) \quad E^2 = I, \quad F^2 = I, \quad \text{where} \quad I \text{ denotes identity map,}
\]

\[
(1.6.11) \quad \|Ex-Ey\| \leq (\frac{\alpha}{2}) \max \left\{ \|Fx-Fy\|, \frac{1}{2} \|Fx-Ex\|, \frac{1}{2} \|Fy-Ey\| \right\}
\]

for every \( xy \in X \), where \( 0 \leq \alpha < 4 \), then there exists at least one fixed point \( x_0 \in X \) such that \( Ex_0 = Fx_0 \).

In this section, we further present another extension of the result due to Khan and Imdad \[42\] which in turn generalizes the main result of Goebel and Zlotkiewicz \[29\]. We prove the following theorem:

Theorem 25. Let \( X \) be a closed and convex subset of \( N \). Let \( E, F \) and \( G : X \rightarrow X \) satisfying the following conditions:

\[
(1.6.12) \quad EF = FE, \quad FG = GF \quad \text{and} \quad EG = GE \;
\]

\[
(1.6.13) \quad E^2 = I, \quad F^2 = I, \quad G^2 = I, \quad \text{where} \quad I \text{ denotes the identity map;}
\]

\[
(1.6.14) \quad \|Ex-Ey\| \leq (\frac{\alpha}{2}) \max \left\{ \|FGx-Fgy\|, \frac{1}{2} \|FGx-Gx\|, \frac{1}{2} \|FGy-Ey\| \right\}
\]

\[
\frac{1}{2} \|FGy-Ey\|, \frac{1}{2} \|FGx-Ey\|, \frac{1}{2} \|FGy-Gx\|
\]
for all $x, y \in X$, where $0 \leq k < 2$, then $E, F$ and $G$ have a unique common fixed point.

In 1986, Pathak [56] established a fixed point theorem for a pair of mappings defined on subset of Banach space satisfying a new contractive type condition and proved the following theorem:

Theorem D3. Let $M$ be a non-empty closed convex subset of a Banach space $B$. Let $F : M \rightarrow M$ and $G : M \rightarrow M$ satisfy the conditions:

(1.6.15) $F$ and $G$ commutes,

(1.6.16) $F^2 = I$, $G^2 = I$, where $I$ denotes the identity map,

(1.6.17) $\|Fx - Fy\|^2 \leq q \max \left\{ \|Gx - Fx\| \|Gy - Fy\|, \|Gx - Fy\| \|Gy - Fx\|, \|Gx - Fy\| \|Gy - Fy\| \right\}$

for all $x, y \in M$, where $q \in (0, 1)$. Let $x_1 \in M$ be arbitrary; $t \in (0, 1)$ and $Gx_{n+1} = (1-t)Gx_n + tFx_n$ for each $n \geq 1$. If the sequence $\{Gx_n\}$ converges to a point $u \in M$, then $u$ is the unique common fixed point of $F$ and $G$.

In our last theorem of this chapter, we improve the above theorem and establish the following theorem:

Theorem 26. Let $M$ be a non-empty closed convex subset of a Banach space $B$. Let $F, G, H : M \rightarrow M$ satisfy the following conditions:

(1.6.18) $F, G$ and $H$ are commutes.
\[(1.6.19) \quad F^2 = I, G^2 = I, H^2 = I \quad \text{where } I \text{ denotes the identity map,} \]
\[(1.6.20) \quad \|Fx - Fy\| \leq q \max \left\{ \|Ghx - Fx\| , \|Ghy - Fy\| , \|Ghx - Fx\| , \|Ghy - Fy\| , \|Ghx - Fx\| \right\} \]
for all \(x, y \in M\), where \(q \in (0, 1)\). Let \(x_1 \in M\) be arbitrary, \(t \in (0, 1)\) and \(Ghx_{n+1} = (1-t) Ghx_n + tFx_n\) for each integer \(n \geq 1\). If the sequence \(\{Ghx_n\}\) converges to a point \(u \in M\), then \(u\) is a unique common fixed point of \(F, G\) and \(H\).

1.7 Fixed point theorems in Pseudocompact Tichonov Space

A topological space \(X\) is said to be pseudocompact iff every real valued continuous function on \(X\) is bounded. It is observed that every compact space is pseudocompact, but the converse is not true (Engleking 1968, Example 5, page 150). However, in a metric space the notion : 'compact' and 'pseudocompact' coincide. A Tichonov space is a completely regular \(T_1\). It is interesting to note that the product of two Tichonov space is again Tichonov space but the product of two pseudocompact space need not be pseudocompact.

In Chapter VI, we establish some fixed point theorems in pseudocompact Tichonov space. These theorems include as a special case of the results of Bohre [8], Fisher [26]. We prove the following theorems:

**Theorem 27.** Let \(P\) be a pseudocompact Tichonov space and \(µ\) be a non-negative real valued continuous
function over $P \times P$ ($P \times P$ is Tichonov; but need not be pseudocompact). Suppose $\mu$ also satisfy

1.7.1 \[ \begin{aligned} \mathcal{M}(x,x) &= 0 \quad \text{for all } x \in P \\
\mathcal{M}(x,y) &\leq \mathcal{M}(x,z) + \mathcal{M}(z,y) \quad \text{for } x, y, z \in P. \end{aligned} \]

If $S$ and $T$ are two continuous self mappings of $P$ satisfying

1.7.2 $ST = TS;$

1.7.3 $[\mathcal{M}(STx, Sy)]^2 \leq \max \left\{ \begin{aligned} \mathcal{M}(Tx, STx) \mathcal{M}(y, Sy), \\
&+ c \mathcal{M}(Tx, Sy) \mathcal{M}(y, STx) \end{aligned} \right\}

for all distinct $x, y \in P,$ where $c \geq 0,$ then $S$ and $T$ have a common fixed point in $P,$ which is unique whenever $c \leq 1.$

Theorem 28. Let $P$ and $\mu$ be the same as defined in theorem 27. Let $S, T : P \to P$ be two continuous maps satisfying

1.7.1 of theorem 27 and

1.7.4 $[\mathcal{M}(STx, Sy)]^2 \leq \frac{1}{2} \left\{ \begin{aligned} \mathcal{M}(Tx, STx) \mathcal{M}(y, Sy) \\
+ \mathcal{M}(Tx, Sy) \mathcal{M}(y, STx) \end{aligned} \right\}

for all distinct $x, y \in P,$ then $S$ and $T$ have a unique common fixed point in $P.$

Theorem 29. Let $P$ and $\mu$ be the same as defined in theorem 27. Let $S, T : P \to P$ be two continuous maps satisfying (1.7.1) of theorem 25 and

1.7.5 $\mathcal{M}(STx, Sy) \leq \frac{\left\{ \begin{aligned} [\mathcal{M}(Tx, STx)]^2 + [\mathcal{M}(y, Sy)]^2 \\
&+ \mathcal{M}(Tx, STx) + \mathcal{M}(y, Sy) \end{aligned} \right\}}{\left\{ \mathcal{M}(Tx, STx) + \mathcal{M}(y, Sy) \right\}}$
for all \(x, y \in P\) for which \(\{\mu(Tx, STx) + \mu(y, Sy)\} \neq 0\),
then \(S\) and \(T\) have a fixed point in \(P\). Further, if

\[\{\mu(Tx, STx) + \mu(y, Sy)\} = 0\]

implies \(\mu(STx, Sy) = 0\), then

the fixed point is unique.

1.8 Fixed point theorems in Hausdorff space

A topological space \(X\) is a Hausdorff space or

\(T_2\)-space iff it satisfies the following axiom:

1.8.1 Each pair of distinct points \(a, b \in X\) belong respectively to disjoint open sets. In other words, there exist open sets \(G\) and \(H\) such that \(a \in G, b \in H\) and \(G \cap H = \emptyset\). Observe that a Hausdorff space is always a \(T_1\)-space.

In 1980, Chatterjee and Ray [13] have established a fixed point theorem in Hausdorff space which generalize the results of Edelstein [22] and Sehgal [80] as follows:

Theorem E1. Let \(T_1\) and \(T_2\) be two self maps of a Hausdorff space \(X\) and let \(F : X \times X \rightarrow [0, \infty)\) be continuous symmetric mapping such that \(F(x, y) = 0\) for \(x = y\) and each pair of distinct \(x, y \in X\) one has

\[F(T_1x, T_2y) \leq \max \left\{ \begin{array}{c} [F(x, y), F(x, T_1x), F(y, T_2y)] \\ \cup \min \left[ F(x, T_2y), F(y, T_1x) \right] \end{array} \right\}

\]

for some \(x_0 \in X\) the sequence \(\{x_n\}\), where \(x_{2n+1} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1}\) has a subsequence converges to a point \(x_1 \in X\). If \(T_1\) and \(T_2T_1\) or \(T_2\) and \(T_1T_2\) are continuous at \(x_1\), then \(x_1\) is a fixed point of \(T_1\) or \(T_2\).
In Chapter VII, we generalize the above theorem to a more general case. In fact we prove the following theorem:

**Theorem 30.** Let $T_1$ and $T_2$ be two continuous mappings of a Hausdorff space $X$ into itself. Let $F$ be a symmetric continuous mapping of $X \times X$ into $\mathbb{R}^+$ satisfying,

1. $F(x, y) \neq 0$ for all $x \neq y$,
2. there is an $h \in H$ such that for all $x, y \in X$,

$$F(T_1 x, T_2 y) \leq h \left( \max \left\{ [F(x, y), F(x, T_1 x), F(y, T_2 y)] \right\} \cup \min \left\{ F(x, T_2 y), F(y, T_1 x) \right\} \right)$$

where $h$ satisfies $h(t, t) < t$ for all $t > 0$. If some $x_0 \in X$, the sequence $\{x_n\}$ where $T_1 x_{2n} = x_{2n+1}$ and $T_2 x_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \ldots$, has a subsequence of the type $\{x_{(2p+1)n}\}$ converges to a point $u$ in $X$, where $p \in \mathbb{N}$ is fixed and $n \in \mathbb{N}$, then $T_1$ and $T_2$ have a common fixed point.

The following is the extension of our own theorem 28:

**Theorem 31.** Let $T_1, T_2, \ldots, T_k$ be the continuous mappings of a Hausdorff space $X$ into itself. Let $F$ be a symmetric continuous mapping of $X \times X$ into $\mathbb{R}^+$ satisfying (1.8.3) of theorem 30 and

1. $F(T_i x, T_{i+1} x) \leq h \left( \max \left\{ [F(x, y), F(x, T_i x), F(y, T_{i+1} y)] \right\} \cup \min \left\{ F(x, T_{i+1} y), F(y, T_i x) \right\} \right)$
where \( h \) satisfies \( h(t, t) < t \) for all \( t > 0 \) and \( T_{k+1} \neq T_k \).

If for some \( x_0 \in X \), the sequence \( \{x_n\} \) defined as:

\[
x_1 = T_1x_0, \quad x_2 = T_2x_1, \quad \ldots \quad x_k = T_kx_{k-1}
\]

\[
x_{k+1} = T_1x_k, \quad x_{k+2} = T_2x_{k+1}, \quad \ldots \quad x_{2k} = T_kx_{2k-1}
\]

\[
x_{n+1} = T_1x_n, \quad x_{n+2} = T_2x_{n+1}, \quad \ldots \quad x_{2n} = T_kx_{2n-1}
\]

for \( n = 0, 1, 2, \ldots \), has a subsequence of the type \( \{ x_{(mk+1)} \} \) converges to a point \( u \) in \( X \), where \( m \in \mathbb{N} \) is fixed and \( n \in \mathbb{N} \), then \( T_1, T_2, \ldots, T_k \) have a common fixed point.

### 1.9 Fixed point theorem in 2-Banach space

By Banach space, we understand that it is a complete normed space. This basic definition is extended to define 2-Banach space.

In 1983, Sharma and Rajput [78] have defined 2-Banach space as follows:

**Definition (1.9.1).** Let \( L \) be a linear space, then a real function on \( L \times L \) is called a 2-norm if \( \| \cdot, \cdot \| \) satisfies the following axioms:

1. \( \|a, b\| = 0 \) exactly when the vectors \( a \) and \( b \) are linearly dependent,
2. \( \|a, b\| = \|b, a\| \)
3. For each real number \( r \), \( \|a, rb\| = r \|a, b\| \)
4. \( \|a, b+c\| \leq \|a, b\| + \|a, c\| \)

Khan and Imdad proved the following theorem which generalizes the theorem of Jungck [39] and Iseki [35]:
Theorem $F_1$. Let $C$ be a closed convex subset of $X$ and let $E : C \rightarrow C$ satisfy the conditions:

\[(1.9.2) \quad E^2 = I, \text{ where } I \text{ denotes the identity map},\]

\[(1.9.3) \quad \|Ex - Ey\| \leq (\frac{\alpha}{2}) \max \left\{\|x - y\|, \frac{1}{2}\|x - Ex\|, \frac{1}{2}\|y - Ey\|\right\},\]

for every $x, y \in C$, where $0 \leq \alpha < 4$, then $E$ has at least one fixed point.

Singh [74] extended Jungck [39] result and proved the following theorem:

Theorem $F_2$. Let $S$ and $T$ be two continuous and commuting self mappings of a complete metric space $(X, d)$ satisfying the following conditions:

\[(1.9.4) \quad S(x) \subseteq T(x),\]

\[(1.9.5) \quad d(Sx, Sy) \leq a d(Tx, Ty) + b \left[d(Tx, Sx) + d(Ty, Sy)\right] + c \left[d(Tx, Sy) + d(Ty, Sx)\right],\]

for all $x, y$ in $X$, where $a, b$ and $c$ are non-negative real numbers satisfying $0 < a + 2b + 2c < 1$, then $S$ and $T$ have a unique common fixed point.

In Chapter VIII, we extend the above theorem $F_1$ and $F_2$ and proved the following:

Theorem 32. Let $E$ be a map of $2$-Banach space $X$ into itself with

\[(1.9.6) \quad E^2 = I, \text{ where } I \text{ denotes the identity map},\]

\[(1.9.7) \quad \|E(x) - E(y), a\| \leq (\frac{\alpha}{2}) \max \left\{\|x - y, a\|, \frac{1}{2}\|x - E(x), a\|, \frac{1}{2}\|y - E(y), a\|, \frac{1}{2}\|x - E(y), a\|, \frac{1}{2}\|y - E(x), a\|\right\},\]

for all $x, y$ and $a$ in $X$, $0 \leq \alpha < 4$, then $E$ has at least one fixed point.
Theorem 33. Let $S$ and $T$ be two continuous and commuting self maps of a 2-Banach space $X$ into itself satisfying the following conditions:

$(1.9.4)$ of theorem $F_2$ and

$(1.9.8)$

$$
\|S(x) - S(y), a\| \leq a_1 \|T(x) - T(y), a\| + a_2 \|T(x) - S(x), a\| + a_3 \|T(y) - S(y), a\| + \|T(y) - S(x), a\|
$$

for all $x, y$ and $a \in X$, where $a_1$, $a_2$ and $a_3$ are non-negative real numbers satisfying $0 < a_1 + 2a_2 + 2a_3 < 1$, then $S$ and $T$ have a unique common fixed point.

1.10 Fixed point theorem in uniform space

In 1978, Mishra [51] proved a fixed point theorem in uniform space for a pair of mappings. He proved the following:

Theorem H$_1$. Let $T_1$ and $T_2$ be two operators on $X$, such that for $V_i \in G (i = 1, 2, \ldots, 5)$ $x, y \in X (x, T_1 x) \in V_1, (y, T_2 y) \in V_2, (x, T_2 y) \in V_3, (y, T_1 x) \in V_4$ and $(x, y) \in V_5$ we have

$(1.10.1)$

$$(T_1 x, T_2 y) \in \preceq_1 V_1 \circ \preceq_2 V_2 \circ \preceq_3 V_3 \circ \preceq_4 V_4 \circ \preceq_5 V_5$$

where $\preceq_1 \geq 0$ $(i = 1, \ldots, 5)$ $\sum_{i=1}^{5} \preceq_1 < 1$

$(1.10.2)$

$\preceq_1 = \preceq_2$ or $\preceq_3 = \preceq_4$, then $T_1$ and $T_2$ have a unique common fixed point.

In Chapter IX, we establish a common fixed point theorem for three maps in uniform space. In a sense our result is a generalization of the result for a pair of mappings obtained by Mishra [51]. We prove the following theorem:
Theorem 34. Let $E$, $F$ and $T$ be three continuous self maps on $X$ such that

(1.10.3) $ET = TE$, $FT = TF$, $E(X) \subseteq T(X)$ and $F(X) \subseteq T(X)$,

(1.10.4) if for any $V_i \in G$ $(i = 1, 2 \cdots 5)$ and $x, y \in X$, $T(x, y) \in V_1$, $(E, T) \in V_2$, $(F, T) \in V_3$, $(E, T) \in V_4$,

(1.10.5) implies $(E, F) \in \alpha_1(x, y) V_1 \circ \alpha_2(x, y) V_2 \circ \alpha_3(x, y) V_3 \circ \alpha_4(x, y) V_4 \circ \alpha_5(x, y) V_5$

for some non-negative functions $\alpha_i = \alpha_i(x, y); i=1, 2 \cdots 5$

satisfying

$$\sup \left\{ \alpha_1(x, y) + \alpha_2(x, y) + \alpha_3(x, y) + \alpha_4(x, y) + \alpha_5(x, y) \right\} = \lambda < 1,$$

then $E$, $F$ and $T$ have a unique common fixed point.