PART - I
FIXED POINT THEOREMS


CHAPTER II

COMPATIBLE MAPPINGS OF TYPE (A-1) AND TYPE (A-2) IN METRIC SPACES AND COMMON FIXED POINTS

2.1 In [89] Jungck introduced the concept of compatible mappings and recently Jungck et al. [92] introduced the concept of compatible mappings of type (A) which is equivalent to compatible mappings under certain conditions, and proved a common fixed point theorem for compatible mappings of type (A) in a metric space. Since then many fixed point theorems have been proved for compatible mappings of type (A) ([31], [124] - [125]).

In this chapter we introduce the concept of compatible mappings of type (A-1) and type (A-2) and show that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions. In the sequel we prove some common fixed point theorem for compatible mappings of type (A-1) and type (A-2) which generalises the results of Kang and Kim [94], Fisher [53], Jungck [90], Khan and Imdad [105], by employing compatible mappings of type (A-1) or type (A-2) instead of commuting mappings and compatible mappings, and also by removing the continuity requirement of the mappings. We have also given an example to illustrate the generality of our theorem and have applied our result to study the existence and uniqueness problems of common solutions for a class of functional equations arising in dynamic programming.

2.2 In this section we show that compatible mappings of type (A-1) and type (A-2) are equivalent to compatible as well as compatible mappings of type (A) under certain conditions.

Definition 2.1[89]: Let S and T be self maps of a metric space (X, d). The mappings S and T are said to be compatible if \(\lim_{n \to \infty} d(STx_n, TSx_n) = 0\), whenever \(\{x_n\}\) is a sequence in X such that

\[\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\] for some t in X

Definition 2.2[92]: Let S and T be self maps of a metric space (X, d). The mappings S and T are said to be compatible of type (A) if

\[\lim_{n \to \infty} d(STx_n, TTx_n) = 0\] and \(\lim_{n \to \infty} d(TSx_n, SSx_n) = 0\)

whenever \(\{x_n\}\) is a sequence in X such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\) for some t in X.

Definition 2.3: Let S and T be self maps of a metric space (X, d). The pair of mappings (S, T) is said to be compatible of type (A-1) if \(\lim_{n \to \infty} d(STx_n, TTx_n) = 0\), whenever
\( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

**Definition 2.1**: Let \( S \) and \( T \) be self-maps of a metric space \((X,d)\). The pair of mappings \((S,T)\) is said to be compatible of type \((A-2)\) if
\[
\lim_{n \to \infty} d(TSx_n, SSx_n) = 0,
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \in X.
\]

Clearly, if a pair of mappings \((S,T)\) is compatible of type \((A-1)\) then the pair \((T,S)\) is compatible of type \((A-2)\). Further from the definitions it's clear that if \( S \) and \( T \) are compatible mappings of type \((A)\) then the pair \((S,T)\) is compatible of type \((A-1)\) as well as type \((A-2)\). The following example illustrates that the implication is not reversible.

**Example 2.1**: Let \( X = \mathbb{R} \), with usual metric \( d \). Consider the mappings \( S \) and \( T \) defined by
\[
Sx = \begin{cases} 
1 & \text{if } x \leq 1 \\
1 + x & \text{if } x > 1
\end{cases}
\]
\[
Tx = \begin{cases} 
1 & \text{if } x = 1 \\
1 & \text{if } x > 1 \\
0 & \text{if } x < 1
\end{cases}
\]

Consider the \( \{x_n\} \) defined by \( x_n = \frac{1}{n} \) for all \( n \). We see that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = 1 \). \( \lim_{n \to \infty} d(STx_n, TSx_n) = 1 \), \( \lim_{n \to \infty} d(STx_n, TTx_n) = 1/2 \), \( \lim_{n \to \infty} d(TSx_n, SSx_n) = 0 \). Hence the pair of mappings \((S,T)\) is compatible of type \((A-2)\) but not of type \((A-1)\). Moreover the mappings \( S \) and \( T \) are neither compatible nor compatible of type \((A)\).

We now cite the following propositions which gives the condition under which definitions 2.1, 2.2, 2.3, and 2.4 become equivalent.

**Proposition 2.1** : Let \( S \) and \( T \) be self-maps of a metric space \((X,d)\).

a) If \( T \) is continuous then the pair of mappings \((S,T)\) is compatible of type \((A-1)\) iff \( S \) and \( T \) are compatible.

b) If \( S \) is continuous then the pair of mappings \((S,T)\) is compatible of type \((A-2)\) iff \( S \) and \( T \) are compatible.

c) If \( S \) and \( T \) are continuous then the pair \((S,T)\) is compatible of type \((A-1)\) iff of type \((A-2)\).

**Proof**: a) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \), and let the pair \((S,T)\) be compatible of type \((A-1)\). Since \( T \) is continuous we have \( \lim_{n \to \infty} TSx_n = Tt \) and \( \lim_{n \to \infty} TTx_n = Tt \). Hence \( d(STx_n, TSx_n) \leq d(STx_n, TTx_n) + d(TTx_n, TSx_n) \). Hence \( \lim_{n \to \infty} d(TSx_n, TSx_n) = 0 \). Now let \( S \) and \( T \) be compatible. Then we have
\[ d(ST_n, TT_n) = d(ST_n, TS_n) + d(TS_n, TT_n) \]
Hence \( \lim_n d(ST_n, TT_n) = 0 \)

b) Let \( \lim_n S_n = \lim_n T_n = t \) for some \( t \in X \), and let the pair \((S, T)\) be compatible of type (A-2). Since \( S \) is continuous we have \( \lim_n ST_n = St \) and \( \lim_n SS_n = St \). Hence \( d(ST_n, TS_n) + d(TS_n, ST_n) + d(SS_n, TS_n) \). Hence \( \lim_n d(ST_n, TS_n) = 0 \) Now let \( S \) and \( T \) be compatible. Then we have \( d(TS_n, SS_n) \leq d(TS_n, ST_n) + d(ST_n, SS_n) \). Hence \( \lim_n d(TS_n, SS_n) = 0 \)

c) Let \( \lim_n S_n = \lim_n T_n = t \) for some \( t \in X \), and let the pair \((S, T)\) be compatible of type (A-1). We have \( d(TS_n, SS_n) - d(TS_n, ST_n) + d(ST_n, TT_n) + d(TT_n, SS_n) \). Since \( S \) and \( T \) are continuous we see that

\[ \lim_n SS_n = \lim_n ST_n \text{ and } \lim_n TS_n = \lim_n TT_n. \]
Hence \( \lim_n d(TS_n, SS_n) = 0 \) Therefore the pair \((S, T)\) is compatible of type (A-2). Next suppose the pair \((S, T)\) is compatible of type (A-2). We have

\[ d(ST_n, TT_n) \leq d(ST_n, TS_n) + d(TS_n, SS_n) + d(SS_n, TT_n). \]
Again using continuity of \( S \) and \( T \) we get \( \lim_n d(ST_n, TT_n) = 0 \). Therefore the pair \((S, T)\) is compatible of type (A-1). As a direct consequence of proposition 2.1 we have the following

**Proposition 2.2.** Let \( S \) and \( T \) be self maps of a metric space \((X, d)\). If \( S \) and \( T \) are continuous then the following statements are equivalent.

a. The pair \((S, T)\) is compatible of type (A-1).

b. The pair \((S, T)\) is compatible of type (A-2).

c. The mappings \( S \) and \( T \) are compatible of type (A).

d. The mappings \( S \) and \( T \) are compatible

Next we give some properties of compatible mappings of type (A-1) and type (A-2) which will be used in our main theorem

**Proposition 2.3.** Let \( S \) and \( T \) be self maps of a metric space \((X, d)\). If the pair \((S, T)\) are compatible of type (A-1) and \( Sz = Tz \) for some \( z \) in \( X \) then \( STz = TTz \).

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) defined by \( x_n = z \) for \( n = 1, 2, \ldots \) and let \( Tz = Sz \). Then we have \( \lim_n Sx_n = Sz \) \( \lim_n Tx_n = Tz \). Since the pair \((S, T)\) is compatible of type (A-1) we have \( d(STz, TTz) = \lim_n d(ST_n, TT_n) = 0 \). Hence \( STz = TTz \).

**Proposition 2.4.** Let \( S \) and \( T \) be self maps of a metric space \((X, d)\). If the pair \((S, T)\) is compatible of type (A-2) and \( Sz = Tz \) for some \( z \) in \( X \) then \( TSz = SSz \).

**Proof.** Proof follows similarly as above. In this section we prove some coincidence
point theorems and common fixed point theorems for compatible mappings of type (A-1) or type (A-2) which improves many known results.

We will require the following lemma for our main results.

Lemma 2.1. ([]). For any \( t > 0 \), \( \phi(t) = t \) if and only if \( \lim_{n \to \infty} \phi^n(t) = 0 \), where \( \phi^n \) denotes the \( n \)-times composition of \( \phi \) with itself.

Let \( A, B, S, T \) be mappings from a metric space \((X, d)\) into itself such that

\[
A(X) \cup B(X) \subseteq S(X) \cup T(X)
\]

(2.3.1) \((A(X) \cup B(X)) \subseteq S(X) \cup T(X)\) is a complete subspace of \( X \)

(2.3.3) \([1 + p \{ d(Ax, Sx) + d(By, Ty) \}] d(Ax, By) \leq p \{ d^2(Ax, Sx) + d^2(By, Ty) \} + h \max \{ d(Ax, Sx), d(By, Ty) \}, 1/2 \{ d(Ax, Ty) + d(By, Sx) \}, d(Sx, Ty) \]

for all \( x, y \) in \( X \), \( 0 \leq h < 1 \) and \( p \geq 0 \). For some arbitrary \( x_n \) in \( X \), by

(2.3.1) we choose \( x_1 \) in \( X \) such that \( Ax_1 = Tx_1 \), and for this \( x_1 \) there exists \( x_2 \) such that

\[
Sx_2 = Bx_1
\]

Continuing this process we define the sequence \( \{ x_n \} \) in \( X \) such that (2.3.4) \( x_{2n} = Ax_{2n} = Tx_{2n+1} \) and \( x_{2n+1} = Bx_{2n+1} = Sx_{2n} \).

Lemma 2.2. Let \( A, B, S, T \) be mappings from a metric space \((X, d)\) into itself satisfying (2.3.1) and (2.3.3). Then \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \), where \( \{ x_n \} \) is the sequence defined by (2.3.4).

Proof. By (2.3.3) we have,

\[
[1 + p \{ d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1}) \}] d(Ax_{2n}, Bx_{2n+1}) \\
\leq p \{ [d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1})] + h \max \{ d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}) \}, 1/2 \{ d(Ax_{2n}, Ty_{2n+1}) + d(By_{2n+1}, Sx_{2n}) \}, d(Sx_{2n}, Tx_{2n+1}) \}
\]

or equivalently we have

\[
[1 + p \{ d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) \}] d(y_{2n}, y_{2n+1}) \\
\leq p \{ [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] + h \max \{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), 1/2 \{ d(y_{2n+1}, y_{2n+2}) \}, d(Sx_{2n}, Tx_{2n+1}) \}
\]

If \( d(y_{2n}, y_{2n+1}) > d(y_{2n+1}, y_{2n+2}) \) in the above inequality then we get \( d(y_{2n}, y_{2n+1}) \leq h \cdot d(y_{2n}, y_{2n+1}) \), a contradiction.

Hence \( d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, y_{2n+1}) \). Hence we get \( d(y_{2n}, y_{2n+1}) \leq h \cdot d(y_{2n}, y_{2n+1}) \) it follows that \( d(y_n, y_{n+1}) \leq h \cdot d(y_n, y_{n+1}) \). This completes the proof.
Lemma 2.3. Let $A$, $B$, $S$ and $T$ be from metric space $(X,d)$ into itself satisfying (2.3.1) and (2.3.3). Then the sequence $\{y_n\}$ defined by (2.3.4) is a Cauchy sequence in $X$.

Proof. In virtue of lemma 2.2 it is sufficient if we can show that a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ is a Cauchy sequence in $X$. Suppose $\{y_n\}$ is not a Cauchy sequence in $X$. Then there exists an $\epsilon > 0$ such that for each even integer $2k$ there exists even integers $2nk$ and $2nk'$ with $2nk' > 2nk > 2k$ such that

$$d(y_{2nk}, y_{2nk'}) > \epsilon.$$ \hfill (2.3.5)

For each even integer $2k$ let $2nk'$ be the least even integer exceeding $2nk$ satisfying (2.3.5), i.e.

$$d(y_{2nk}, y_{2nk'}) > \epsilon \quad \text{and} \quad d(y_{2nk'}, y_{2nk}) > \epsilon.$$ \hfill (2.3.6)

Then for each even integer $2k$, we have

$$r < d(y_{2nk}, y_{2nk'}) \leq d(y_{2nk}, y_{2nk''}) + d(y_{2nk''}, y_{2nk'}) + d(y_{2nk'}, y_{2nk}).$$

Hence from lemma 2.2 and (2.3.6) it follows that

$$d(y_{2nk}, y_{2nk'}) --> \epsilon \quad \text{as} \quad k --> \infty.$$ \hfill (2.3.7)

By triangle inequality we have

$$\left| d(y_{2nk}, y_{2nk'}) - d(y_{2nk}, y_{2nk''}) \right| \leq d(y_{2nk}, y_{2nk''})$$

and

$$\left| d(y_{2nk}, y_{2nk'}) - d(y_{2nk}, y_{2nk}) \right| \leq d(y_{2nk}, y_{2nk''}) + d(y_{2nk}, y_{2nk})$$

From lemma 3.1 and (2.3.7) we see that as $k --> \infty$.

$$d(y_{2nk}, y_{2nk'}) --> \epsilon \quad \text{and} \quad d(y_{2nk}, y_{2nk'}) --> \epsilon.$$ \hfill (2.3.8)

By (2.3.3) we have,

$$[1 + p \{ d(Ax_{2nk} Sy_{2nk}) + d(Bx_{2nk} Ty_{2nk}) \} ] d(As_{2nk}, Bx_{2nk})$$

$$\leq p \{ d(Ax_{2nk} Sy_{2nk}) + d(Bx_{2nk} Ty_{2nk}) \} \max \{ d(As_{2nk}, Bx_{2nk}), d(Bx_{2nk} Ty_{2nk}) \}. $$

By using (2.3.4) we get

$$[1 + p \{ d(y_{2nk}, y_{2nk'}) + d(y_{2nk'}, y_{2nk''}) \} ] d(y_{2nk}, y_{2nk'})$$

$$\leq p \{ d(y_{2nk}, y_{2nk'}) + d(y_{2nk'}, y_{2nk''}) \} \max \{ d(y_{2nk}, y_{2nk'}), d(y_{2nk'}, y_{2nk''}) \}, d(y_{2nk}, y_{2nk'})$$

$$1/2 \{ d(y_{2nk}, y_{2nk'}) + d(y_{2nk}, y_{2nk'}) \}, d(y_{2nk}, y_{2nk'})$$

By using (2.3.7) and (2.3.8) and lemma 2.2 we see that as $k --> \infty$ the above inequality reduces to.
(2.3.10) \( d(y_{n}, y_{m}) \leq h \varepsilon \)

Also we have, \( d(y_{n}, y_{m}) \leq d(y_{n}, y_{n+1}) + d(y_{n+1}, y_{m}) \)

Hence from Lemma 2.2, (2.3.7), (2.3.8), and (2.3.10), we see that as \( n \to \infty \), the above inequality reduces to \( \varepsilon \leq h \varepsilon \), which is a contradiction. Hence \( \{y_{n}\} \) is a Cauchy sequence in \( X \).

**Theorem 2.1.** Let \( A, B, S \) and \( T \) be from metric space \( (X, d) \) into itself satisfying (2.3.1), (2.3.2), and (2.3.3). Then

a) \( A \) and \( S \) have a coincidence point in \( X \)

b) \( B \) and \( T \) have a coincidence point in \( X \).

**Proof.** By Lemma 2.3 sequence \( \{y_{n}\} \) defined by (2.3.4) is a Cauchy sequence in \( S(X) \cap T(X) \). Since \( S(X) \cap T(X) \) is a complete subspace of \( X \), the sequence \( \{y_{n}\} \) must converge to some point say \( w \) in \( S(X) \cap T(X) \). On the other hand since subsequences \( \{y_{n}\} \) and \( \{y_{m}\} \) are also Cauchy sequences in \( S(X) \cap T(X) \), they should also converge to the same point \( w \) in \( X \). Hence there should exist two points \( u \) and \( v \) in \( X \) such that \( Su = w \) and \( Tv = w \). Using (2.3.3) we get,

\[
[1 + p \{d(Au, Su) + d(Bx_{2n+1}, Tx_{2n+1})\}] d(Ax, Bw) \\
\leq p [d'(Au, Su) + d'(Bx_{2n+1}, Tx_{2n+1})] + h \max\{d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1})\} \\
1/2 \{d(Ax, Bw) + d(Bx_{2n+1}, Su)\} + d(Su, Tw)
\]

as \( n \to \infty \) we get

\[
[1 + p \{d(Au, w)\}] d(Au, w) \leq p d'(Au, w) + h \max\{d(Au, w), 0\}, 0 \}
\]

i.e. \( d(Au, w) \leq h d(Au, w) \) a contradiction. Therefore \( Au = w \).

Hence \( Su = Au = w \). Similarly it can be shown that \( Bv = Tv = w \).

**Theorem 2.2.** Let \( A, B, S \) and \( T \) be from metric space \( (X, d) \) into itself satisfying (2.3.1), (2.3.2), (2.3.3), and (2.3.11).

(2.3.11) The pairs \( (A, S) \) and \( (B, T) \) are compatible of type (A-1) or type (A-2).

Then \( A, B, S \), and \( T \) have a unique common fixed point.

**Proof.** By Theorem 2.1, \( Au = Su = w \) and \( Bv = Tv = w \). If the pairs \( (A, S) \) and \( (B, T) \) are compatible of type (A-1) then by proposition (2.3) we get \( ASu = SSu \) and \( BTv = TTv \), i.e. \( Aw = Sw \) and \( Bw = Tw \). If the pairs \( (A, S) \) and \( (B, T) \) are compatible of type (A-2) then by proposition (2.4) we get \( SAu = AAu \) and \( TBv = BBv \), i.e. \( Sw = Aw \) and \( Tw = Bw \). i.e. in both the cases we get \( Aw = Sw \) and \( Bw = Tw \). Using (2.3.3) we get

\[
[1 + p \{d(Ax_{n}, Sx_{n}) + d(Bw, Tw)\}] d(Ax_{n}, Bw) \\
\leq p [d'(Ax_{n}, Sx_{n}) + d'(Bw, Tw)] + h \max\{d(Ax_{n}, Sx_{n}), d(Bw, Tw)\} \\
1/2 \{d(Ax_{n}, Tw) + d(Bw, Sx_{n})\} + d(Sx_{n}, Tw)
\]
as $n \to \infty$ we get $d(w,Bw) = h d(w,Bw)$ a contradiction. Hence $Bw \neq w$. Again by (2.13), we have

$$[1 + p \cdot d(Aw,Sw) + d(Bw,Tw)] d(Aw,Bw)$$

$$\leq p [d(Aw,Sw) + d(Bw,Tw)] + h \max \{d(Aw,Sw), d(Bw,Tw),$$

$$1/2 \cdot [d(Ax,Tx) + d(Bx,Sx)] + d(Sx,Tx)]$$

i.e. $d(Aw,w) \leq h d(Aw,w)$, a contradiction.

Hence $Aw = w$. Therefore $w$ is common fixed point of $A,B,S$ and $T$. Taking $p = 0$ in (2.3.3) we get the following.

**Corollary 2.3.** Let $A,B,S$ and $T$ be from metric space $(X,d)$ into itself satisfying (2.3.1), (2.3.2), (2.3.11) and (2.3.12)

(2.3.12) $d(Ax,By) \leq h \max \{d(Ax,Sx), d(By,Ty), 1/2 \cdot [d(Ax,Ty) + d(By,Sx)] + d(Sx,Ty)]$

for all $x, y \in X$, and $0 \leq h < 1$. Then $A,B,S$, and $T$ have a unique common fixed point.

**Remark 1.** Corollary 2.3 generalizes the result of G Jungck [90] by removing the continuity requirement of the mappings.

**Remark 2.** Theorem 2.2 and corollary 2.3 generalizes the results of B. Fisher [53] by employing compatible mappings of type (A-1) or type (A-2) instead of commuting mappings.

**Remark 3.** The condition (2.3.3) is more general than the condition of Kang and Kim [94].

The following example shows that our result is more general than that of G. Jungck [90], B. Fisher [53] and Kang and Kim [94].

**Example 2.2.** Let $X = [-1,2]$ with usual metric $d$ Consider the mappings $A,B,S$ and $T$ in $X$ defined by

$$Ax = Bx = \begin{cases} 1 & \text{if } -1 \leq x < 1 \\ 1 & \text{if } x = 1 \\ 3/4 & \text{if } 1 < x < 5/4 \\ 1 + x^2/32 & \text{if } 5/4 \leq x \leq 2 \\ 1 + x^2/4 & \text{if } -1 \leq x < -1 \\ 1 & \text{if } x = 1 \\ 2 & \text{if } 1 < x < 5/4 \\ 1 - x^2/8 & \text{if } 5/4 \leq x \leq 2 \end{cases}$$

$$Sx = Tx = \begin{cases} 1 & \text{if } -1 \leq x < 1 \\ 1 & \text{if } x = 1 \\ 3/4 & \text{if } 1 < x < 5/4 \\ 1 + x^2/32 & \text{if } 5/4 \leq x \leq 2 \\ 1 + x^2/4 & \text{if } -1 \leq x < -1 \\ 1 & \text{if } x = 1 \\ 2 & \text{if } 1 < x < 5/4 \\ 1 - x^2/8 & \text{if } 5/4 \leq x \leq 2 \end{cases}$$
Clearly the mappings \( A, B, S \) and \( T \) are not continuous. Considering the sequence \( \{x_n\} \) defined by \( x_n = \frac{1}{n} \), it can be easily verified that the pairs \((A, S)\) and \((B, T)\) are neither compatible nor compatible of type \((A)\), but they are compatible of type \((A-2)\).

Also \( A(X) \cup B(X) \cup S(X) \cup T(X) \) being the union of finite number of closed sets is closed and hence complete. It can also be easily verified that for all \( x, y \) in \( X \), \( d(Ax, By) \leq h \cdot d(Sx, Ty) \) where \( h = 1/2 \). Therefore the mappings satisfies all conditions of Theorem 2.2. Clearly 1 is common fixed point of \( A, B, S \) and \( T \).

**Lemma 2.4.** Let \( A, B, S \) and \( T \) be from metric space \((X, d)\) into itself satisfying (2.3.1) and

\[
(2.3.13) [1 + p \cdot \{d(Ax, Sx) + d(By, Ty)\}] \cdot d(Ax, By) \\
\leq p \cdot \{d(Ax, Sx) + d(By, Ty)\} + \phi(\max[d(Ax, Sx), d(By, Ty)] , \\
1/2 \cdot \{d(Ax, Ty) + d(By, Sx)\} , d(Sx, Ty))
\]

for all \( x, y \) in \( X \), where \( \phi : [0, \infty) \to [0, \infty) \) is a nondecreasing and upper semicontinuous function and \( \phi(t) < t \) for all \( t > 0 \).

Then \( \lim_{n \to \infty} d(y_n, y_{n+1}) = 0 \), where \( \{y_n\} \) is the sequence defined by (2.3.4).

**Proof** By (2.3.13) we have,

\[
[1 + p \cdot \{d(Ax^m, Sx^m) + d(By^m, Ty^m)\}] \cdot d(Ax^m, By^m) \\
\leq p \cdot \{d(Ax^m, Sx^m) + d(By^m, Ty^m)\} + \phi(\max[d(Ax^m, Sx^m), d(By^m, Ty^m)] , \\
1/2 \cdot \{d(Ax^m, Ty^m) + d(By^m, Sx^m)\} , d(Sx^m, Ty^m))
\]

or equivalently we have

\[
[1 + p \cdot \{d(y^m_n, y^m_n) + d(y^m_{n+1}, y^m_{n+1})\}] \cdot d(y^m_n, y^m_{n+1}) \\
\leq p \cdot \{d(y^m_n, y^m_n) + d(y^m_{n+1}, y^m_{n+1})\} + \phi(\max[d(y^m_n, y^m_n), d(y^m_{n+1}, y^m_{n+1})] , \\
1/2 \cdot \{d(y^m_n, y^m_{n+1}) + d(y^m_{n+1}, y^m_n)\} , d(y^m_{n+1}, y^m_n))
\]

If \( d(y^m_n, y^m_{n+1}) \geq d(y^m_n, y^m_{n+1}) \) in the above inequality then we get

\[d(y^m_n, y^m_{n+1}) \leq \phi(d(y^m_n, y^m_{n+1})) \), a contradiction. Hence \( d(y^m_n, y^m_{n+1}) \leq d(y^m_n, y^m_{n+1})\).

Hence we get \( d(y^m_n, y^m_{n+1}) \leq \phi(\max[d(y^m_n, y^m_n), d(y^m_{n+1}, y^m_{n+1})]) \) it follows that \( d(y_n, y_{n+1}) \leq \phi(\max[d(y_n, y_n), d(y_{n+1}, y_{n+1})]) \).

Hence in view of lemma 2.1 the proof is complete.

**Lemma 2.5.** Let \( A, B, S \) and \( T \) be from metric space \((X, d)\) into itself satisfying (2.3.1) and (2.3.13). Then the sequence \( \{y_n\} \) defined by (2.3.4) is a Cauchy sequence in \( X \).

**Proof** In virtue of lemma 2.4 it is sufficient if we can show that a subsequence \( \{y^m_{2k}\} \)

of \( \{y_n\} \) is a cauchy sequence in \( X \). Suppose \( \{y^m_{2k}\} \) is not a cauchy sequence in \( X \). Then there exists \( \epsilon > 0 \) such that for each even integer \( 2k \) there exists even integers \( 2m \) and \( 2nk \) with \( 2nk > 2nk \) such that
For each even integer 2k let 2nk be the least even integer exceeding 2nk satisfying (2.3.14), i.e.

(2.3.15) \[ d(y_{2k}, y_{2k}) \leq \varepsilon \text{ and } d(y_{2k}, y_{2k}) \leq \varepsilon. \]

Then for each even integer 2k, we have

(2.3.16) \[ d(y_{2k}, y_{2k}) \leq d(y_{2k}, y_{2k}) + d(y_{2k}, y_{2k}) + d(y_{2k}, y_{2k}) \]

Hence from lemma 2.4 and (2.3.15) it follows that

(2.3.17) \[ d(y_{2k+1}, y_{2k+1}) \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \]

By triangle inequality we have \[ |d(y_{2k}, y_{2k}) - d(y_{2k+1}, y_{2k+1})| \leq d(y_{2k+1}, y_{2k+1}) \]
and

(2.3.18) \[ d(y_{2k+1}, y_{2k+1}) \rightarrow \varepsilon \text{ and } d(y_{2k+1}, y_{2k+1}) \rightarrow \varepsilon. \]

Hence from (2.3.13) we get

\[ [1 + p] \{d(Ax_{2k}, Sx_{2k}) + d(Bx_{2k}, Tx_{2k})\} \leq p \{d(Ax_{2k}, Sx_{2k}) + d(Bx_{2k}, Tx_{2k})\} + \phi(\max\{d(Ax_{2k}, Sx_{2k}), d(Bx_{2k}, Tx_{2k})\}), \]

By using (2.4.4) we get

\[ [1 + p] \{d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+1})\} \leq p \{d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+1})\} + \phi(\max\{d(y_{2k}, y_{2k+1}), d(y_{2k+1}, y_{2k+1})\}, d(y_{2k+1}, y_{2k+1})) \]

By using (2.3.16), (2.3.17) and lemma 2.4, we see that as \( k \rightarrow \infty \), the above inequality reduces to

(2.3.18) \[ d(y_{2k+1}, y_{2k+1}) \leq \phi(\varepsilon) \]

Also we have \( d(y_{2k+1}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k+1}) + d(y_{2k+1}, y_{2k+1}) \)

Hence from (2.3.16), (2.3.18) and lemma 2.4 we see that as \( k \rightarrow \infty \), the above inequality reduces to \( \varepsilon \leq \phi(\varepsilon) \), which is a contradiction. Hence \( \{y_{2k}\} \) is a cauchy sequence in \( X \).

**Theorem 2.4.** Let \( A, B, S \) and \( T \) be self-mapping of a complete metric space \( (X, d) \) into itself satisfying (2.3.1), (2.3.2) and (2.3.13). Then
a) \( A \) and \( S \) have a coincidence point in \( X \)
b) \( B \) and \( T \) have a coincidence point in \( X \)
Proof. By Lemma 2.5 sequence \( \{ y_n \} \) defined by (2.1.4) is a Cauchy sequence in \( S(X) \otimes T(X) \). Since \( S(X) \otimes T(X) \) is a complete subspace of \( X \), the sequence \( \{ y_n \} \) must converge to some point say \( w \) in \( S(X) \otimes T(X) \). On the other hand since subsequences \( \{ y_{n_k} \} \) and \( \{ y_{m_l} \} \) are also Cauchy sequences in \( S(X) \otimes T(X) \), they should also converge to the same point \( w \) in \( X \). Hence there should exist two points \( u \) and \( v \) in \( X \) such that \( Su = w \) and \( Tv = w \). Using (2.3.1) we get

\[
[1 + p \cdot \{ (d(Au, Su) + d(Bx_{n_1}, Tx_{n_1})) \} \} d(Au, Bx_{n_1})]
\leq p \cdot [d'(Au, Su) + d'(Bx_{n_1}, Tx_{n_1})] + \phi(\max\{d(Au, Su), d(Bx_{n_1}, Tx_{n_1})\),
1/2 \{d(Au, Tx_{n_1}) + d(Bx_{n_1}, Su)\}, d(Su, Tx_{n_1})]
\]

as \( n \to \infty \), we get

\[
[1 + p \cdot \{ d(Au, w) \} \} d(Au, w)
\leq p \cdot d'(Au, w) + \phi(\max\{d(Au, w), 0\}, 1/2 \{d(Au, w) + 0\} \cdot 0) \]
\]

i.e. \( d(Au, w) \leq \phi(d(Au, w)) \) a contradiction. Therefore \( A w = w \) Hence \( Su = Au = w \).

Similarly it can be shown that \( Bw = w \).

**Theorem 2.5.** Let \( A, B, S \) and \( T \) be four metric space \( (X, d) \) into itself satisfying (2.3.1), (2.3.2), (2.3.11) and (2.3.13). Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** By Theorem 2.4, \( Au = Su = w \) and \( By = Tv = w \). If the pairs \( (A, S) \) and \( (B, T) \) are compatible of type (A-1) then by proposition (2.3) we get \( ASu = SSu \) and \( BTv = TTv \), i.e. \( Aw = Sw \) and \( Bw = Tw \). If the pairs \( (A, S) \) and \( (B, T) \) are compatible of type (A-2) then by proposition (2.3) we get \( SAu = AAw \) and \( TBv = BBv \), i.e. \( Sw = Aw \) and \( Tw = Bw \). i.e. in both the cases we get, \( Aw = Sw \) and \( Bw = Tw \).

Using (2.3.13) we get

\[
[1 + p \cdot \{ d(Ax_{n_1}, Sw_{n_1}) + d(Bw, Tw) \} \} d(Ax_{n_1}, Bw)
\leq p \cdot [d'(Ax_{n_1}, Sw_{n_1}) + d'(Bw, Tw)] + \phi(\max\{d(Ax_{n_1}, Sw_{n_1}), d(Bw, Tw)\),
1/2 \{d(Ax_{n_1}, Tw) + d(Bw, Sw_{n_1})\}, d(Sw_{n_1}, Tw))]
\]

as \( n \to \infty \), we get, \( d(w, Bw) \leq \phi(d(w, Bw)) \) a contradiction. Hence \( Bw = w \). Again by (2.3.13), we have

\[
[1 + p \cdot \{ d(Aw, Sw) + d(Bw, Tw) \} \} d(Aw, Bw)
\leq p \cdot [d'(Aw, Sw) + d'(Bw, Tw)] + \phi(\max\{d(Aw, Sw), d(Bw, Tw)\),
1/2 \{d(Ax_{n_1}, Sw) + d(Bw, Sw_{n_1})\}, d(Sw_{n_1}, Tw))]
\]

i.e. \( d(Aw, w) \leq \phi(d(Aw, w)) \), a contradiction. Hence \( Aw = w \). Therefore \( w \) is common fixed point of \( A, B, S \) and \( T \). This completes the proof.

Taking \( p = 0 \) in (2.3.13) we get the following.

**Corollary 2.6.** Let \( A, B, S \) and \( T \) be four metric space \( (X, d) \) into itself satisfying (2.3.1), (2.3.2), (2.3.11) and
for all \( x, y \) in \( X \), where \( \phi : [0, \infty) \to [0, \infty) \) is a nondecreasing and upper semicontinuous function and \( \phi(t) < t \) for all \( t > 0 \). Then \( A, B, S, t \) and \( T \) have a unique common fixed point.

2.4 In this section we establish a common fixed point theorem for five maps in a metric space which extends the result of Chatterjee and Singh [29].

Our result is as follows.

**Theorem 2.7** Let \((X, d)\) be a complete metric space and let \( T_i : X \to X, i = 1, 2, 3, 4, 5 \) satisfy the following conditions

\[
(2.4.1) \quad d(T_1x, T_2x, T_3x, T_4x, T_5x) \leq \alpha_1 d(T_1x, T_2x)^2 + \alpha_2 d(T_1x, T_3x) d(T_2x, T_4x) + \alpha_3 d(T_1x, T_4x) d(T_2x, T_5x) + \alpha_4 d(T_1x, T_5x) d(T_2x, T_3x)
\]

for all \( x, y \in X \), where \( \alpha_1 \geq 0, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1 \) and \( \alpha_1 + \alpha_4 < 1 \). Further assume that \( T_1T_2 = T_2T_1, T_3T_4 = T_4T_3, T_5T_1 = T_1T_5, T_1T_3 = T_3T_1, T_1T_4 = T_4T_1, T_1T_5 = T_5T_1 \) and \( T_iT_j = T_jT_i \) for \( i = 1, 2, 3, 4, 5 \). If \( T_i \) is continuous in \( X \) then \( T_i, i = 1, 2, 3, 4, 5 \) have a unique common fixed point in \( X \).

**Proof.** Consider the sequence \( \{y_n\} \) such that for an arbitrary \( x_0 \in X \)

\[ T_1T_2x_{2n} = T_3x_{2n} = y_{2n} \quad \text{and} \quad T_4T_5x_{2n} = T_1x_{2n+1} = y_{2n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots \]

Now,
\[
d(y_{2n+1}, y_{2n})^2 = d(T_1x_{2n+1}, T_2x_{2n+1})^2 = d(T_1x_{2n}, T_2x_{2n})^2 = \alpha_1 d(T_1x_{2n}, T_2x_{2n})^2 + \alpha_2 d(T_1x_{2n}, T_3x_{2n}) d(T_2x_{2n}, T_4x_{2n}) + \alpha_3 d(T_1x_{2n}, T_4x_{2n}) d(T_2x_{2n}, T_5x_{2n}) + \alpha_4 d(T_1x_{2n}, T_5x_{2n}) d(T_2x_{2n}, T_3x_{2n})
\]

\[
= \alpha_1 d(y_{2n}, y_{2n})^2 + \alpha_2 d(y_{2n}, y_{2n}) d(y_{2n+1}, y_{2n+1}) + \alpha_3 d(y_{2n+1}, y_{2n+1}) d(y_{2n}, y_{2n}) + \alpha_4 d(y_{2n+1}, y_{2n+1}) d(y_{2n}, y_{2n})
\]

\[
= \alpha_1 d(y_{2n}, y_{2n+1})^2 + \alpha_2 d(y_{2n}, y_{2n+1})^2 + (\alpha_2/2 + \alpha_4/2) \{d(y_{2n}, y_{2n+1})^2 + d(y_{2n}, y_{2n})^2\}.
\]

Therefore
\[
d(y_{2n+1}, y_{2n}) \leq \left[\frac{\alpha_1 + \alpha_2 + (\alpha_2/2) + (\alpha_4/2)}{1 - (\alpha_2/2 - (\alpha_4/2))}\right]^{1/2} d(y_{2n}, y_{2n+1}).
\]

Since,
\[
\frac{\alpha_1 + \alpha_2 + (\alpha_2/2) + (\alpha_4/2)}{1 - (\alpha_2/2 - (\alpha_4/2))} < 1
\]

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the sequence \( \langle y_n \rangle \) is a Cauchy sequence, in the Complete Metric space \( X \) and hence \( \langle y_n \rangle \) converges to a point say \( y \) in \( X \).

Consider, \( d(Ty, Ty_Ty) \leq (d(Ty, Ty_Ty_Ty) + d(Ty_Ty, Ty_Ty_Ty) + d(Ty_Ty_Ty, Ty_Ty_Ty)) \)

Now, \( d(Ty_Ty_Ty, Ty_Ty) \leq (d(Ty_Ty_Ty, Ty_Ty_Ty) + d(Ty_Ty_Ty, Ty_Ty_Ty) + d(Ty_Ty_Ty, Ty_Ty_Ty)) \)

Since, \( T, T, T = T \), and \( T \) is continuous, we have as \( n \to \infty \)

\( d(Ty, Ty_Ty) \to 0 \). Hence, \( T \to Ty_Ty \).

Again, \( d(Ty, Ty_Ty) \leq (d(Ty, Ty_Ty_Ty) + d(Ty_Ty, Ty_Ty_Ty)) \)

Now, \( d(Ty_Ty_Ty, Ty_Ty) \leq (d(Ty_Ty_Ty, Ty_Ty_Ty) + d(Ty_Ty_Ty, Ty_Ty_Ty) + d(Ty_Ty_Ty, Ty_Ty_Ty)) \)

Hence, we have as \( n \to \infty \)

\( d(Ty, Ty_Ty) \to 0 \). Similarly, we can prove that \( Ty = Ty_Ty \).

Again, \( d(Ty, Ty_Ty) \leq (d(Ty, Ty_Ty_Ty) + d(Ty_Ty, Ty_Ty_Ty)) \)

Then, \( (1 - \alpha, - \alpha,) \)\( d(Ty, Ty_Ty) \to 0 \). Hence, \( Ty = Ty_Ty \). Therefore, from above we have

\( Ty_Ty = Ty = y \) & \( Ty_Ty = Ty = y \).

Again, \( d(Ty, Ty_Ty) \leq (d(Ty, Ty_Ty_Ty) + d(Ty_Ty, Ty_Ty_Ty)) \)

Then, \( (1 - \alpha, - \alpha,) \)\( d(Ty, Ty_Ty) \to 0 \). Hence, \( Ty = y \). Therefore, from above we have

\( Ty = Ty_Ty = y \). Similarly, we can show that \( Ty = y \). Therefore, from above we have

\( Ty = Ty_Ty = y \).

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Remark 4. Taking $T_1 = 1$, (2.4.1) reduces to Chatterjee and Singh [29].

Remark 5. Taking $T_1 - T_2 = T_3 - 1, \alpha_1 = \alpha_3 = 0$, (2.4.1) reduces to Fisher [52].

Remark 6. Taking $T_1 = 1, \alpha_2 = \alpha_4 = 0$ and if $T_1 = 1, \theta$, then (2.4.1) reduces to Rao and Rao's [153].
CHAPTER III

FIXED POINT THEOREMS IN NORMED SPACES

3.1. Naimpally and Singh [128] extended some fixed point theorems of Rhoades [157A] for a mapping $T$ satisfying certain contractive conditions, if the sequence of Ishikawa iterates converges to a fixed point of $T$. We recall that the G-iterative process associated with two self mappings $T_1$ and $T_2$ of a normed space $N$ is defined in the following manner.

Let $x_n \in N$ and set

$$x_{2n+1} = (\mu_{2n} - \lambda_{2n})x_{2n} + \lambda_{2n}T_1x_{2n} + (1 - \mu_{2n})T_2x_{2n}$$

and

$$x_{2n+2} = (\mu_{2n+1} - \lambda_{2n+1})x_{2n+1} + \lambda_{2n+1}T_2x_{2n+1} + (1 - \mu_{2n+1})T_1x_{2n+1}$$

for $n = 0, 1, 2, \ldots$, where (a) $\mu_0 = \lambda_0 = 0$, (b) $0 < \lambda_n < 1$, $0 \leq \mu_n < 1$, such that $\mu_n \geq \lambda_n$, $n \to 0$ (c) $\lim \lambda_n = 0$, and (d) $\lim \mu_n = 1$.

When $\langle \mu_n \rangle = 1$, the G-iterative process reduces to Mann iteration. Pathak and Dubey [144] extending the result of Naimpally and Singh [128] proved the following

**Theorem 1.** Let $X$ be a closed, convex, bounded subset of a normed space $N$ and let $T_1$ and $T_2$ be self mappings of $X$ satisfying

$$\langle \parallel T_1x - T_2y \parallel \leq \max(c, \parallel x - y \parallel, \parallel x - T_1x \parallel + \parallel y - T_2y \parallel, \parallel x - T_2y \parallel + \parallel y - T_1x \parallel)$$

for every $x, y \in X$ where $c > 0$, $0 < q < 1$. Let the sequence $\langle x_n \rangle$ be defined in accordance with the G-iterates associated with $T_1$ and $T_2$ as defined above:

If $\langle x_n \rangle$ converges to $z$ in $X$, then $z$ is a common fixed point of $T_1$ and $T_2$.

The object of this chapter is to extend the results of Naimpally and Singh [128] and Pathak and Dubey [144] by altering the norm conditions of contractive condition.

The technique of the proof differs from that of Pathak and Dubey [144]. Further we have used our contractive condition for the solvability of certain non linear functional equations.

**Remark 1.** If $\parallel \cdot \parallel$ is a norm defined in any normed space $X$, then for any real number $p > 0$, $\parallel \cdot \parallel^p$ is not necessarily a norm.

**Example 3.1.** Consider the $l_1$ space. For some $x = (1/2, 0, 0, \ldots)$ and $y = (0, 1/2, 0, \ldots)$
we see that $||x + y||^p \leq ||x||^p + ||y||^p$.

**Theorem 3.1.** Let $X$ be a closed, convex, bounded subset of a normed space $N$ and let $T_1$ and $T_2$ be self mappings of $X$ satisfying

\[(3.1.1) \quad ||T_1 x - T_2 y||^p + a_1 (||x - T_1 x||^p + ||x - T_2 y||^p) + a_2 (||y - T_1 x||^p + ||y - T_2 y||^p) \leq q \max\{ c||x - y||^p, ||x - T_1 x||^p + ||y - T_2 y||^p \}
\]

for every $x, y \in X$ where $c > 0 < q < (a_1 + a_2) < 1$. Let the sequence $<x_n>$ be defined in accordance with Mann iteration process associated with two mappings $T_1$ and $T_2$ as follows

\[(3.1.2) \quad x_{n+1} = (1 - c_{n+1}) x_n + c_{n+1} T_1 x_n
\]

\[(3.1.3) \quad x_{n+1} = (1 - c_{n+1}) x_n + c_{n+1} T_2 x_n
\]

for $n > 0$ where $c_0 = 1, 0 < c_n < 1$ for $n > 0$, and $\lim_{n \to \infty} c_n = 0$. If $<x_n>$ converges to $z$ in $X$ then $z$ is a common fixed point of $T_1$ and $T_2$. Moreover if $\max \{c_n, 2q\} < 1 + a_1 + a_2$, then $z$ is a unique common fixed point of $T_1$ and $T_2$.

**Proof:** Case I: When $p$ is a positive integer.

If possible let $T_1 z \neq z$, then $||z - T_1 z||^p = ||z - x_{2n+1} + x_{2n+1} - T_1 z||^p$

\[
\leq ( ||z - x_{2n+1}||^p + ||x_{2n+1} - T_1 z||^p )^p
\]

\[
= ||z - x_{2n+1}||^p + pC_1 ||z - x_{2n+1}||^p ||x_{2n+1} - T_1 z|| + ... + ||x_{2n+1} - T_1 z||^p
\]

\[
= ||z - x_{2n+1}||^p + pC_1 ||z - x_{2n+1}||^p ||x_{2n+1} - T_1 z|| + ... + ((1 - c_{2n+1}) ||x_{2n} - T_1 x_{2n}|| + c_{2n+1} ||T_1 x_{2n} - T_1 z||)^p
\]

\[
\leq ||z - x_{2n+1}||^p + pC_1 ||z - x_{2n+1}||^p ||x_{2n+1} - T_1 z|| + ... + ((1 - c_{2n+1}) ||x_{2n} - T_1 z|| + c_{2n+1} ||T_1 x_{2n} - T_1 z||)^p
\]

\[
= ||z - x_{2n+1}||^p + pC_1 ||z - x_{2n+1}||^p ||x_{2n+1} - T_1 z|| + ... + (1 - c_{2n+1}) ||x_{2n} - T_1 z||^p +
\]

\[
pC_1 (1 - c_{2n+1}) ||x_{2n} - T_1 z||^p c_n ||T_1 x_{2n} - T_1 z|| + pC_2 (1 - c_{2n+1}) ||x_{2n} - T_1 z||^p c_n ||T_1 x_{2n} - T_1 z||^p c_n ||T_1 x_{2n} - T_1 z||^p
\]

\[
+ ... + c_{2n+1} ||T_1 x_{2n} - T_1 z||^p
\]

\[(3.1.4) \quad = ||z - x_{2n+1}||^p + pC_1 ||z - x_{2n+1}||^p ||x_{2n+1} - T_1 z|| + (1 - c_{2n+1}) ||x_{2n} - T_1 z||^p
\]

\[
+ pC_1 (1 - c_{2n+1}) ||x_{2n} - T_1 z||^p c_n ||T_1 x_{2n} - T_1 z|| + pC_2 (1 - c_{2n+1}) ||x_{2n} - T_1 z||^p c_n ||T_1 x_{2n} - T_1 z||^p
\]

\[
+ ... + c_{2n+1} ||T_1 x_{2n} - T_1 z||^p
\]

\[
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\]
From (3.1) we have:

\[
\| T_{x_{n+1}} - T_z \| = \max(q, \| x_{n+1} - T_{x_{n+1}} \| + \| z - T_{x_{n+1}} \| - \| x_n - T_{x_n} \|) ^ p
\]

and from (1.1) we see that as \( n \to \infty \), \( \| x_{n+1} - T_{x_{n+1}} \| \to 0 \) and \( \| z - T_{x_{n+1}} \| \to 0 \).

Therefore \( \| T_{x_{n+1}} - T_z \| \leq h^p (q - (a_1 + a_2)) \| z - T_z \| \).

Hence, as \( n \to \infty \), (3.1.4) reduces to

\[
\| z - T_z \| \leq (1 - h)^p \| z - T_z \| + h^p (q - (a_1 + a_2))^p \| z - T_z \| ^ p
\]

\[
= \| z - T_z \| ^ p ((1 - h) + h (q - (a_1 + a_2))^p) \]

\[
= \| z - T_z \| ^ p (1 - h) + h (q - (a_1 + a_2))^p \]

\[
= \| z - T_z \| ^ p \text{ a contradiction.}
\]

Hence \( z = T_z \), i.e. \( z \) is a fixed point of \( T_z \).

Case II: When \( p \) is any fraction

\[
\| z - T_z \| \leq \left( \| z - x_{n+1} \| + \| x_{n+1} - T_z \| \right) ^ p
\]

\[
\leq \| x_{n+1} - T_z \| ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
= \left( 1 - c_n \right) x_n + c_n T_z + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
\leq \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
= \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
= \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
\leq \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
= \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
= \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]

\[
= \left( 1 - c_n \right) x_n + c_n T_z x_{n+1} - T_z \ | ^ p \left( 1 + \frac{\| z - x_{n+1} \| \} {\| x_{n+1} - T_z \| ^ p} \right)
\]
By the same argument as given in case 1, we see that as \( n \to \infty \)

\[
\|T_1 x_n - T_2 z\|^p \leq (q - (a_1 + a_2)) \|z - T_1 z\|^p.
\]

Therefore, as \( n \to \infty \), (3.1.5) reduces to

\[
\|z - T_1 z\|^p \leq \|z - T_2 z\|^p \left( (1 - h) + h (q - (a_1 + a_2))^p \right) \text{ a contradiction.}
\]

Hence \( z = T_1 z \), i.e. \( z \) is a fixed point of \( T_1 \). Similarly, one can prove that \( z \) is a fixed point of \( T_2 \).

\( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

Remark 2: It is not known whether the result of theorem 3.1 follows without boundedness of the space.

Remark 3: If we put \( a_1 = a_2 = 0 \) and \( p = 1 \), in theorem 3.1 we get the result of [157A].

Remark 4: If we put \( T_1 = T_2 = T \), \( a_1 = a_2 = 0 \) and \( p = e = 1 \) in Theorem 2.1, we obtain the following:

Corollary 3.2: Let \( X \) be a closed, convex, bounded subset of a normed space \( N \) and let \( T \) be self mapping of \( X \) satisfying

\[
\|T x - Ty\| \leq q \max (\|x - y\|, \|x - T x\| + \|y - T y\|, \|x - T y\| + \|y - T x\|).
\]
for every $x, y \in X$ where $0 < q < 1$. Let the sequence $\{x_n\}$ be defined in accordance with Mann iteration process associated with $T$ as follows $x_{n+1} = (1 - c_n) x_n + c_n T x_n$ for $n > 0$ where $c_n = 1, 0 < c_n < 1$ for $n > 0$, and $\lim c_n = 0$. If $\langle x_n \rangle$ converges to $z$ in $X$ then $z$ is a fixed point of $T$.

**Remark 5:** The above corollary is a generalization of Theorem 1.2 and 1.4 II (D) of [128], as can be seen from the following example.

**Example 3.2.** Let $X = [0, 1]$ with usual metric. Define $T : X \to X$ by $T x = 1 - x$. Then clearly $T$ satisfies (2.7). Indeed,

$$\|T x - T y\| = \|x - y\| = \frac{1}{2}. (\|2x - 2y\| + \|2x - 1\|)$$

$$= \frac{1}{2}. (\|x - T x\| + \|y - T y\|).$$

Let $q = \frac{1}{2}$. Then $\|T x - T y\| \leq q. (\|x - T x\| + \|y - T y\|)$. On the other hand $T$ does not satisfy condition (I) and II (D) of [114] for $x = 0$ and $y = 1$.

**Theorem 3.3.** Let $X$ be a closed, convex, bounded subset of a normed space $N$ and let $T_1$ and $T_2$ be self mappings of $X$ satisfying any one of the following. For all $x$ and $y$ in $X$

(I) $\|T_1 x - T_2 y\| \leq q. \max \{c \|x - y\|, \|x - T_1 x\|, \|y - T_1 y\|, \|x - T_2 y\| + \|y - T_1 x\|\} \quad 0 < q < 1.$

(II) $\|x - T_1 x\| + \|y - T_2 y\| \leq a. \|x - y\|,$

(III) $\|x - T_1 x\| + \|y - T_2 y\| \leq b. \min \{\|x - T_1 x\|, \|y - T_1 y\|, \|x - y\|\} \quad 1 \leq a < 2.$

(IV) $\|x - T_1 x\| + \|y - T_2 y\| \leq c. \min \{\|T_1 x\|, \|y - T_1 y\|, \|x - y\|\} \quad 1 \leq b < 2/3.$

(V) $\|T_1 x - T_2 y\| \leq k. \max \{c \|x - y\|, \|x - T_1 x\|, \|y - T_2 y\|, \|T_1 x - T_1 y\|/2\} \quad 0 \leq k < 1.$

Let the sequence $\langle x_n \rangle$ be defined in accordance with Mann iteration process associated with two mappings $T_1$ and $T_2$ as follows

(3.1.8) $x_{n+1} = (1 - c_{n+1}) x_n + c_{n+1} T_1 x_n.$

(3.1.9) $x_{n+2} = (1 - c_{n+2}) x_{n+1} + c_{n+1} T_2 x_{n+1}.$

for $n > 0$ where $c_n = 1, 0 < c_n < 1$ for $n > 0$, and $\lim c_n = h > 0$. If $\langle x_n \rangle$ converges to $z$ in $X$ then $z$ is a common fixed point of $T_1$ and $T_2$.

**Proof.** $\|z - T_2 z\|^p \leq (\|z - x_{2n+1}\| + \|x_{2n+1} - T_2 z\|)^p$

$\leq \|x_{2n+1} - T_2 z\|^p \cdot \left[1 + \frac{\|z - x_{2n+1}\|}{\|x_{2n+1} - T_2 z\|}\right]^p$
\[
\begin{align*}
&\| (1 - c_n) x_{2n} + c_n T_{x_{2n}} - T_z \|_p \\
&\quad \geq \left( 1 + \frac{\| x_{2n} - z \|_p}{\| x_{2n} - T_z \|_p} \right)^p \left( 1 + \frac{\| x_{2n} - x_{2n+1} \|_p}{\| x_{2n+1} - T_z \|_p} \right)
\end{align*}
\]

\[
\begin{align*}
&\| (1 - c_n) x_{2n} - (1 - c_n) T_{x_{2n}} + c_n T_{x_{2n}} - c_n T_z \|_p \\
&\quad \geq \left( 1 + \frac{\| x_{2n} - z \|_p}{\| x_{2n} - T_z \|_p} \right)^p \left( 1 + \frac{\| x_{2n} - x_{2n+1} \|_p}{\| x_{2n+1} - T_z \|_p} \right)
\end{align*}
\]

\[
\begin{align*}
&\leq (1 - c_n) \| x_{2n} - T_z \| + c_n \| T_{x_{2n}} - T_z \|_p \\
&\quad \geq \left( 1 + \frac{\| x_{2n} - z \|_p}{\| x_{2n} - T_z \|_p} \right)^p \left( 1 + \frac{\| x_{2n} - x_{2n+1} \|_p}{\| x_{2n+1} - T_z \|_p} \right)
\end{align*}
\]

\[
(3.1.10) \leq (1 - c_n)^p \| x_{2n} - T_z \|_p
\]

If \( x_{2n}, z \) satisfy (I), then

\[
\| T_{x_{2n}} - T_z \|_p \leq q \max(\| x_{2n} - z \|_p, \| x_{2n} - x_{2n+1} \|_p, \| z - T_z \|_p, \| x_{2n} - T_z \|_p + \| z - T_{x_{2n}} \|_p)
\]

from (3.1.8) we see that as \( n \to \infty \), \( \| x_{2n} - T_{x_{2n}} \|_p \to 0 \) and \( \| z - T_{x_{2n}} \|_p \to 0 \).

Therefore \( \| T_{x_{2n}} - T_z \|_p \leq q \| z - T_z \|_p \) as \( n \to \infty \).

If \( x_{2n}, z \) satisfy (II), then

\[
\| T_{x_{2n}} - T_z \|_p \leq 1/2 \left( a \| x_{2n} - z \|_p + T C_i \| T_{x_{2n}} - x_{2n+1} \|_p \| x_{2n} - T_z \|_p + \| z - T_{x_{2n}} \|_p \right)
\]

from (3.1.8) we see that as \( n \to \infty \), \( \| x_{2n} - T_{x_{2n}} \|_p \to 0 \) and \( \| z - T_{x_{2n}} \|_p \to 0 \).

Therefore, \( \| T_{x_{2n}} - T_z \|_p \to 0 \) as \( n \to \infty \).

If \( x_{2n}, z \) satisfy (III), then

\[
\| T_{x_{2n}} - T_z \|_p \leq 1/2 \left( \sum_{i=1}^{n} T C_i \| T_{x_{2n}} - x_{2n+1} \|_p \| x_{2n} - T_z \|_p + \| z - T_{x_{2n}} \|_p \right)
\]

from (3.1.8) we see that as \( n \to \infty \), \( \| x_{2n} - T_{x_{2n}} \|_p \to 0 \) and

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Therefore, \( \| z - T_2 x_{n} \| \to 0 \).

Hence, substituting the value of \( \| T_1 x_{n} - T_2 x_{n} \| \) we see that as \( n \to \infty \), (3.1.10) reduces to

\[
\| z - T_1 z \| \leq \| z - T_2 z \|, (1 - h) + h \max(\epsilon \| x - z \|, \| x - T_1 x \|, \| z - T_2 z \|, \| x - T_2 z \|, \| z - T_2 z \|, \| k \|) \]

Hence \( z = T_1 z \), i.e. \( z \) is a fixed point of \( T_1 \). Similarly, we can prove that \( z \) is a fixed point of \( T_2 \). Hence \( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

**Remark 6.** It is not known whether the result of theorem 5.1 follows without boundedness of the space.

### 3.2

In this section we use our contraction principle to investigate the solvability of certain non-linear functional equations in a banach space.

**Theorem 3.4.** Let \( \{ f_n \} \) and \( \{ g_n \} \) be sequence of elements in a Banach space \( B \). Let \( u \) and \( v \) be the unique solution of the equation \( u - T_1 u = f_n \) and \( v - T_2 v = g_n \) where \( T_1 \) and \( T_2 \) are mappings of \( B \) into itself satisfying condition (3.1.1) of Theorem 3.1.

If \( \| f_n \| \to 0 \) and \( \| g_n \| \to 0 \) as \( n \to \infty \) and \( \max(c q, 2q) < 14a + a_j \), then \( \{ u_n \} \) and \( \{ v_n \} \) converges to a common solution of the equation \( x = T_1 x = T_2 x \).

**Proof:** First of all we prove that as \( n \to \infty \), \( < u_n > \) and \( < v_n > \) tends to the same limit \( u \). Consider

\[
\| u_n - v_n \| = \| u_n - T_1 u_n + T_1 u_n - T_1 v_n + T_1 v_n - v_n \|
\]

(3.2.1) \( \leq \| u_n - T_1 u_n \| + \| T_1 u_n - T_1 v_n \| + \| T_1 v_n - v_n \|)\)

Now, from (3.1.1) we get, \( \| T_1 u_n - T_1 v_n \| < q \max(\| u_n - v_n \|, \| u_n - T_1 u_n \|, \| v_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|, \| u_n - T_2 v_n \|, \| v_n - T_2 v_n \|)\)

- \( a_j(\| v_n - T_2 v_n \| + \| v_n - T_2 v_n \|)\)

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we see that as $n \to \infty$, (3.2.1) reduces to

$$
\|u_n - v_n\| \leq q/(1 + a_n + a_n) \max \left( \|u_n - u_n\|, \|v_n - v_n\|, 2 \|u_n - v_n\| \right)
$$

Hence, as $n \to \infty$, (3.2.2) reduces to

$$
\|u_n - v_n\| \leq q/(1 + a_n + a_n) \max \left( \|u_n - u_n\|, \|v_n - v_n\|, 2 \|u_n - v_n\| \right)
$$

which is a contradiction. Hence $\lim_{n \to \infty} u = \lim_{n \to \infty} v = u$ say.

Now, $\|u - T_1 u\|^p = \|u - v_n + v_n - T_2 v_n + T_2 v_n - T_1 u\|^p$

(3.2.2) $\leq (\|u - v_n\| + \|v_n - T_2 v_n\|) \cdot \|T_2 v_n - T_1 u\|$

Again, $\|T_1 u - T_2 v_n\|^p \leq q \max (\|u - v_n\|, \|u - T_1 u\| + \|v_n - T_2 v_n\|^p, \|u - T_1 u\|^p + \|v_n - T_2 v_n\|^p, \|u - T_1 u\|^p, \|v_n - T_2 v_n\|^p)$

$\leq q \max (\|u - v_n\|^p, \|u - T_1 u\|^p + \|v_n - T_2 v_n\|^p, \|u - T_1 u\|^p, \|v_n - T_2 v_n\|^p, \|u - T_1 u\|^p, \|v_n - T_2 v_n\|^p, \|u - T_1 u\|^p, \|v_n - T_2 v_n\|^p)$

+ $\|v_n - v_n\| + \|u - T_1 u\|^p)$

Hence as argued before we see that as $n \to \infty$ $\|u - T_1 u\|^p \leq q/(1 + a_n + a_n) \max (\|u - T_1 u\|^p, \|v_n - T_1 u\|^p)$

Hence as $n \to \infty$, (3.2.2) reduces to $\|u - T_1 u\|^p \leq q/(1 + a_n + a_n) \max (\|u - T_1 u\|^p, \|v_n - T_1 u\|^p)$, which is a contradiction. Hence $u = T_1 u$. Similarly one can prove that $u = T_2 u$. Hence $u = T_1 T_2 u$.
Chapter IV

Common Fixed Points for Compatible Mappings of Type (A-1) and Type (A-2) in Fuzzy Metric Spaces

4.1 Zadeh’s introduction [210] of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. In the last two decades, there were a tremendous development and growth in fuzzy mathematics ([1], [21], [51], [65], [107], [178]). In [65], Grabiec extended the well-known fixed point theorems of Banach [4] and Edelsten [49] to fuzzy metric spaces in the sense of Kramosil and Michalek [107]. Moreover, it appears that the study of Kramosil and Michalek [107] of fuzzy metric spaces paved the way for developing a soothing machinery in the field of fixed point theorems, in particular, for the study of contractive type maps. In [51], Fang proved some fixed point theorems in fuzzy metric spaces, which improve, generalise, extend and unify some main results of [4], [49], [86], and [105]. Following Grabiec [65] and Kramosil-Michalek [107], Mishra-Sharma-Singh [120] obtained common fixed point theorems for compatible maps and asymptotically commuting maps on fuzzy metric spaces, which generalise, extend and fuzzify several fixed point theorems for contractive type maps on metric spaces and other spaces.

In this chapter, we introduce the concept of compatible mappings of type (A-1) and type (A-2) in fuzzy metric space and show that they are equivalent to compatible mappings under certain conditions. In the sequel, we prove some common fixed point theorem for compatible mappings of type (A-1) and type (A-2) on fuzzy metric spaces which generalise, extend, unify and fuzzify several well-known fixed point theorems for contractive type maps on metric spaces, Menger spaces, uniform spaces and fuzzy metric spaces.

4.2. Following Grabiec [65] and Kramosil-Michalek [107], we have the following notations and definitions.

A fuzzy metric space (shortly an FM-space) is an ordered triplet (X, M, *) consisting of a nonempty set X, a fuzzy set M in X x (0, ∞) and a continuous T-norm * . The functions M(x, y, .) : (0, ∞) → (0, ∞) are left continuous and are assumed to satisfy the following conditions:

1. (FM-1) M(x, y, t) = 1 for all t > 0 if and only if x = y.
2. (FM-2) M(x, y, 0) = 0.
3. (FM-3) M(x, y, t) = M(y, x, t).
4. (FM-4) M(x, y, t) M(y, z, s) ≤ M(x, z, t+s) for all x, y, z in X and t, s ≥ 0.

Grabiec [65] has shown that M(x, y, .) is monotonically nondecreasing for all x, y in X.

In all that follows, N denotes the set of natural numbers, and X an FM-space (X, M, *) with the following condition:

5. (FM-5) lim t→∞ M(x, y, t) = 1 for all x, y in X.
The proof of the following lemma is given in [65]

**Lemma 4.1** Let \( \{ y_n \} \) be a sequence in an FM-space \( X \). If there exists a positive number \( k < 1 \) such that \( M(y_n, y_{n+1}, k) \geq M(y_n, y_{n+1}, 1) \cdot 0.1 \), then \( \{ y_n \} \) is a Cauchy sequence in \( X \).

From (FM-5), the following lemma follows immediately

**Lemma 4.2** If for two points \( x, y \) in \( X \) and a positive number \( k < 1 \), \( M(x, y, k) \geq M(x, y, 1) \) then \( x \sim y \).

4.3. In this section we show that compatible mappings of type (A-1) and type (A-2) in fuzzy metric space are equivalent to compatible mappings under certain conditions.

**Definition 4.1** Let \( S \) and \( T \) be self maps of an FM-space \( X \). The mappings \( S \) and \( T \) are said to be compatible if \( \lim_{n \to \infty} M(Sx_n, Tx_n, 1) = 1 \), whenever \( \{ x_n \} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} L(Sx_n) = \lim_{n \to \infty} L(Tx_n) = z \) for some \( z \) in \( X \).

**Definition 4.2** Let \( S \) and \( T \) be self maps of an FM-space \( X \). The pair of mappings \( \langle S, T \rangle \) is said to be compatible of type (A-1) if \( \lim_{n \to \infty} M(STx_n, TTx_n, 1) = 1 \), whenever \( \{ x_n \} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} L(Sx_n) = \lim_{n \to \infty} L(Tx_n) = z \) for some \( z \) in \( X \).

**Definition 4.3** Let \( S \) and \( T \) be self maps of an FM-space \( X \). The pair of mappings \( \langle S, T \rangle \) is said to be compatible of type (A-2) if \( \lim_{n \to \infty} M(TSx_n, SSx_n, 1) = 1 \), whenever \( \{ x_n \} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} L(Sx_n) = \lim_{n \to \infty} L(Tx_n) = z \) for some \( z \) in \( X \).

Clearly if a pair of mappings \( \langle S, T \rangle \) is compatible of type (A-1) then the pair \( \langle T, S \rangle \) is compatible of type (A-2). Further from the definitions it clear that if \( S \) and \( T \) compatible mappings of type (A) then the pair \( \langle S, T \rangle \) is compatible of type (A-1) as well as type (A-2).

We now cite the following propositions which gives the condition under which definitions 4.1, 4.2, and 4.3 becomes equivalent.

**Proposition 4.1** Let \( S \) and \( T \) be self maps of an FM-space \( X \).

a) If \( T \) is continuous then the pair of mappings \( \langle S, T \rangle \) is compatible of type (A-1) if and only if \( S \) and \( T \) are compatible.

b) If \( S \) is continuous then the pair of mappings \( \langle S, T \rangle \) is compatible of type (A-2) if and only if \( S \) and \( T \) are compatible.

**Proof**: a) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \), and let the pair \( \langle S, T \rangle \) be compatible of type (A-1). Since \( T \) is continuous we have \( \lim_{n \to \infty} TSx_n = Tz \) and \( \lim_{n \to \infty} TTx_n = Tz \).

Therefore it follows that \( M(STx_n, TSx_n, 1) \geq M(STx_n, TTx_n, 1/2) * M(TTx_n, TSx_n, 1/2) \), yields \( \lim_{n \to \infty} M(STx_n, TSx_n, 1) \geq 1 \times 1 = 1 \) and so the mappings \( S \) and \( T \) are compatible.
Now let \( S \) and \( T \) be compatible. Therefore it follows that \( M(STx_n,TTx_n,t) \geq M(STx_n,TSx_n,t/2) \cdot M(TSx_n,TTx_n,t/2) \) yields \( \lim_{n \to \infty} M(STx_n,TTx_n,t) \geq 1 \) and so the pair of mappings \((S, T)\) are compatible of type \((A-1)\).

b) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \), and let the pair \((S, T)\) be compatible of type \((A-2)\). Since \( S \) is continuous we have \( \lim_{n \to \infty} STx_n = Sx \) and \( \lim_{n \to \infty} SSx_n = Sx \). It follows that \( M(STx_n,TSx_n,t) \geq M(STx_n,SSx_n,t/2) \cdot M(SSx_n,TSx_n,t/2) \) yields \( \lim_{n \to \infty} M(STx_n,TSx_n,t) \geq 1 \) and so the mappings \( S \) and \( T \) are compatible.

Now let \( S \) and \( T \) be compatible. Then we have \( M(TSx_n,SSx_n,t) \geq M(STx_n,STx_n,t/2) \cdot M(STx_n,SSx_n,t/2) \) yields \( \lim_{n \to \infty} M(TSx_n,SSx_n,t) = 1 \) and so the pair \((S, T)\) are compatible of type \((A-2)\).

Next we give some properties of compatible mappings of type \((A-1)\) and type \((A-2)\) which will be used in our main theorem.

**Proposition 4.2:** Let \( S \) and \( T \) be self-maps of an FM-space \( X \). If the pair \((S, T)\) are compatible of type \((A-1)\) and \( Sz = Tz \) for some \( z \) in \( X \) then \( STz = TTz \).

**Proof:** Let \( \{x_n\} \) be a sequence in \( X \) defined by \( x_n = z \) for \( n = 1, 2, \ldots \) and let \( Tz = Sz \). Then we have \( \lim_{n \to \infty} Sx_n = Sz \) \( \lim_{n \to \infty} Tx_n = Tz \). Since the pair \((S, T)\) is compatible of type \((A-1)\) we have \( M(STx_n,TTx_n,t) = \lim_{n \to \infty} M(STx_n,TTx_n,t) = 1 \). Hence \( STz = TTz \).

**Proposition 4.3:** Let \( S \) and \( T \) be self-maps of an FM-space \( X \). If the pair \((S, T)\) is compatible of type \((A-2)\) and \( Sz = Tz \) for some \( z \) in \( X \) then \( TSz = SSz \).

**Proposition 4.4:** Let \( S \) and \( T \) be self-maps of an FM-space \( X \) with \( t^*t > t \) for all \( t \) in \([0, 1]\). If the pair \((S, T)\) are compatible of type \((A-1)\) and \( Sx_n, Tx_n \to z \) for some \( z \) in \( X \) and a sequence \( \{x_n\} \) in \( X \) then \( TTx_n \to Sz \) if \( S \) is continuous at \( z \).

**Proof** Since \( S \) is continuous at \( z \) we have \( STx_n \to Sz \). Since the pair \((S, T)\) are compatible of type \((A-1)\), we have \( M(STx_n,TTx_n,t) \to 1 \) as \( n \to \infty \). It follows that \( M(Sz,TTx_n,t) \geq M(Sz,STx_n,t/2) \cdot M(STx_n,TTx_n,t/2) \) yields \( \lim_{n \to \infty} M(Sz,TTx_n,t) \geq 1 \) and so we have \( TTx_n \to Sz \) as \( n \to \infty \).

**Proposition 4.5:** Let \( S \) and \( T \) be self-maps of an FM-space \( X \) with \( t^*t > t \) for all \( t \) in \([0, 1]\). If the pair \((S, T)\) are compatible of type \((A-2)\) and \( Sx_n, Tx_n \to z \) for some \( z \) in \( X \) and a sequence \( \{x_n\} \) in \( X \) then \( SSx_n \to Tz \) if \( T \) is continuous at \( z \).

**Proposition 4.4.** Let \( (X, M, *) \) be an FM-space with \( t^*t > t \) for all \( t \) in \([0, 1]\) and \( p \in [0, 1] \). Let \( S, T \) be self-maps of \( X \) such that \( (4.4.1) \) \( F(X) \subseteq T(X) \) and \( Q(X) \subseteq S(X) \)

\[
(4.4.2) \quad [1 + pM(Sx, Ty, kt)] \cdot M(Px, Qy, kt) \geq p[M(Px, Sx, kt) \cdot M(Qy, Ty, kt) + M(Px, Ty, kt) \cdot M(Qy, Sx, kt)] + M(Sx, Ty, t) \cdot M(Px, Sx, t) \cdot M(Qy, Ty, t) \cdot M(Px, Ty, at) \cdot M(Qy, Sx, (2-a)t)
\]

for all \( x, y \) in \( X \), \( p \geq 0 \), \( t > 0 \) and \( a \in (0, 2) \).
For some arbitrary \( x_0 \) in \( X \), by (4.4.1) we choose \( x_1 \) in \( X \) such that \( P x_0 = T x_0 \) and for this \( x_1 \) there exists \( x_2 \) such that \( S x_1 = Q x_1 \). Continuing this process we define the sequence \( \{ y_n \} \) in \( X \) such that

\[(4.4.3) \quad y_{2n-1} = P x_{2n-1} \quad \text{and} \quad y_{2n} = Q x_{2n} \quad \text{for all} \quad n \in \mathbb{N}.
\]

**Lemma 4.3.** Let \( P, Q, S \) and \( T \) be self maps of an FM-space \((X, M, * )\) with \( t * t < t \) for all \( t \) in \([0, 1]\) satisfying (4.4.1) and (4.4.2). Then the sequence \( \{ y_n \} \) defined by (4.4.3) is a Cauchy sequence.

**Proof.** By (4.4.2) for \( \alpha = 1 + q \), \( q \in (0, 1) \) we have,

\[
[1 + p \cdot M(y_{2n}, y_{2n-1}, kt)] * M(y_{2n-1}, y_{2n}, kt) \\
= [1 + p \cdot M(T x_{2n-1}, S x_{2n-1}, kt)] * M(Q x_{2n-1}, P x_{2n-1}, kt) \\
\geq p [M(P x_{2n}, S x_{2n-1}, kt)] * M(Q x_{2n-1}, T x_{2n-1}, kt) + M(P x_{2n}, T x_{2n}, kt) \\
M(Q x_{2n-1}, S x_{2n-1}, kt)] + M(S x_{2n-1}, T x_{2n-1}, y, t) * M(P x_{2n}, S x_{2n}, t) \\
M(Q x_{2n}, T x_{2n-1}, 1) * M(P x_{2n}, T x_{2n-1}, (1+q)t) * M(Q x_{2n}, S x_{2n-1}, (1-q)t)
\]

\[
= p [M(y_{2n}, y_{2n}, kt)] \cdot M(y_{2n}, y_{2n}, kt) + M(y_{2n}, y_{2n}, kt) \cdot M(y_{2n}, y_{2n}, kt) \\
+ M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, (1+q)t) \\
* M(y_{2n}, y_{2n}, (1-q)t)
\]

\[
= p [M(y_{2n}, y_{2n}, kt)] \cdot M(y_{2n}, y_{2n}, kt) + M(y_{2n}, y_{2n}, kt) \cdot M(y_{2n}, y_{2n}, kt) \cdot 1 + M(y_{2n}, y_{2n}, t) \\
* M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, (1+q)t) \cdot 1
\]

\[
\geq p [M(y_{2n}, y_{2n}, kt)] \cdot M(y_{2n}, y_{2n}, kt) + M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, t) \cdot 1 \\
* M(y_{2n}, y_{2n}, t) \cdot M(y_{2n}, y_{2n}, t)
\]

Thus it follows that

\[
M(y_{2n+1}, y_{2n+1}) \geq M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n}, qt)
\]

Since the \( t \)-norm \( * \) is continuous and \( M(x, y) \) is left continuous, letting \( q \to 1 \), we have\( M(y_{2n+1}, y_{2n+1}) \geq M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n}, t) \).

Similarly we get \( M(y_{2n}, y_{2n+1}) \geq M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) \). In general we have

\[
M(y_{2n+1}, y_{2n+1}) \geq M(y_{2n+1}, y_{2n}, t)
\]

Consequently it follows that

\[
M(y_{2n}, y_{2n}) \geq M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n}, t)
\]

By noting that \( M(y_{m}, y_{n}, t) \to 1 \) as \( n \to \infty \), we have \( M(y_{m}, y_{n}, t) \geq M(y_{m}, y_{n}, t) \). Hence by lemma 4.1 \( \{ y_n \} \) is a Cauchy sequence.

**Theorem 4.1.** Let \((X, M, *)\) be a complete FM-space with \( t * t < t \) for all \( t \) in \([0, 1]\), and let \( P, Q, S \) and \( T \) be self maps of \( X \) satisfying (4.4.1), (4.4.2) and
(4.4.1) the pairs \((P,S)\) and \((Q,I)\) are compatible of type \((A-1)\) or type \((A-2)\)

(4.4.5) one of \(S\) and \(T\) is continuous

Then \(P,Q,S\) and \(T\) have a unique common fixed point in \(X\)

**Proof** Let the pairs \((P,S)\) and \((Q,T)\) be compatible of type \((A-1)\) and suppose \(T\) is continuous. By lemma 4.1 the sequence \(\{y_n\}\) defined by (4.4.3) is a cauchy sequence in \(X\) and since \(X\) is complete it should converge to some point say \(z\) in \(X\). Consequently the subsequences \(\{P_{x_n}\},\{Q_{x_n}\}\) and \(\{S_{x_n}\}\) also converge to the same point \(z\). Since \(T\) is continuous \(TQ_{x_n}\) \(\rightarrow Tz\) as \(n\) \(\rightarrow \infty\), and also since the pair \((Q,T)\) are compatible of type \((A-1)\), by proposition (4.4) we have \(QT_{x_n}\) \(\rightarrow Tz\) as \(n\) \(\rightarrow \infty\).

Now for \(\alpha = 1\) (4.4.2) yields \([1 + pM(Sx_{x_n},TTx_{x_n},kt)] \geq M(z,Tz,kt)\) which gives \(M(z,Tz,kt) \geq M(z,Tz,kt)\). This shows that \(Tz = z\). Again by (4.4.2) with \(\alpha = 1\) we have

\([1 + pM(Sx_{x_n},TTx_{x_n},kt)] \geq M(z,Tz,kt)\)

which yields \(M(z,Tz,kt) \geq M(z,Tz,kt)\) and so \(Qz = z\). Hence by (4.4.1) there exists some \(u\) in \(X\) such that \(Qz = Su\). By (4.4.2) with \(\alpha = 1\) we have

\([1 + pM(Su,Tz,kt)] \geq M(Pu,Qz,kt)\)

which yields \(M(Pu,z,kt) \geq M(Pu,z,kt)\) and so \(Pu = z\).

Therefore \(z = Pu = Su\) and since the pair \((P,S)\) are compatible of type \((A-1)\) by proposition 4.2 we have \(PSu = SSu\), i.e. \(Pz = Sz\). Hence

\([1 + pM(Sz,Tz,kt)] \geq M(Pz,Qz,kt)\)

which yields \(M(Pz,z,kt) \geq M(Pz,z,kt)\) and so \(Pz = z\). Hence \(z\) is a common fixed point of \(P,Q,S\) and \(T\). Finally the uniqueness of \(z\) as a common fixed point of \(P,Q,S\) and \(T\) follows easily from (4.4.2)

If we take \(p = 0\) in (4.4.2) we have the following

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Corollary 4.2. Let \((X,M,\ast)\) be a complete FM-space with \(t^*t, \ t \in [0,1]\), and let \(P, Q, S\) and \(T\) be self maps of \(X\) satisfying (4.4.1), (4.4.4), (4.4.5) and (4.4.6). Then
\[
M(P(x,y), t) \leq M(Sx, Ty, t) + M(Px, Sx, t) + M(Py, Sy, \alpha t) + M(Sx, Sy, \alpha t)
\]
for all \(x,y\) in \(X\), \(t > 0\), and \(\alpha \in (0,2)\). Then \(P, Q, S\) and \(T\) have a unique common fixed point in \(X\).

If we take \(S = T\) and \(P = Q\) in Theorem 4.1, we have the following.

Corollary 4.3. Let \((X,M,\ast)\) be a complete FM-space with \(t^*t, \ t \in [0,1]\), and let \(P\) and \(S\) be self maps of \(X\) such that the pair \((P,S)\) are compatible of type (A-1) or type (A-2) and \(P(X) \subseteq S(X)\).

If \(S\) is continuous and there exists a constant \(k \in (0,1)\) such that (4.4.7)
\[
[1 + pM(Sx, Sy, kt)]M(Px, Py, kt) \geq p[M(Sx, Sy, kt) + M(Px, Sy, kt) + M(Sx, Sx, t) + M(Px, Sx, t) + M(Py, Sy, \alpha t) + M(Py, Sx, (2- \alpha)t)]
\]
for all \(x,y\) in \(X\), \(p > 0\), \(t > 0\) and \(\alpha \in (0,2)\). Then \(P\) and \(S\) have a unique common fixed point in \(X\).

In [65] Grabiec presented the fuzzy version of the Banach contraction theorem as follows:

Corollary 4.4. Let \((X,M,\ast)\) be a complete FM-space with \(t^*t, \ t \in [0,1]\), and let \(P\) be self map of \(X\). If there exists a constant \(k \in (0,1)\) such that
\[
M(P(x,y), t) \geq M(x,y, t)
\]
for all \(x,y\) in \(X\) and \(t > 0\), then \(P\) has a unique common fixed point in \(X\).

Proof. This corollary follows from corollary 4.3 since (4.4.7) with \(P = Q\) and \(p = 0\) includes (4.4.8). However, Grabiec [6] does not require \(t^*t > t\) in his proof. In this section we wish to apply corollaries (4.3) and (4.4) to establish the following result.

Theorem 4.5. Let \((X,M,\ast)\) be a complete FM-space with \(t^*t, \ t \in [0,1]\), and let \(P\) and \(Q\) be two maps on the product \(X \times X\) with values in \(X\). If there exists a constant \(k \in (0,1)\) such that
\[
[1 + pM(x,u, kt)]M(P(x,y), Q(u,v), kt) \geq p[M(P(x,y), x, kt) + M(Q(x,y), u, kt) + M(P(x,y), u, kt) + M(Q(x,y), v, t) + M(P(x,y), v, t) + M(Q(u,v), x, (2- \alpha) t)]
\]
for all \(x,y\) in \(X\), \(p > 0\), \(t > 0\) and \(\alpha \in (0,2)\), then there exists exactly one point \(w\) in \(X\) such that \(P(w,w) = w = Q(w,w)\).
Proof By (4.4.9) we have

\[ 1 + p M(x,u,kt) * M(P(x,y),Q(u,y),kt) \]

\[ \geq p[M(P(x,y),x,kt) * M(Q(u,y),u,kt) + M(P(x,y),u,kt) * M(Q(u,y),x,kt)] \]

\[ + M(P(x,y),x,t) * M(Q(u,y),u,t) * M(x,u,t) * M(P(x,y),u,at) \]

\[ * M(Q(u,y),x,(2 - \alpha)t) \]

for all \( x, y, u \) in \( X \). Therefore by corollary 4.3 for each \( y \) in \( X \), there exists one and only one \( z(y) \) in \( X \) such that (4.4.10) \( P(z(y),y) = z(y) = Q(z(y),y) \). Now for any \( y, y' \) in \( X \) by (4.4.9) with \( \alpha = 1 \), we have

\[ 1 + p M(z(y),z(y'),kt) * M(P(z(y),y),Q(z(y'),y'),kt) \]

\[ \geq p[M(P(z(y),y),z(y),kt) * M(Q(z(y'),y'),z(y'),kt) + M(P(z(y),y),z(y'),kt) \]

\[ * M(Q(z(y'),y'),z(y),kt)] + M(P(z(y),y),z(y),t) * M(Q(z(y'),y'),z(y'),t) \]

\[ * M(z(y),z(y'),t) * M(y,y',t) * M(z(y),z(y'),t) \]

\[ * M(z(y'),z(y),t) * M(z(y'),z(y'),t) \]

i.e.

\[ M(z(y),z(y'),kt) \]

\[ \geq 1 \]

\[ * 1 \]

\[ * M(z(y),z(y'),t) * M(y,y',t) * M(z(y),z(y'),t) * M(z(y'),z(y),t) \]

i.e.

\[ M(z(y),z(y'),kt) \geq M(z(y),z(y'),t) * M(y,y',t) \]

\[ \geq M(z(y),z(y'),t/\alpha) * M(y,y',t) \rightarrow M(y,y',t) . \]

Therefore corollary 4.4 yields that the map \( z(.) \) of \( X \) into itself has exactly one fixed point \( w \) in \( X \), i.e. \( z(w) = z \). Hence by (4.4.10)

\[ w = z(w) = P(w,w) = Q(w,w) . \]

This completes the proof.

Remark Inspiring from the work of Iseki [82] we may observe that if \( x, y \) e \( X \) are such that \( x = P(x,y) \) and \( y = Q(x,y) \), then (4.4.9) yields \( x = y \).
CHAPTER V

COORDINATEWISE COMPATIBLE MAPPINGS AND COMMON FIXED POINTS

5.1. The well known Banach contraction principle has been generalised in two directions in the galaxy of contractive principles by Gerald Jungck [88] and J. Matkowski [115]-[116]. Due to simplicity and elegant nature, Jungck's result leads a massive group of fixed point theorems for contractive type maps. For instance, see [32], [35], [41], [56], [69], [87], [140], [160]-[162], [167], [168], [169], [190], [195], [197], [204], [209].

On the other hand Matkowski's result is somewhat tedious in nature and consequently only a few researchers inspired from his work in applicable mathematics (see [39], [40], [155], [156], [184], [185], [187], [189]).

In this chapter we have generalised the results of Singh and Gairola [186], under appreciably weaker conditions. In the sequel our result unify the results of Jungck [op. cit.] and Matkowski [op. cit.] which provides the sound footing of results of Jungck and Matkowski which are apparently diverse in nature. In our main result, the utility of the concept of coordinatewise compatibility of maps, in the contest of fixed point theory has been demonstrated by generalizing the result of Singh and Gairola [186].

5.2. We shall make use of the following notations and definitions in the sequel

Let $\{X_i, d_i\}, i = 1, 2, \ldots, m$, be metric spaces, $X = X_1 \times X_2 \times \cdots \times X_m$, and $P_1, S_i : X \to X_i, i = 1, 2, \ldots, m$. Two system of maps $(P_i, P_2, \ldots, P_m)$ and $(G_1, G_2, \ldots, G_m)$ are said to be coordinatewise commuting if and only if

$$F_i(P_i(x_1, \ldots, x_m)) = G_i(P_{i-1}(x_1, \ldots, x_m)).$$

Two system of maps $(P_i, P_2, \ldots, P_m)$ and $(S_1, S_2, \ldots, S_m)$ are said to be coordinatewise weakly commuting if and only if

$$d_i(P_i(S_i(x_1, \ldots, x_m)), S_i(P_i(x_1, \ldots, x_m))) \leq d_i(P_i(x_1, \ldots, x_m), S_i(x_1, \ldots, x_m)), \text{ for all } (x_1, \ldots, x_m) \in X.$$ 

Clearly two system of coordinatewise commuting mappings are coordinatewise weakly commuting but the converse is not necessarily true. For details refer to Singh and Gairola [23].

**Definition 5.1** Two system of maps $(P_1, P_2, \ldots, P_m)$ and $(S_1, S_2, \ldots, S_m)$ are said to be coordinatewise compatible if

$$\lim d_i(P_i(S_i(x_1^{(n)}, \ldots, x_m^{(n)}), S_2(x_1^{(n)}, \ldots, x_m^{(n)}), \ldots, S_m(x_1^{(n)}, \ldots, x_m^{(n)})), S_i(P_i(x_1^{(n)}, \ldots, x_m^{(n)}), P_2(x_1^{(n)}, \ldots, x_m^{(n)})),$$

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when \((x_1^{(m)}, x_2^{(m)})\) is a sequence of points of \(X\) such that \(\lim_{n \to \infty} P(x_1^{(m)}, x_2^{(m)}) \to t\) for some \(t \in X\).

Clearly, two system of coordinatewise weakly commuting mappings are coordinatewise compatible but the converse is not necessarily true as shown in the following

**Example 5.1.** Let \(X_1 = \mathbb{R}, X_2 = [0, 2]\), \(X_1 \times X_2 \times X_1 \times X_2 \to X_1\) and \(P_1, S_1 : X \to X_2\) such that \(P_1(x_1, x_2) = 5x_1^2, S_1(x_1, x_2) = 2x_1^2 \)

\[
P_2(x_1, x_2) = \begin{cases} 
  x_2^4/128 & \text{for } 0 \leq x_2 \leq 1/2 \\
  x_2^4/16 & \text{for } 1/2 < x_2 < 2
\end{cases}
\]

and \(S_2(x_1, x_2) = x_1^4/8\)

Now, \(d_1(P_1(x_1, x_2), S_1(x_1, x_2)) = d_1(5x_1^2, 2x_1^2) \to 0 \text{ iff } x_1 \to 0\)
\(d_1(P_1(S_1(x_1, x_2)), S_1(P_1(x_1, x_2)), P_2(x_1, x_2)) = d_1(40x_1^4, 250x_1^4) \to 0 \text{ iff } x_1 \to 0\).

For \(0 \leq x_1 \leq 1/2\),
\[
d_1(P_2(x_1, x_2), S_2(x_1, x_2)) = d_1(x_2^4/128, x_1^4/8) \to 0 \text{ iff } x_2 \to 0 \text{ or } x_2 \to \infty.
\]

For \(1/2 < x_2 < 2\),
\[
d_1(P_2(x_1, x_2), S_2(x_1, x_2)) = d_1(x_2^4/16, x_1^4/8) \to 0 \text{ iff } x_2 \to \infty.
\]

Hence, the system of maps \((P_1, P_2)\) and \((S_1, S_2)\) are coordinatewise compatible. On the other hand we see that \(d_1(P_1(S_1(x_1, x_2)), S_1(x_1, x_2), P_2(x_1, x_2)) = d_1(40x_1^4, 250x_1^4) = d_1(P_1(x_1, x_2), S_1(x_1, x_2))\) for all \(x_1\) in \(X_1\). Therefore, the system of maps \((P_1, P_2)\) and \((S_1, S_2)\) are coordinatewise weakly commuting.

We now prove a necessary and sufficient condition for two system of maps to be coordinatewise compatible. For the sake of convenience we assume that \((v_1, \ldots, v_m) = v(1, m)\) and \((v_m, \ldots, v_m) = v(1, m)\)

**Theorem 5.1.** Let \(P_i : X \to X, i = 1, \ldots, m\) be continuous. If \(P_i, i = 1, \ldots, m\) are proper maps then the systems \((P_1, \ldots, P_m)\) and \((S_1, \ldots, S_m)\) are coordinatewise compatible if
\[
d_1(P_i(x(1, m)), S_i(x(1, m))) = 0 \text{ implies }
\]
Proof. Let the systems \( (P_i, S_i) \) and \( (P'_i, S'_i) \) be coordinatewise compatible and \( d_i(P_i(x(1,m))), S_i(x(1,m))) = 0 \). Now considering the sequence 
\((x_i^{(n_{11})}, \ldots, x_i^{(n_m)})\) for all \( n \), we see that the necessity part is proved.

Now for sufficiency part let \((x_i^{(n_{11})}, \ldots, x_i^{(n_m)})\) be a sequence in \( X \) such that

\[(5.2.1) \lim d_i(P_i(x_i^{(n_{11})}, \ldots, x_i^{(n_m)})) = \lim S_i(x_i^{(n_{11})}, \ldots, x_i^{(n_m)})) = t_i \text{ for some } t_i \in X_i.

Then \( S_i = (P_i(x_i^{(n_{11})}, \ldots, x_i^{(n_m)})) \text{ is compact for all } i \) and \( P'_i(S_i) \) is compact since \( P'_i \) is proper for all \( i \). Consequently the sequence \((x_i^{(n_{11})}, \ldots, x_i^{(n_m)})\) has a subsequence \((x_i^{(n_{11}')} \ldots, x_i^{(n_m')})\) which converges to an element \((c_i, \ldots, c_m)\) of \( X \). Since \( P'_i \) and \( S'_i \) are continuous for all \( i \)

\[(5.2.2) \lim S_i(x_i^{(n_{11}'} \ldots, x_i^{(n_m')})) --\rightarrow S_i(c_i, \ldots, c_m).

Therefore using (5.2.1) we get

\[(5.2.3) P'_i(c_i, \ldots, c_m) = S'_i(c_i, \ldots, c_m) \text{ for all } i.

Hence from the hypothesis we get

\[P_i(S_i(c_i, \ldots, c_m), \ldots, S_m(c_i, \ldots, c_m)) = S_i(P'_i(c_i, \ldots, c_m), \ldots, P'_m(c_i, \ldots, c_m)).

Therefore since \( P'_i \)'s and \( S'_i \)'s are continuous, by using (5.2.3) and (5.2.3) we get

\[\lim P_i(S_i(x_i^{(n_{11}'} \ldots, x_i^{(n_m')})), \ldots, S_m(x_i^{(n_{11}'} \ldots, x_i^{(n_m')})) = \lim S_i(P'_i(x_i^{(n_{11}'} \ldots, x_i^{(n_m')})), \ldots, P'_m(x_i^{(n_{11}'} \ldots, x_i^{(n_m')}))

as desired.

Let \((c_i^k)\) be a square matrix wherein \( c_i^k \) are real numbers, for \( i, k = 1, \ldots, m \).

Following Malkowski [153], matrices \((c_i^k)\) are recursively as follows:

\[(5.2.4) c_i^k = \begin{bmatrix}
    c_{i1}^{(0)} & c_{i1}^{(1)} & \ldots & c_{i1}^{(m_1)} & c_{i2}^{(0)} & \ldots & c_{i2}^{(m_2)} & \ldots & c_{ik}^{(0)} & \ldots & c_{ik}^{(m_k)} \\
    & & &  & & & & & & & \\
    & & &  & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
\end{bmatrix}
\text{ for } i \neq k
\]

\[c_i^k = \begin{bmatrix}
    c_{i1}^{(0)} & c_{i1}^{(1)} & \ldots & c_{i1}^{(m_1)} & c_{i2}^{(0)} & \ldots & c_{i2}^{(m_2)} & \ldots & c_{ik}^{(0)} & \ldots & c_{ik}^{(m_k)} \\
    & & &  & & & & & & & \\
    & & &  & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
    & & & & & & & & & & \\
\end{bmatrix}
\text{ for } i = k
\]

Malkowski's theorem [115], [116] is as follows

**Theorem 5.2.** Let \((X, d), i = 1, \ldots, m\) be complete metric spaces and \(T_i : X \rightarrow X\), \(i = 1, \ldots, m\). If there exist numbers \(a_{i,k}, i, k = 1, \ldots, m\) such that

\[(5.2.5) d_i(T_i(x(1,m))), T_i(y(1,m))) \leq \sum a_{i,k} d_i(x, y_i)
\]

for every \(x, y_i \in X_i, i, k = 1, \ldots, m\), and the numbers

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fulfill the conditions

\[(5.2.7) \quad \varepsilon_n^{(i)} > 0, \quad i = 1, \ldots, m; \quad a_i > 0, \quad m = 1, \ldots, m\]

where \(\varepsilon_n^{(i)}\) are defined by (1.4), then the system of equations

\[(5.2.8) \quad x_i = T_i(x(1,m)) \quad i = 1, \ldots, m\]

has exactly one solution \(x_1, \ldots, x_m\) such that \(x_i \in X_i, i = 1, \ldots, m\). For any arbitrary fixed \(x \in X_i, i = 1, \ldots, m\) the sequence of successive approximations

\[(5.2.9) \quad x^{p+1}_i = T_i(x^p(1,m)) \quad p = 0, 1, \ldots, i = 1, \ldots, m\]

converges and

\[(5.2.10) \quad x_i = \lim_{p \to \infty} x_i^p, \quad i = 1, \ldots, m\]

5.3. In this section we prove a common fixed point theorem which improves the result of Singh and Gairola [186].

First of all we state the following Lemma, which is a direct consequence of Singh and Gairola’s [186] result.

**Lemma 5.1.** Let \((X, d)\) be a metric space and \(P, Q, S_i, T_i : X \to X, i = 1, \ldots, m\) such that

\[(5.3.1) \quad P(X) \cup Q(X) \subseteq S(X) \cap T(X)\]

and \(S(X) \cap T(X)\) is a complete subspace of \(X_i, i = 1, \ldots, m\). If there exists non-negative numbers \(b, a_i, i, k = 1, \ldots, m\) such that (5.2.4), (5.2.6), (5.2.7) and the following hold

\[(5.3.2) \quad 0 \leq b < 1-h \quad \text{and} \quad h = \max_i r_i \quad \text{s.t.} \quad r_i = \frac{\varepsilon_i}{\sum_{i=1}^{m} \varepsilon_i}\]

\[(5.3.3) \quad d(P(x(1,m)), Q(y(1,m))) \leq \max_i \sum_{i=1}^{m} a_i d_i(S_i(x(1,m)), T_i(y(1,m))) + b \max_i d_i(S_i(x(1,m)), P_i(x(1,m))) + d_i(T_i(y(1,m)), Q_i(y(1,m))) + \frac{b}{2} \max_i d_i(S_i(x(1,m)), Q_i(y(1,m))) + d_i(T_i(y(1,m)), P_i(x(1,m))))\]

for all \(x(1,m), y(1,m)\) in \(X\) then the sequence \(z \to z^n\) such that for some \(x_n \in X_i\),
Theorem 5.3. Let $(X, d)$ be a metric space and $P_i, Q_i, S_i, T_i : X \to X$, $i = 1, ..., m$, satisfy all conditions of Lemma 5.1. If there exists non-negative numbers $b, a_k, i, k = 1, 2, ..., m$ such that (5.2.4), (5.2.6), (5.2.7) and (5.2.8) hold and if the system of maps $(P_1, ..., P_m)$ and $(S_1, ..., S_m)$ as well as $(Q_1, ..., Q_m)$ and $(T_1, ..., T_m)$ are coordinatewise compatible then the system of equations

\[(5.3.5) \quad P_1(x(1, m)) = Q_1(x(1, m)) \quad S_1(x(1, m)) = T_1(x(1, m))\]

has a unique common solution $x_1, ..., x_m$ such that $x_i \in X$, $i = 1, ..., m$.

**Proof.** From the above lemma we see that the sequence $x^p$ defined by (5.3.4) converges to some $u_i$ in $S_i(X) \cap T_i(X)$, $i = 1, ..., m$.

Let $v(1, m)$ be an element of $S_i u_i$. Then $S_i(v(1, m)) = u_i$, $i = 1, ..., m$. Now proceeding as in the proof of Theorem 2 of Singh and Gairola [186] we see that as $p \to \infty$

\[d(S_i(v(1, m)), P_i(v(1, m))) \leq b d(S_i(v(1, m)), P_i(v(1, m)))\]

which is a contradiction, hence $S_i(v(1, m)) = P_i(v(1, m)) = u_i$, $i = 1, ..., m$. Similarly, there exists a point $w(1, m)$ in $T_i u_i$ such that $T_i(w(1, m)) = Q_i(w(1, m)) = u_i, i = 1, ..., m$.

Now since the system of maps $(P_1, ..., P_m)$ and $(S_1, ..., S_m)$ are coordinatewise compatible we get

\[d(P_i(S_i(v(1, m))), ..., S_i(v(1, m))) = S_i(P_i(v(1, m)), ..., P_i(v(1, m))) = 0\]

Therefore, $P_i(u(1, m)) = S_i(u(1, m))$.

Similarly we see that since the systems $(Q_1, ..., Q_m)$ and $(T_1, ..., T_m)$ are coordinatewise compatible $Q_i(u(1, m)) = T_i(u(1, m))$.

Rest of the proof follows from that of Singh and Gairola [186].

The following example illustrates the superiority of our main theorem over that of Singh and Gairola [186].

Example 5.2. Let $X_1 = [0, 1]$, $X_2 = [0, 1]$, and $X \times X \times X$. Let $P_i, S_i, Q_i, T_i : X \to X$ be defined as follows,

\[
P_i(x, y, z) = 3x, \quad P_i(x, y, z) = x/16, \quad S_i(x, y, z) = 15x, \quad S_i(x, y, z) = x, \quad Q_i(x, y, z) = x/16, \quad Q_i(x, y, z) = x/8, \quad T_i(x, y, z) = x/2, \quad T_i(x, y, z) = x/2.
\]

Now,

\[d(P_i(x, y, z), Q_i(x, y, z)) = ||3x - y/16|| + 15 ||15x - y/16|| \leq 2 \cdot 15 ||15x - y/16|| \leq 2 \cdot 15 (||15x - y/16|| - y/16) \leq 15 (||15x - y/16|| - y/16) \leq 125 (||15x - y/16|| - y/16) \leq 125 (||15x - y/16|| - y/16)
\]
we see that condition (5.3.3) of our main theorem is satisfied. Also it can be easily verified that the system of maps \((P,P)\) and \((S,S)\) are coordinatewise compatible, \((Q,Q)\) and \((T,T)\) are coordinatewise compatible. Hence our theorem ensures the existence of unique common solution \((0,0)\) of the equation

\[(5.3.6) P(x,y) = Q(x,y) - S(x,y) - T(x,y) \quad 0 \text{ for } i = 1,2\]

On the other hand we see that the system of maps \((P,P)\) and \((S,S)\) are not coordinatewise weakly commuting and hence Theorem 2 of Singh and Gairola [186] does not ensure the existence of any solution of (5.3.6).

**Theorem 5.4.** Let \((X,d)\) be a complete metric space and \(P,Q,S,T: X \to X, i = 1, \ldots, m\). If there exist nonnegative numbers \(b\) and \(a_k\) \(i, k = 1, \ldots, n\) such that (5.2.4), (5.2.6), (5.2.7), (5.3.2), (5.3.3) and the following hold

\[(5.3.7) \quad P(X) \subseteq T(X), Q(X) \subseteq S(X), i = 1, \ldots, m,\]

\[(5.3.8) \quad S_i \text{ or } T_i \text{ is continuous, } i = 1, \ldots, m\]

and if the system of maps \((P_1, P_2)\) and \((S_1, S_2)\) as well as \((Q_1, Q_2)\) and \((T_1, T_2)\) are coordinatewise compatible then the system of equations (5.3.5) has a unique common solution \(x_1, \ldots, x_m\) such that \(x_i \in X_i, i = 1, \ldots, m\).

**Proof.** Proof follows from that of Theorem 5.2, [189, Th.3] and [87, Th. 3.2].

**Remark 1.** If we assume \((M,d) = (X,d)\), \(P = P_1, Q = Q_2, T = T_3, \) and \(S = S_2, i = 1, \ldots, m\) and \(i = 1, \ldots, m\), then (5.3.3) with \(a_{ii} = k\) may be written as

\[
d(Px,Qy) \leq k \max(d(Sx,Ty), b \max(d(Sx,Px), d(Ty,Qy)) \leq \frac{1}{2} (d(Sx,Qy) + d(Ty,Px)) \),\]

if \(x,y \in X\).

This inequality may also be written as

\[(5.3.9) \quad d(Px,Qy) \leq \beta \max(d(Sx,Ty), d(Sx,Px), d(Ty,Qy)),\]

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\[ \frac{1}{2} (d(Sx, Qy) + d(Ty, Px)) \]

where \( x, y \in X \) and \( 0 < \beta = \max(k, h) < 1 \), which is the same contractive condition of Jungck [90]. Consequently, [Th 3.1, 90] follows as corollary.

**Remark 2.** Many fixed point theorems generalize the Jungck contraction principle have been established under condition (5.3.9) and its particular cases (see, [32], [35], [56], [69], [87], [108], [160], [168], [169], [190], [199], [204], [209]).

In particular, Kubaik's main result [108] proved exactly under condition (5.3.9) uses the commutativity of \( P \) with \( S \) and \( Q \) with \( T \).
6.1. The notion of probabilistic metric spaces was introduced and studied by K. Menger [118], which is a generalisation of metric space, and the study of these spaces was expanded rapidly with the pioneering works of B. Schweizer and A. Sklar [150], [164]. The theory of probabilistic metric space is of fundamental importance in probabilistic functional analysis. For details we refer to [50], [70], [166], [175] - [177], [199] and [200].

Some fixed point theorems in probabilistic metric spaces have been proved by Bharucha and Reid [5], Bocsan [20], Chang [28], Ciric [34], Hadzic [67], [68], [71], Hicks [72], Singh and Punt [190] - [192], Stojakovic [200] - [202], Tian [203] and many others [222], [242], [72], [119] and [207]. Fixed point theorems in Menger spaces were proved by Cho et al [31], Dedeic and Sarappa [42], Radu [149] - [151], Stojakovic [200] - [202] and many others.

The existence of fixed points for compatible mappings in metric spaces and probabilistic metric spaces is shown by Jungck ([87], [89], [90]), Mishra [119] and Sessa et al [170].

Recently Jungck et al [92] introduced the concept of compatible mappings of type (A) in metric spaces and Cho et al [31] introduced the concept of compatible mappings of type (A) in Menger spaces and proved some fixed point theorems for these mappings in Menger spaces which extend, generalise and improve many known results.

In this chapter we introduce the concept of compatible mappings of type (A-1) and type (A-2) in Menger spaces and show that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions. We also prove some common fixed point theorems which improves the result of Cho et al [31].

6.2. Let \( \mathbb{R} \) denote the set of reals and \( \mathbb{R}^+ \) be the set of non negative reals. A mapping \( F : \mathbb{R} \rightarrow \mathbb{R}^+ \) is called a distribution function if it is nondecreasing left continuous with \( \inf F = 0 \) and \( \sup F = 1 \). Let \( L \) denote the set of all distribution functions.

A probabilistic metric space (briefly PM-space) is a pair \((X, F)\) where \( X \) is a non empty set and \( F \) is a mapping from \( X \times X \) to \( L \). For any \((u, v) \in X \times X\), the distribution function
$F(u,v)$ is denoted by $F_{uv}$. The functions $F_{uv}$ are assumed to satisfy the following conditions:

(P1) $F_{uv}(x) = 1$ for all $x > 0$ if and only if $u = v$

(P2) $F_{uv}(0) = 0$ for all $u, v \in X$

(P3) $F_{uv}(x) = F_{vu}(x)$ for all $u, v \in X$

(P4) If $F_{uv}(x) = 1$ and $F_{vw}(y) = 1$, then $F_{uw}(x+y) = 1$ for all $u, v, w \in X$.

A function $t : [0,1] \times [0,1] \rightarrow [0,1]$ is called a $T$-norm if it satisfies the following conditions:

(t1) $t(a, 1) = a$ for all $a \in [0,1]$ and $t(0, 0) = 0$.

(t2) $t(a, b) = t(b, a)$ for all $a, b \in [0, 1]$

(t3) If $c \geq a$ and $d \geq b$, then $t(c, d) \geq t(a, b)$.

(t4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c \in [0, 1]$.

A Menger space is a triplet $(X, F, t)$ where $(X, F)$ is a PM-space and $t$ is a $T$-norm with the following condition:

(P5) $F_{uv}(x + y) \geq t(F_{uv}(x), F_{vw}(y))$ for all $u, v, w \in X$ and $x, y \in R^+$.

We require the following well known definitions and theorems ([22]):

**Definition 6.1.** Let $(X, F, t)$ be a menger space. A mapping $S$ from $X$ into itself is said to be continuous at a point $p \in X$, if for all $\varepsilon > 0$ and $\lambda > 0$, there exists $\delta > 0$ such that if $q \in U_p(\delta, \lambda)$, then $S_q \in U_p(\varepsilon, \lambda)$, that is, if $F_{pq}(\delta) > 1 - \lambda$, then $F_{sp}(\varepsilon) > 1 - \lambda$.

**Definition 6.2.** Let $(X, F, t)$ be a menger space with the continuous $T$-norm $t$. A sequence $(p_n)$ in $X$ is said to be convergent to a point $p \in X$, if for all $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $p_n \in U_p(\varepsilon, \lambda)$ for all $n \geq N$ i.e., $F_{pn}(\varepsilon) > 1 - \lambda$ for all $n \geq N$. 

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Theorem 6.1. Let \((X, F, t)\) be a Menger space with the continuous T-norm \(t\) and \(S\) be a mapping from \(X\) into itself. Then \(S\) is continuous at a point \(p\) if and only if for every sequence \(\{p_n\}\) converging to \(p\), the sequence \(\{S(p_n)\}\) converges to the point \(S(p)\).

Theorem 6.2. Let \((X, F, t)\) be a Menger space with the continuous T-norm \(t\). Then \(F\) is a lower semicontinuous function of points in \(X\), that is, for any fixed \(x \in X\), if \(q_n \rightarrow q\) and \(p_n \rightarrow p\) as \(n \rightarrow \infty\), then \(\lim_{n \rightarrow \infty} \inf F_{q_n, p_n}(x) = F_{q, p}(x)\).

Definition 6.3. Let \((X, F, t)\) be a Menger space with the continuous T-norm \(t\). A sequence \(\{p_n\}\) in \(X\) is said to be a Cauchy sequence, if for all \(\varepsilon > 0\) and \(\lambda > 0\), there exists an integer \(N = N(\varepsilon, \lambda)\) such that \(F_{p_n, p_m}(\varepsilon) > 1 - \lambda\) for all \(m, n \geq N\).

Definition 6.4. A Menger space \((X, F, t)\) with continuous T-norm \(t\) is said to be complete if every Cauchy sequence in \(X\) converges to a point in \(X\).

The following theorem establishes the relations between metric spaces and Menger spaces.

Theorem 6.3. Let \(t\) be a T-norm defined by \(t(a, b) = \min\{a, b\}\). Then the induced Menger space \((X, F, t)\) is complete if a metric space \((X, d)\) is complete.

Theorem 6.4. Let \((X, F, t)\) be an induced Menger space by the metric \(d\). Let \(\{p_n\}\) be a sequence in \(X\) and \(S\) be a mapping from \(X\) into itself. Then for all \(\varepsilon > 0\) and \(\lambda > 0\), \(F_{p_n, p_m}(\varepsilon) > 1 - \lambda\) if and only if there exists an integer \(N\) such that \(d(p_n, p_m) < \varepsilon\) for all \(n, m \geq N\), and \(S\) is continuous at \(p\) in the sense of Menger space if and only if \(S\) is continuous at \(p\) in the sense of the metric space.

6.3. In this section we show that compatible mappings of type (A-1) and type (A-2) in Menger space are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions.

Definition 6.5[24]: Let \((X, F, t)\) be a Menger space such that the T-norm \(t\) is continuous and \(S, T\) be self maps of \(X\). The mappings \(S\) and \(T\) are said to be compatible if \(\lim_{n \rightarrow \infty} F_{S(x), T(x)} = 1\) for all \(x > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t\) for some \(t\) in \(X\).

Definition 6.6[31]: Let \((X, F, t)\) be a Menger space such that the T-norm \(t\) is continuous and \(S, T\) be self maps of \(X\). The mappings \(S\) and \(T\) are said to be compatible of type (A) if \(\lim_{n \rightarrow \infty} F_{S(x), T(x)} = 1\) and \(\lim_{n \rightarrow \infty} F_{1S(x), Sx_n} = 1\) for all \(x > 0\) whenever...
{x_n} is a sequence in X such that \( \lim_{n \to \infty} S x_n \neq \lim_{n \to \infty} T x_n = t \) for some \( t \) in X.

**Definition 6.7:** Let \((X,F,t)\) be a Menger space such that the T-norm \( t \) is continuous and \( S, T \) be self maps of \( X \). The pair of mappings \((S,T)\) is said to be compatible of type (A-1) if \( \lim_{n \to \infty} F_{\geq 1} S_{\geq 1} T (x) = 1 \) for all \( x > 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = t \) for some \( t \) in X.

**Definition 6.8:** Let \((X,F,t)\) be a Menger space such that the T-norm \( t \) is continuous and \( S, T \) be self maps of \( X \). The pair of mappings \((S,T)\) is said to be compatible of type (A-2) if \( \lim_{n \to \infty} F_{\geq 1} S_{\geq 1} T (x) = 1 \) for all \( x > 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = t \) for some \( t \) in X.

Clearly if a pair of mappings \((S,T)\) is compatible of type (A-1) then the pair \((T,S)\) is compatible of type (A-2). Further from the definitions it clear that if \( S \) and \( T \) compatible mappings of type (A) then the pair \((S,T)\) is compatible of type (A-1) as well as type (A-2).

We now cite the following propositions which gives the condition under which definitions 6.5, 6.6, 6.7 and 6.8 becomes equivalent.

**Proposition 6.1:** Let \((X,F,t)\) be a Menger space such that the T-norm \( t \) is continuous and \( t(x,x) > x \) for all \( x \in [0,1] \) and \( S, T \) be self maps of \( X \).

(a) If \( T \) is continuous then the pair of mappings \((S,T)\) is compatible of type (A-1) iff \( S \) and \( T \) are compatible.

(b) If \( S \) is continuous then the pair of mappings \((S,T)\) is compatible of type (A-2) iff \( S \) and \( T \) are compatible.

(c) If \( S \) and \( T \) are continuous then the pair \((S,T)\) is compatible of type (A-1) iff the pair \((S,T)\) is compatible of type (A-2).

**Proof:** (a) Let \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T x_n = t \) for some \( t \) in \( X \), and let the pair \((S,T)\) be compatible of type (A-1). Since \( T \) is continuous we have \( \lim_{n \to \infty} T S x_n = T t \) and \( \lim_{n \to \infty} T T x_n = T t \) and so for positive reals \( \varepsilon, \lambda \) there exists an integer \( M(\varepsilon, \lambda) \) such that \( F_{\geq 1} S_{\geq 1} T (x) > 1 - \lambda \) and \( F_{\geq 1} T_{\geq 1} S (x) > 1 - \lambda \) for all \( n \geq M(\varepsilon, \lambda) \).

Hence \( F_{\geq 1} S_{\geq 1} T (x) \geq t(F_{\geq 1} S_{\geq 1} T , F_{\geq 1} T_{\geq 1} S (x)) \). Hence \( \lim_{n \to \infty} F_{\geq 1} S_{\geq 1} T (x) = 1 \).

Now let \( S \) and \( T \) be compatible.

Then we have \( F_{\geq 1} S_{\geq 1} T (x) \geq t(F_{\geq 1} S_{\geq 1} T , F_{\geq 1} T_{\geq 1} S (x)) \).
Let \( \lim n \rightarrow \infty Sx_{n} = \lim n \rightarrow \infty Tx_{n} = t \) for some \( t \) in \( X \), and let the pair \((S, T)\) be compatible of type (A-2). Since \( S \) is continuous we have \( \lim n \rightarrow \infty STx_{n} = St \) and \( \lim n \rightarrow \infty SSx_{n} = St \).

Hence \( F_{STx_{n}}(\epsilon) \geq \epsilon \) \( F_{Sx_{n}}, F_{Tx_{n}} \) \( F_{x_{n}} \). Hence \( \lim n \rightarrow \infty F_{STx_{n}}(\epsilon) = 1 \).

Now let \( S \) and \( T \) be compatible. Then we have \( F_{SSx_{n}}(\epsilon) \geq \epsilon \) \( F_{Sx_{n}}, F_{x_{n}} \). Hence \( \lim n \rightarrow \infty F_{SSx_{n}}(\epsilon) = 1 \).

(c) The proof follows from (a) and (b). As a direct consequence of proposition 2.1 we have the following

**Proposition 6.2.** Let \((X, F, t)\) be a Menger space such that the T-norm \( t \) is continuous and \( t(x, x) > x \) for all \( x \in [0, 1] \) and \( S, T \) be self maps of \( X \). If \( S \) and \( T \) are continuous then the following statements are equivalent.

(a). The pair \((S, T)\) is compatible of type (A-1).
(b). The pair \((S, T)\) is compatible of type (A-2).
(c). The mappings \( S \) and \( T \) are compatible of type (A).
(d). The mappings \( S \) and \( T \) are compatible.

Next we give some properties of compatible mappings of type (A-1) and type (A-2) which will be used in our main theorem.

**Proposition 6.3.** Let \((X, F, t)\) be a Menger space such that the T-norm \( t \) is continuous and \( t(x, x) \geq x \) for all \( x \in [0, 1] \) and \( S, T \) be self maps of \( X \). If the pair \((S, T)\) are compatible of type (A-1) and \( Sz = Tz \) for some \( z \) in \( X \) then \( STz = TTz \).

**Proof:** Let \( \{x_{n}\} \) be a sequence in \( X \) defined by \( x_{n} = z \) for \( n = 1, 2, \ldots \) and let \( Tz = Sz \). Then we have \( \lim n \rightarrow \infty Sx_{n} = Sz \) and \( \lim n \rightarrow \infty Tx_{n} = Tz \). Since the pair \((S, T)\) is compatible of type (A-1) we have

\[
F_{STz}(\epsilon) = \lim n \rightarrow \infty F_{Sx_{n}Tz}(\epsilon) = 1 \quad \text{Hence} \quad STz = TTz.
\]

**Proposition 6.4:** Let \((X, F, t)\) be a Menger space such that the T-norm \( t \) is continuous and \( t(x, x) \geq x \) for all \( x \in [0, 1] \) and \( S, T \) be self maps of \( X \). If the pair \((S, T)\) is compatible of type (A-2) and \( Sz = Tz \) for some \( z \) in \( X \) then \( TSz = SSz \).
Proposition 6.5: Let \((X,F,t)\) be a Menger space such that the T-norm \(t\) is continuous and \(t(x,x) > x\) for all \(x \in [0,1]\) and \(S, T\) be self maps of \(X\). If the pair \((S, T)\) is compatible of type \((A-1)\) and \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z\) for some \(z \in X\) then

(a) \(\lim_{n \to \infty} STx_n = Tz\) if \(T\) is continuous
(b) \(\lim_{n \to \infty} TTx_n = Sz\) if \(S\) is continuous.

Proof
(a) Let \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\) for some \(t \in X\), and let the pair \((S, T)\) be compatible of type \((A-1)\). Since \(T\) is continuous we have \(\lim_{n \to \infty} TTx_n = Tt\) and so for positive reals \(\varepsilon, \lambda\) there exists an integer \(M(\varepsilon, \lambda)\) such that \(F_{1,1,1}(T(t/2)) > 1 - \lambda\) and \(F_{1,1,1}(T(t/2)) > 1 - \lambda\) for all \(n > M(\varepsilon, \lambda)\). Hence for all \(\varepsilon > 0\) we have

\[
F_{1,1,1}(\varepsilon) > t(F_{1,1,1}(\varepsilon/2), F_{1,1,1}(\varepsilon/2)) > 1 - \lambda
\]

for all \(n \geq M(\varepsilon, \lambda)\) which means that \(STx_n \to Tz\) as \(n \to \infty\).

(b) Proof follows on similar lines as argued in (a). As a direct consequence of Proposition (2.5) we have the following.

Proposition 6.6: Let \((X,F,t)\) be a Menger space such that the T-norm \(t\) is continuous and \(t(x,x) > x\) for all \(x \in [0,1]\) and \(S, T\) be self maps of \(X\). If the pair \((S, T)\) is compatible of type \((A-2)\) and \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z\) for some \(z \in X\) then

(a) \(\lim_{n \to \infty} SSx_n = Tz\) if \(T\) is continuous
(b) \(\lim_{n \to \infty} TSx_n = Sz\) if \(S\) is continuous.

6.4. In this section we prove a common fixed point theorem in menger space which improves the results of Cho et al [31] and many others. We require the following Lemmas ([163] and [192]) for our main theorem.

Lemma 6.1. Let sequence \(\{x_n\}\) be a sequence in a menger space \((X,F,t)\), where \(t\) is a continuous T-norm and \(t(x,x) > x\) or all \(x \in [0,1]\). If there exists a constant \(k \in (0,1]\) such that \(F_{x_n,x_n}(kx) \leq F_{x_n,x_n}(x)\) for all \(x > 0\) and \(n \geq N\) then sequence \(\{x_n\}\) is a cauchy sequence in \(X\).

Now we are ready to prove our main theorem.
Theorem 6.5 Let \((X, F, t)\) be a complete Menger space with \(t(x, y) = \min(x, y)\) for all \(x, y \in (0, 1)\) and \(A, B, S\) and \(T\) be self maps of \(X\) such that

\[(6.4.1) A(X) \subseteq T(X) \quad \text{and} \quad B(X) \subseteq S(X)\]

(6.4.2) the pairs \((A, S)\) and \((B, T)\) are compatible of type \((A-1)\) or type \((A-2)\)

\[(6.4.3) \left[F_{\alpha, 1}(kx) \right] \geq \min\{F_{\alpha, 1}(x), F_{\alpha, A}(x), F_{\alpha, 1}(x), F_{\alpha, A}(x), F_{\alpha, 1}(2x), F_{\alpha, A}(2x), F_{\alpha, 1}(2x), F_{\alpha, A}(2x)\}\]

\[(6.4.4) \text{One of } A, B, S \text{ and } T \text{ is continuous}\]

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof: Since \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\) for some arbitrary \(x_0\) in \(X\), we choose \(x_1\) in \(X\) such that \(Ax_0 = Tx_1\), and for this \(x_1\) there exists \(x_2\) such that \(Sx_2 = Bx_1\).

Continuing this process we define the sequence \(\{y_n\}\) in \(X\) such that

\[(6.4.5) \quad y_{2n} = Ax_{2n} = Tx_{2n-1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n}\]

Now we shall prove that \(F_{y_{2n+1}}(2kx) \geq F_{y_{2n+1}}(y_{2n+1}(x))\) for all \(x > 0\) where \(k \in (0, 1)\). Suppose \(F_{y_{2n+1}}(2kx) < F_{y_{2n+1}}(y_{2n+1}(x))\). Then by using (6.4.3) and \(F_{y_{2n+1}}(2kx) \leq F_{y_{2n+1}}(y_{2n+1}(x))\), we have

\[F_{y_{2n+1}}(2kx) \geq \min\{F_{y_{2n+1}}(y_{2n}(x)), F_{y_{2n+1}}(y_{2n+1}(x)), F_{y_{2n+1}}(y_{2n}(x)), F_{y_{2n+1}}(y_{2n+1}(x)), F_{y_{2n+1}}(y_{2n}(x)), F_{y_{2n+1}}(y_{2n+1}(x))\}\]
\[
F_{y_n(y_n(x))} = [F_{y_n(y_n(x))}^{(kx)}]^2
\]
which is a contradiction. Hence \(F_{y_n(y_n(x))}^{(kx)} \neq F_{y_n(y_n(x))}^{(kx)}\) for all \(n \geq N\).

Similarly, we also have \(F_{y_n(y_n(x))}^{(kx)} \neq F_{y_n(y_n(x))}^{(kx)}\). Hence by Lemma 6.1 \(\{y_n\}\) is a Cauchy sequence in \(X\). Since the Menger space \((X,F,t)\) is complete, the sequence \(\{y_n\}\) converges to a point \(z\) in \(X\), and consequently the subsequences \(\{Ax_{k_n}\}, \{Bx_{k_n}\}, \{Sx_{k_n}\}\) and \(\{Tx_{k_n}\}\) of \(\{y_n\}\) also converge to the same point \(z\).

Now suppose the pairs \((A,S)\) and \((B,T)\) are compatible of type \((A^{-1})\) and let \(A\) be continuous. By proposition (6.5) it follows that

\[
ASx_n \text{ and } SSx_n \text{ converge to } Az \text{ as } n \to \infty. \quad \text{By (6.4.3) we have}
\]

\[
[F_{ASx_n}^{(kx)}] \geq \min\{F_{ASx_n}^{(kx)}, F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \}
\]

as \(n \to \infty\) we have

\[
[F_{Az}^{(kx)}] \geq \min\{F_{Az}^{(kx)} \cdot F_{Az}^{(kx)} \cdot F_{Az}^{(kx)} \cdot F_{Az}^{(kx)} \cdot F_{Az}^{(kx)} \}
\]

which is a contradiction. Hence \(Az = z\). Since \(A(X) \subseteq T(X)\) there exists a point \(p\) in \(X\) such that \(z = Az = Tp\). Again by (6.4.3) we have

\[
[F_{ASx_n}^{(kx)}] \geq \min\{F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \cdot F_{ASx_n}^{(kx)} \}
\]

as \(n \to \infty\) we get.
\[
\{F_{z,lp}(kx)\}^2 \geq \min\{\{F_{Az,lp}(x)\}^2, F_{Az,lp}(x) F_{Az,lp}(x) F_{Az,lp}(x), F_{Az,lp}(x) F_{Az,lp}(x) F_{Az,lp}(x), F_{Az,lp}(x) F_{Az,lp}(x) F_{Az,lp}(x)\}.
\]

Since \(B(X) \subseteq S(X)\), there exists a point \(q\) in \(X\) such that \(z = Bz = Sq\). By (6.4.3) we have

\[
\{F_{z,Az}(kx)\}^2 = \{F_{z,Az}(kx)\}^2 \geq \min\{\{F_{z,Az}(x)\}^2, F_{z,Az}(x) F_{z,Az}(x) F_{z,Az}(x), F_{z,Az}(x) F_{z,Az}(x) F_{z,Az}(x), F_{z,Az}(x) F_{z,Az}(x) F_{z,Az}(x)\}.
\]

As we have

\[
\{F_{z,Az}(kx)\}^2 = \{F_{z,Az}(kx)\}^2 \geq \min\{\{F_{z,Az}(x)\}^2, F_{z,Az}(x) F_{z,Az}(x) F_{z,Az}(x), F_{z,Az}(x) F_{z,Az}(x) F_{z,Az}(x), F_{z,Az}(x) F_{z,Az}(x) F_{z,Az}(x)\}.
\]

So that \(z = Az\). Since the pair \((A,S)\) are
compatible of type \((A-1)\) by proposition \((6.3)\) we have \(\text{AS} = \text{SS}^q\), i.e. \(\text{Az} = \text{Sz}\). Hence \(z\) is a common fixed point of \(A, B, S\) and \(T\). Now suppose \(S\) is continuous. By proposition \((6.5)\) we have \(\text{AS}_n \to \text{Sz}\) and also \(\text{SS}_n \to \text{Sz}\) By \((6.4.3)\) we have

\[
[F_{\text{AS}n, \text{Sz}}(kx)]^2 \geq \min\{ [F_{\text{SS}n, \text{Sz}}(x)], F_{\text{AS}, \text{Sz}}(x), F_{\text{AS}n, \text{Sz}}(x) \}
\]

\[
F_{\text{SS}n, \text{Sz}}(x) = F_{\text{AS}, \text{Sz}}(x) = F_{\text{AS}n, \text{Sz}}(x)
\]

as \(n \to \infty\) we get

\[
[F_{\text{Sz}, \text{Sz}}(kx)]^2 \geq \min\{ [F_{\text{Sz}, \text{Sz}}(x)], F_{\text{Sz}, \text{Sz}}(x), F_{\text{Sz}, \text{Sz}}(x) \}
\]

\[
F_{\text{Sz}, \text{Sz}}(x) = F_{\text{Sz}, \text{Sz}}(x) = F_{\text{Sz}, \text{Sz}}(x)
\]

which is a contradiction. Hence \(z = \text{Sz}\). Again by \((6.4.3)\)

\[
[F_{\text{Az}, \text{Sz}}(kx)]^2 \geq [F_{\text{Az}, \text{Sz}}(x)]^2
\]

which is a contradiction. Hence \(z = \text{Az}\). Now since \(A(X) \subseteq T(X)\) there exists a point \(p'\) in \(X\) such that \(z = \text{Az} = Tp'\). By \((6.4.3)\) we have \([F_{\text{T}, \text{Tp'}}(kx)]^2\)

\[
[F_{\text{Az}, \text{Bp'}}(kx)]^2 \geq [F_{\text{Az}, \text{Bp'}}(x)]^2
\]

which means that \(z = \text{Bp'}\).
Rest of the proof follows on a similar line as before. Similarly it can be shown that \( z \) is a common fixed point of \( A, B, S \) and \( T \) if \( B \) or \( T \) is continuous. Proof in case the pair of mappings \( (A, S) \) and \( (B, T) \) are compatible of type (A-2) follows similarly and the proof of uniqueness follows easily from (6.4.3).

As a consequence of Theorem 6.3 and Theorem 6.5 we have the following

**Theorem 6.6** Let \( A, B, S \) and \( T \) be mappings from a complete metric space \( (X,d) \) into itself such that

\[
(A) \quad A(X) \subseteq T(X) \quad \text{and} \quad B(X) \subseteq S(X),
\]
\[
(B) \quad \text{The pairs} \ (A, S) \text{ and} \ (B, T) \text{ are compatible of type} \ (A-1) \text{ or type} \ (A-2)
\]
\[
(C) \quad \text{one of} \ A, B, S \text{ and} \ T \text{ is continuous and}
\]
\[
(D) \quad \left| d(Au, Bv) \leq k \max \{d(Su, Tv), d(Su, Au), d(Tv, Bv), d(Su, Tv), d(Su, Au),ight.
\]
\[
\left. d(Tv, Bv), d(Su, Tv), d(Su, Bv), d(Tv, Au), d(Su, Bv), d(Tv, Au), d(Su, Bv), d(Tv, Bv) \right]\}
\]

for all \( u, v \) in \( X \), and \( k \in (0, 1) \). Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Remark 1** Theorems 6.5 and 6.6 generalize and improves the results of Cho et al. [31].


7.1 In this chapter, inspired from a recent work of Camargo [23], we obtain the locally convex version of a fixed point theorem of Naimpally and Singh [128]. Our main result is as follows.

**Theorem 7.1.** Let $X$ be a locally convex Hausdorff topological vector space, whose topology is generated by a family $U$ of continuous seminorms. Let $M$ be a nonempty sequentially complete subset of $X$, and $T_1$ and $T_2$ be two self maps of $M$ such that for each $P \in U$ and $x,y \in M$, there exists a number $a(p)$, satisfying any one of the following conditions:

I. For $0 \leq a(p) < 1/2$ at least one of the following conditions hold.
   
   (A) $P(T_1 x - T_2 y) \leq a(p). \max \{ P(x - y), P(x - T_1 x), P(y - T_2 y), P(x - T_2 y), P(y - T_1 x) \}$
   
   (B) $P(T_1 x - T_2 y) \leq a(p). \max \{ P(x - y), P(x - T_1 x), P(y - T_2 y), [P(x - T_2 y) + P(y - T_1 x)] / 2 \}$
   
   (C) $P(T_1 x - T_2 y) \leq a(p). \max \{ P(x - y), P(x - T_1 x), P(y - T_2 y), P(x - T_2 y) + P(y - T_1 x) \}$
   
   (D) $P(T_1 x - T_2 y) \leq a(p). \max \{ P(x - y), P(x - T_1 x) + P(y - T_2 y), P(x - T_2 y) + P(y - T_1 x) \}$

II. $P(x - T_1 x) + P(y - T_2 y) + P(T_1 x - T_2 y) \leq a(p). \max \{ P(x - T_2 y) + P(y - T_1 x), 1 \} \leq a(p) < 3/2$ ;

III. $P(x - T_1 x) + P(y - T_2 y) \leq a(p). \max \{ P(x - T_2 y) + P(y - T_1 x), P(x - y) \}$

IV. $P(x - T_1 x) + P(y - T_2 y) \leq a(p). \max \{ P(x - T_2 y) + P(y - T_1 x), 1/2 \} \leq a(p) < 2$.

Then $T_1$ and $T_2$ has a unique common fixed point.

**Proof:** For some arbitrary $x_0$ in $M$, consider the sequence $<x_n>$ defined by $x_{n+1} = T_1 x_n$ for $n = 0, 2, 4, \ldots$ and $x_{n+1} = T_2 x_n$ for $n = 1, 3, 5, \ldots$. We now claim that $<x_n>$ is a cauchy sequence. Case I. $T_1$ and $T_2$ satisfy condition I(D).

For some arbitrary $P$ in $U$ we have,

$P(x_{n+1} - x_n) = P(T_1 x_n - T_2 x_n)$

\[ \leq a(p). \max \{ P(x_0 - x_1), P(x_n - T_1 x_n), P(x_{n+1} - T_2 x_n) \} + P(x_0 - T_1 x_0) \]

$P(x_0 - T_2 x_0) + P(x_{n+1} - T_1 x_0)$

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\[ a(p) \max \{ P(x_0 - x_1), P(x_0 - x_2) + P(x_1 - x_2), P(x_0 - x_1) + P(x_1 - x_2) \} \leq a(p) \{ P(x_0 - x_1) + P(x_1 - x_2) \}. \]

i.e, \( P(x_1 - x_2) \leq (a(p)/1-a(p))) \cdot P(x_0 - T_1 x_2) \)

Similarly, \( P(x_2 - x_3) \leq (a(p)/1-a(p))) \cdot P(x_1 - T_2 x_3) \)

Continuing this process we have

\[ P(x_n - x_{n+1}) \leq (a(p)/1-a(p))) \cdot P(x_0 - T_{n} x_n). \]

Now for any positive integer \( r \) we have

\[ P(x_{n+r} - x_{n+r+1}) \leq P(x_n - x_{n+1}) + P(x_{n+1} - x_{n+2}) + \ldots + P(x_{n+r-1} - x_{n+r}) \]

\[ \leq h^r/(1-h) \cdot P(x_n - T_{n} x_n) \text{ where } h = a(p)/(1-a(p)). \]

Also \( 0 \leq a(p) < 1/2 \) implies \( 0 < h < 1 \)

Hence we see that \( P(x_n - x_{n+r}) \rightarrow 0 \) as \( n \rightarrow \infty \). Therefore \( \langle x_n \rangle \) is a cauchy sequence

Case 2. \( T_1 \) and \( T_2 \) satisfy condition II.

\[ P(x_1 - x_2) = P(T_0 x_0 - T_1 x_1) \]

\[ \leq a(p) \{ P(x_0 - T_1 x_1) + P(x_0 - T_0 x_0) \cdot P(x_1 - T_1 x_1) \}

\[ \leq a(p) \cdot P(x_0 - x_1) \cdot P(x_1 - x_2) \cdot P(x_0 - x_1). \]

Hence \( P(x_1 - x_2) \leq \frac{a(p) \cdot 1}{2 - a(p)} \cdot P(x_0 - T_1 x_1) \)

If we take \( (a(p) - 1)/(2 - a(p)) = h \), we get \( P(x_1 - x_2) \leq h \cdot P(x_0 - T_1 x_1) \). Similarly we get

\[ P(x_2 - x_3) \leq h \cdot P(x_1 - x_2) \leq h^2 \cdot P(x_0 - T_1 x_1) \]

Continuing this process \( n \) times we get \( P(x_n - x_{n+1}) \leq h^n \cdot P(x_0 - T_1 x_1) \).

If \( r \) is any positive integer then

\[ P(x_n - x_{n+r}) \leq P(x_n - x_{n+r-1}) + P(x_{n+r-1} - x_{n+r-2}) + \ldots + P(x_{n+r-r} - x_{n+r}) \]

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We see that \( 1 \geq a(p) \cdot \frac{3}{2} \) and so \( 0 < h < 1 \). Hence from (7.1.1) we get
\[
P(x_n - x_{n+1}) \to 0 \text{ as } n \to \infty, \text{ and so } \{x_n\} \text{ is a Cauchy sequence.}
\]

Case 3. If \( T_1 \) and \( T_2 \) satisfy condition (III)
\[
P(x_n - T_1 x_n) + P(x_n - T_2 x_n) \leq a(p) [P(x_n - T_1 x_n) + P(x_n - T_2 x_n)]
\]
i.e. \( P(x_n - x_n) + P(x_n - x_n) \leq a(p) [P(x_n - x_n) + P(x_n - x_n)] \)
and so \( P(x_n - x_n) \leq h.P(x_n - T_n x_n) \) where \( h = \frac{2 \cdot a(p) - 1}{1 - a(p)} \).

Similarly we get,
\[
P(x_n - T_1 x_n) + P(x_n - T_2 x_n) \leq a(p) [P(x_n - T_1 x_n) + P(x_n - T_2 x_n)]
\]
i.e \( P(x_n - x_n) + P(x_n - x_n) \leq a(p) [P(x_n - x_n) + P(x_n - x_n)] \)
and so \( P(x_n - x_n) \leq h.P(x_n - T_n x_n) \) where \( h = \frac{2 \cdot a(p) - 1}{1 - a(p)} \).

Continuing this process \( n \) times we get \( P(x_n - x_{n+1}) \leq h^n.P(x_n - T_n x_n) \).

If \( r \) is any positive integer then
\[
P(x_n - x_{n+r}) \leq P(x_n - x_{n+1}) + P(x_{n+1} - x_{n+2}) + \ldots + P(x_{n+r} - x_{n+r})
\]
(7.1.2) \[ \leq \frac{h^n}{(1-h)}P(x_n - T_n x_n) \]
We see that \( 1/2 \leq a(p) < 2/3 \) and so \( 0 < h < 1 \). Hence from (7.1.2) we get
\[
P(x_n - x_{n+r}) \to 0 \text{ as } n \to \infty, \text{ and so } \{x_n\} \text{ is a Cauchy sequence.}
\]

Case 4. If \( T_1 \) and \( T_2 \) satisfy condition (IV).
\[
P(x_n - T_1 x_n) + P(x_n - T_2 x_n) \leq a(p) [P(x_n - x_n)] \text{ i.e. } P(x_n - x_n) + P(x_n - x_n) \leq a(p) [P(x_n - x_n)]
\]
and so \( P(x_n - x_n) \leq h.P(x_n - T_n x_n) \) where \( h = a(p) - 1 \).

Similarly we get,
\[
P(x_n - T_1 x_n) + P(x_n - T_2 x_n) \leq a(p) [P(x_n - x_n)]
\]
i.e \( P(x_n - x_n) + P(x_n - x_n) \leq a(p) [P(x_n - x_n)] \)
and so \( P(x_n - x_n) \leq h.P(x_n - T_n x_n) \) where \( h = a(p) - 1 \).
Continuing this process \( n \) times we get
\[
P(x_n - x_{n+1}) \leq h^n.P(x_n - T_n x_n)
\]
If $r$ is any positive integer then

\[ P(x_n - x_{n+1}) \leq P(x_n - x_{n+1}) + P(x_{n+1} - x_{n+2}) + \cdots + P(x_{n+r-1} - x_n) \]

(7.1.3) \[ \leq h^n (1 - h) P(x_n - T_n x_n) \]

We see that $1 \leq a(p) < 2$ and so $0 \leq h < 1$. Hence from (7.1.3) we get $P(x_n - x_{n+1}) \to 0$ as $n \to \infty$, and so $\langle x_n \rangle$ is a Cauchy sequence. Hence in all the cases we see that the sequence $\langle x_n \rangle$ is a Cauchy sequence. But $M$ is sequentially complete and so $\langle x_n \rangle$ converges to some $x$ in $M$. Now we claim that $x$ is a common fixed point of $T_1$ and $T_2$.

Case 1. If $T_1$ and $T_2$ satisfy condition (I).

\[ P(x - T_1 x) \leq P(x - x_n) + P(T_1 x_n - T_1 x) \]

\[ \leq P(x - x_n) + a(p) \max\{P(x - x_{n-1}), P(x - T_1 x) + P(x_{n-1} - T_1 x_n)\} + P(x_{n-1} + T_1 x) \]

\[ \leq P(x - x_n) + a(p) \max\{P(x - x_{n-1}), P(x - T_1 x) + P(x_{n-1} - x_n) + P(x_{n-1} - T_1 x)\} \]

\[ \leq P(x - x_n) + a(p) \max\{P(x - x_{n-1}), P(x - T_1 x) + P(x_{n-1} - x_n) + P(x_{n-1} - x_n) + P(x_{n-1} - T_1 x)\} \]

\[ = P(x - x_n) + a(p) \max\{P(x - x_{n-1}) + P(x_{n-1} - x_n) + P(x_{n-1} - T_1 x)\}. \]

Hence we have,

(7.1.4) \[ P(x - T_1 x) \leq \left( \frac{1}{1 - a(p)} \right) \{ (1 + a(p)) P(x - x_n) + 3 a(p) P(x - x_{n+1}) \}. \]

We observe that if $\epsilon > 0$ and $P \in U$ are arbitrary then for sufficiently large values of $n$ we have

\[ P(x - x_n) < \epsilon \left( \frac{1 - a(p)}{1 + 4a(p)} \right) \text{ and } P(x - x_{n+1}) < \epsilon \left( \frac{1 - a(p)}{1 + 4a(p)} \right). \]

Hence from (7.1.4) we see that for all sufficiently large values of $n$, and arbitrary $P$ in $U$, $P(x - T_1 x) < \epsilon$. Since $\epsilon > 0$ is arbitrary we have $P(x - T_1 x) = 0$. Since $X$ is Hausdorff we conclude that $x - T_1 x = 0$, i.e. $x = T_1 x$. Similarly it can be shown that $x = T_2 x$.

Case 2. If $T_1$ and $T_2$ satisfy condition (II).

\[ 3 P(x - T_2 x) + 3 P(x - T_2 x_{n+1}) + P(T_2 x_n - T_2 x) \]
\[ 3 \cdot \text{P}(x-T_1x_1) + \text{P}(x-T_1x_2) + \text{P}(x_n - T_1x) + a(p) \{ \text{P}(x_n - T_2x) + \text{P}(x - T_1x_n) \} \]

and so

\[(7.1.5) \quad \text{P}(x-T_2x) \leq \{4+a(p)\}/\{2-a(p)\} \text{P}(x-x_n_1) + \{a(p)+1\}/\{2-a(p)\} \text{P}(x-x_n) \]

we see that for \( \varepsilon > 0 \), \( P \in U \) arbitrary, and sufficiently large values of \( n \),

\[ \text{P}(x-x_n_1) \leq \{2-a(p)\}/\{4+a(p)\} \text{P}(x-x_n_1) \text{ and } \]

\[ \text{P}(x-x_n) \leq \{1-a(p)\}/\{2-a(p)\} \text{P}(x-x_n) \]

Hence from (7.1.5) we see that for all sufficiently large values of \( n \), and arbitrary \( P \) in \( U \)

\[ \text{P}(x-T_2x) < \varepsilon \]. Since \( \varepsilon > 0 \) is arbitrary we have \( \text{P}(x-T_2x) = 0 \). Since \( X \) is Hausdorff, we conclude that \( x-T_2x = 0 \) i.e \( x = T_2x \). Similarly it can be shown that \( x = T_1x \).

Case 3. If \( T_1 \) and \( T_2 \) satisfy condition (III).

\[ 2 \cdot \text{P}(x-T_2x) \leq 2 \cdot \text{P}(x-T_1x_n) + 2 \cdot \text{P}(x_n - T_2x) \]

\[ \leq 2 \cdot \text{P}(x-T_1x_n) + \text{P}(x-T_1x) + \text{P}(x_n - T_2x) + a(p) \{ \text{P}(x_n - T_2x) + \text{P}(x-T_1x_n) + \text{P}(x-x_n) \} \]

and so

\[(7.1.6) \quad \text{P}(x-T_2x) \leq \{3+a(p)\}/\{1-a(p)\} \text{P}(x-x_n_1) + \{1+2.a(p)\}/\{1-a(p)\} \text{P}(x-x_n) \]

we see that for \( \varepsilon > 0 \), \( P \in U \) arbitrary, and sufficiently large values of \( n \)

\[ \text{P}(x-x_n_1) \leq \{1-a(p)\}/\{3+a(p)\} \text{P}(x-x_n_1) \text{ and } \]

\[ \text{P}(x-x_n) \leq \{1-a(p)\}/\{1+2.a(p)\} \text{P}(x-x_n) \]

Hence from (7.1.6) we see that for all sufficiently large values of \( n \), and arbitrary \( P \) in \( U \), \( \text{P}(x-T_2x) < \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary we have \( \text{P}(x-T_2x) = 0 \). Since \( X \) is Hausdorff, we conclude that \( x-T_2x = 0 \) i.e \( x = T_2x \). Similarly it can be shown that \( x = T_1x \).

Case 4. If \( T_1 \) and \( T_2 \) satisfy condition (IV).

\[ 2 \cdot \text{P}(x-T_2x) \leq 2 \cdot \text{P}(x-T_1x_n) + 2 \cdot \text{P}(x_n - T_2x) \]

\[ \leq 2 \cdot \text{P}(x-T_1x_n) + \text{P}(x-T_1x) + \text{P}(x_n - T_2x) + a(p) \{ \text{P}(x_n - T_2x) + \text{P}(x-T_1x_n) + \text{P}(x-x_n) \} \]

and so

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\[\begin{align*}
1.7 & \quad P(x - T_2 x) \leq \{3 + a(p)\} P(x - x_{n+1}) + \{1 + a(p)\} P(x - x_n)
\end{align*}\]

we see that for \( c = 0 \), \( P \in U \) arbitrary, and sufficiently large values of \( n \)

\[ P(x - x_{n+1}) \leq \frac{1}{\{3 + a(p)\}} P(x - x_{n+1}) \quad \text{and} \quad P(x - x_n) \leq \frac{1}{\{1 + a(p)\}} P(x - x_n) \]

Hence from (7.1.7) we see that for all sufficiently large values of \( n \), and arbitrary \( P \)
in \( U \) \( P(x - T_2 x) < c \). Since \( c > 0 \) is arbitrary we have \( P(x - T_2 x) = 0 \). Since \( X \) is Hausdorff
we conclude that \( x - T_2 x = 0 \) i.e. \( x = T_2 x \). Similarly it can be shown that \( x = T_1 x \).

**Example 7.1.**

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and \( X = C(\Omega) \) be the space of continuous real valued functions on \( \Omega \). Let \( \Delta \) be the family of closed subsets of \( \Omega \). For \( \gamma \in \Delta \) define \( P_\gamma(f) = \max \{|f(x)|, \ f \in X \} \)

Then \( P \) is a seminorm and the family \( \{ P_\gamma : \gamma \in \Delta \} \) generates a topology under which
\( X \) is locally convex space.

In particular let \( \Omega = [-4,4] \) and \( X = C([-4,4]) \).

Let \( C = \{ f \in X : f : [0,22/7] \to [0,22/7] \} \). Then \( C \) is a nonempty sequentially complete
subset of \( X \). Let \( T_1 \) and \( T_2 \) be self maps of \( C \) defined by \( (T_1 f)x = (1/2) \sin^2 x f(x) \) and
\( (T_2 f)x = 0 \) for all \( f(x) \in C \).

Then \( P((T_1 f)x - (T_2 g)x) = P((1/2) \sin^2 x f(x) - 0) = \max \{|(1/2) \sin^2 x f(x)|

\begin{align*}
& \leq (1/2) \max |f(x)| \\
& \leq (1/2) \max |f(x) - (T_2 g)x| + (1/2) \max |g(x) - (T_1 f)x| \\
& = (1/2) \{ P(f(x) - (T_2 g)x) + P(g(x) - (T_1 f)x) \}
\end{align*}\]

Hence \( T_1 \) and \( T_2 \) satisfies condition I(D). Clearly \( f = 0 \) in \( C \) is the common fixed point
of \( T_1 \) and \( T_2 \).
CHAPTER VII

COINCIDENCE AND FIXED POINT THEOREMS IN 2-METRIC SPACES

8.1. In this chapter we introduce the concept of compatible mappings of type (A-1) and type (A-2) in 2-metric spaces and show that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions. We also prove coincidence point theorem and common fixed point theorem for compatible mappings of type (A-1) and type (A-2) in 2-metric spaces which improves and generalizes the results of Murthy et al [124].

A 2-metric space is a set $X$ with a real valued function $d$ on $X \times X \times X$ satisfying the following conditions:

(M1) For distinct points $x$ and $y$ in $X$ there exists a point $z$ in $X$ such that $d(x,y,z) = 0$.

(M2) $d(x,y,z) = 0$ if at least two of $x,y,z$ are equal.

(M3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$

(M4) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $u$ in $X$.

8.2. In this section first we introduce the concepts of compatible mappings of type (A-1) and type (A-2) in 2-metric spaces and show that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions.

We now state the following definitions:

Definition 8.1. A sequence $\{x_n\}$ in a 2-metric space $(X,d)$ is said to converge to a point $x \in X$, denoted by $\lim_{n \to \infty} x_n = x$, if $\lim_{n \to \infty} d(x_n, x, z) = 0$ for all $z$ in $X$.

Definition 8.2. A sequence $\{x_n\}$ in a 2-metric space $(X,d)$ is called a cauchy sequence if $\lim_{n,m \to \infty} d(x_n, x_m, z) = 0$ for all $z$ in $X$.

Definition 8.3. A mapping $S$ from a 2-metric space $(X,d)$ into itself is said to be sequentially continuous at $x$ if for every sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} d(x_n, x, z) = 0$ for all $z$ in $X$, $\lim_{n \to \infty} d(Sx_n, Sx, z) = 0$.
Definition 8.4/124: Let \( S \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself. The mappings \( S \) and \( T \) are said to be compatible if \( \lim_{n \to \infty} d(STx_n, TSx_n, t) = 1 \) for all \( t \in X \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \).

Definition 8.5/124: Let \( S \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself. The mappings \( S \) and \( T \) are said to be compatible of type \((A)\) if \( \lim_{n \to \infty} d(STx_n, TTx_n, t) = 1 \) and \( \lim_{n \to \infty} d(TSx_n, Sx_n, t) = 1 \) for all \( t \) in \( X \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \).

Definition 8.6: Let \( S \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself. The pair of mappings \((S, T)\) is said to be compatible of type \((A-1)\) if \( \lim_{n \to \infty} d(STx_n, TTx_n, t) = 1 \), for all \( t \) in \( X \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \).

Definition 8.7: Let \( S \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself. The pair of mappings \((S, T)\) is said to be compatible of type \((A-2)\) if \( \lim_{n \to \infty} d(TSx_n, Sx_n, t) = 1 \), for all \( t \) in \( X \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \).

Clearly if a pair of mappings \((S, T)\) is compatible of type \((A-1)\) then the pair \((T, S)\) is compatible of type \((A-2)\). Further from the definitions it's clear that if \( S \) and \( T \) are sequentially continuous mappings of type \((A)\) then the pair \((S, T)\) is compatible of type \((A-1)\) as well as type \((A-2)\).

We now cite the following propositions which gives the condition under which definitions 8.4, 8.5, 8.6 and 8.7 becomes equivalent.

Proposition 8.1: Let \( S \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself.

(a) If \( T \) is sequentially continuous then the pair of mappings \((S, T)\) is compatible of type \((A-1)\) if \( S \) and \( T \) are compatible.

(b) If \( S \) is sequentially continuous then the pair of mappings \((S, T)\) is compatible of type \((A-2)\) if \( S \) and \( T \) are compatible.

(c) If \( S \) and \( T \) are sequentially continuous then the pair \((S, T)\) is compatible of type \((A-1)\) if the pair \((S, T)\) is compatible of type \((A-2)\).

Proof: (a) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \), and let the pair \((S, T)\) be compatible of type \((A-1)\). Then
\[d(ST_{x_n}, TS_{x_n}, t) \leq d(ST_{x_n}, TS_{x_n}, TX_{x_n}) + d(ST_{x_n}, TX_{x_n}, t) + d(TX_{x_n}, TS_{x_n}, t)\]

Hence, since \( T \) is sequentially continuous and the pair \((S, T)\) are compatible of type (A-1) we get \( \lim_{n \to \infty} d(ST_{x_n}, TS_{x_n}, t) = 0 \).

Now let \( S \) and \( T \) be compatible. Then we have

\[d(ST_{x_n}, TX_{x_n}, t) \leq d(ST_{x_n}, TX_{x_n}, TS_{x_n}) + d(ST_{x_n}, TS_{x_n}, t) + d(TS_{x_n}, TT_{x_n}, t)\]

Hence, since \( S \) is sequentially continuous and \( S \) and \( T \) are compatible we get \( \lim_{n \to \infty} d(ST_{x_n}, TT_{x_n}, t) = 0 \).

(b) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \) for some \( z \) in \( X \), and let the pair \((S, T)\) be compatible of type (A-2). Then

\[d(ST_{x_n}, TS_{x_n}, t) \leq d(ST_{x_n}, TS_{x_n}, SS_{x_n}) + d(ST_{x_n}, SS_{x_n}, t) + d(SS_{x_n}, TS_{x_n}, t)\]

Hence, since \( T \) is sequentially continuous and the pair \((S, T)\) are compatible of type (A-2) we get \( \lim_{n \to \infty} d(ST_{x_n}, TS_{x_n}, t) = 0 \).

Now let \( S \) and \( T \) be compatible. Then we have

\[d(TS_{x_n}, SS_{x_n}, t) \leq d(TS_{x_n}, SS_{x_n}, ST_{x_n}) + d(TS_{x_n}, ST_{x_n}, t) + d(ST_{x_n}, SS_{x_n}, t)\]

Hence, since \( S \) is sequentially continuous and \( S \) and \( T \) are compatible we get \( \lim_{n \to \infty} d(TS_{x_n}, SS_{x_n}, t) = 0 \).

(c) The proof follows from (a) and (b)

As a direct consequence of proposition 8.1 we have the following

\textit{Proposition 8.2.} Let \( S \) and \( T \) be mappings from a 2-metric space \((X, d)\) into itself. If \( S \) and \( T \) are sequentially continuous then the following statements are equivalent.

(a) The pair \((S, T)\) is compatible of type (A-1).
(b) The pair \((S, T)\) is compatible of type (A-2).
(c) The mappings \( S \) and \( T \) are compatible of type (A).
(d) The mappings \( S \) and \( T \) are compatible.
Next we give some properties of compatible mappings of type (A-1) and type (A-2) which will be used in our main theorem.

**Proposition 8.3:** Let $S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself. If the pair $(S, T)$ are compatible of type (A-1) and $S^{n} = T^{n}$ for some $n$ in $X$ then $ST^{n} = TT^{n}$.

**Proof:** Let $\{x_{n}\}$ be a sequence in $X$ defined by $x_{n} = z$ for $n = 1, 2, \ldots$ and let $Tz = Sz$. Then we have $\lim_{n \to \infty} Sx_{n} = Sz$ and $\lim_{n \to \infty} Tx_{n} = Tz$. Since the pair $(S, T)$ is compatible of type (A-1) we have $d(ST^{n}x, TT^{n}z, t) = \lim_{n \to \infty} d(STx_{n}, TTx_{n}, t) = 0$.

Hence $STz = TTz$.

**Proposition 8.4:** Let $S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself. If the pair $(S, T)$ is compatible of type (A-2) and $S^{n} = T^{n}$ for some $z$ in $X$ then $TS^{n} = SS^{n}$.

**Proposition 8.5:** Let $S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself. If the pair $(S, T)$ is compatible of type (A-1) and $\lim_{n \to \infty} Sx_{n} = \lim_{n \to \infty} Tx_{n} = z$ for some $z$ in $X$ then

(a) $\lim_{n \to \infty} STx_{n} = Tz$ if $T$ is sequentially continuous at $z$.
(b) $\lim_{n \to \infty} TTx_{n} = Sz$ if $S$ is sequentially continuous at $z$.

**Proof (a):** Let $\lim_{n \to \infty} Sx_{n} = \lim_{n \to \infty} Tx_{n} = z$ for some $z$ in $X$, and let the pair $(S, T)$ be compatible of type (A-1). Since $T$ is sequentially continuous at $z$ we have $\lim_{n \to \infty} TTx_{n} = Tz$. By (M4) we have

$$d(STx_{n}, Tz, t) \leq d(STx_{n}, Tz, TTx_{n}) + d(TTx_{n}, Tz, t).$$

Hence since the pair $(S, T)$ are compatible of type (A-1) we get

$$\lim_{n \to \infty} d(STx_{n}, Tz, t) = 0.$$ Hence $\lim_{n \to \infty} STx_{n} = Tz$.

(b) Proof follows on similar lines as argued in (a).

As a direct consequence of Proposition 8.5 we have the following.

**Proposition 8.6:** Let $S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself. If the pair $(S, T)$ is compatible of type (A-2) and $\lim_{n \to \infty} Sx_{n} = \lim_{n \to \infty} Tx_{n} = z$ for some $z$ in $X$ then
(a) \( \lim_{n \to \infty} S S x_n = T z \) if \( T \) is sequentially continuous at \( z \).

(b) \( \lim_{n \to \infty} T S x_n = S z \) if \( S \) is sequentially continuous at \( z \).

8.3. In this section we prove a coincidence point theorem and some common fixed point theorem in 2-metric spaces which improves the results of Murthy et al [124] and many others.

Let \( N \) and \( R' \) be the set of all natural numbers and set of non negative real numbers respectively and \( F \) be the family of mappings from \((R')^2\) into \( R' \) such that each \( \phi \) in \( F \) is upper semicontinuous , non-decreasing in each coordinate variable and for any \( t > 0 \),

\[
\phi(t,t,t,t,0,0,0,0) \leq \beta t \text{ and } \phi(t,t,t,t,0,0,0,0) \leq \beta t
\]

where \( \beta = 1 \) for \( \alpha = 2 \) and \( \beta < 1 \) for \( \alpha < 2 \). \( \gamma(t) = \phi(t,t,t,t,1,1,1,1) \) \( t < 1 \) where

\[
\gamma : R' \to R' \text{ is a mapping and } a_1 + a_2 + a_3 + a_4 \leq 7.
\]

Let \( A, B, S \) and \( T \) be mappings from a 2-metric space \((X,d)\) into itself such that

\[
(8.3.1) \quad A(X) \cup B(X) \subseteq S(X) \cap T(X)
\]

\[
(8.3.2) \quad d^2(Ax,By,z) \leq \phi\{ d^2(Sx,Ty,z), d(Sx,Ax,z) d(Ty,By,z), d(Sx,Ty,z)d(Sx,Ax,z) \\
+ d(Sx,Ty,z)d(Ty,By,z), d(Sx,Ty,z)d(Sx,By,z), d(Ty,Ax, z), d(Sx,Ty,z)d(Ty,Ax,z), d(Sx, Ax, z) \}
\]

for all \( x, y, z \) in \( X \) and \( \phi \in F \).

By (8.3.1) we see that \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \) and hence for some arbitrary \( x_0 \) in \( X \) there exists a point \( x_1 \) in \( X \) such that \( Ax_0 = Tx_1 \), and for this \( x_1 \) there exists \( x_2 \) such that \( S x_2 = B x_1 \).

Continuing this process we define the sequence \( \{y_n\} \) in \( X \) such that

\[
(8.3.3) \quad y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}
\]
for all $n \in \mathbb{N} \cup \{0\}$

We need the following Lemmas for our main theorems

**Lemma 8.1.** [117] For any $t > 0$, $\phi(t) < t$ if and only if $\lim_{n \to \infty} y^n(t) = 0$, where $y^n$ denotes the $n$ times composition of $y$ with itself.

**Lemma 8.2.** Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself satisfying (8.3.1) and (8.3.2). Then we have the following:

1. For every $n \in \mathbb{N}$, $d(y_n, y_{n-1}, y_{n-2}) = 0$
2. For every $i, j, k \in \mathbb{N}$, $d(y_i, y_j, y_k) = 0$, where $\{y_n\}$ is the sequence defined in (8.3.3).

**Proof.** (1) By (8.3.2) we have,

$$d^2(y_{2n+2}, y_{2n+1}, y_{2n}) = d^2(Ax_{2n+2}, Bx_{2n+1}, y_{2n})$$

$$\leq \phi\{d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}, y_{2n}), d(y_{2n+1}, y_{2n+2}, y_{2n})\}$$

$$= \phi(0, 0, 0, 0, 0, 0, 0, 0, 0) \leq 0,$$

and so $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$. Similarly we have $d(y_{2n+1}, y_{2n+2}, y_{2n}) = 0$. Thus it follows that $d(y_n, y_{n-1}, y_{n-2}) = 0$.

(2) For all $z \in X$ let $d_n(z) = d(y_n, y_{n+1}, z)$. Then, by (M4) and lemma 8.2(1) we have

$$d(y_n, y_{n+1}, z) \leq d(y_n, y_{n+2}, z) + d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z)$$

$$= d(y_n, y_{n+1}, z) + d(y_n, y_{n+1}, z) = d_n(z) + d_{n+1}(z).$$

By (8.3.2) we have

$$d^2_{2n+1}(z) = d^2(y_{2n+2}, y_{2n+1}, z) = d^2(Ax_{2n+2}, Bx_{2n+1}, z)$$

$$\leq \phi\{d^2(y_{2n+1}, y_{2n+2}, z), d(y_{2n+1}, y_{2n+2}, z), d(y_{2n+2}, y_{2n+2}, z), d(y_{2n+2}, y_{2n+2}, z),$$

$$d(y_{2n+1}, y_{2n+2}, z), d(y_{2n+2}, y_{2n+2}, z), d(y_{2n+2}, y_{2n+2}, z), d(y_{2n+2}, y_{2n+2}, z)\}$$

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Now we shall prove that \( \{d_n(z)\} \) is a non-increasing sequence in \( \mathbb{R}^+ \). In fact suppose that \( d_{n+1}(z) > d_n(z) \) for some \( n \). Then for some \( \alpha < 2 \), \( d_{n+1}(z) + d_n(z) = d_{n+1}(z) \). Since \( \phi \) is non-decreasing in each variable and \( \beta < 1 \) for some \( \alpha < 2 \), by (8.3.2) we get,

\[
d_{2n+1}(z) \leq \phi(d_{2n+1}(z), d_{2n+1}(z), d_{2n+1}(z), d_{2n+1}(z), 0, \alpha d_{2n+1}(z), 0, \alpha d_{2n+1}(z), 0) \leq \beta\cdot d_{2n+1}(z)
\]

\[
< d_{2n+1}(z)
\]

and

\[
d_{2n+2}(z) \leq \phi(d_{2n+2}(z), d_{2n+2}(z), d_{2n+2}(z), d_{2n+2}(z), \alpha d_{2n+2}(z), 0, 0, 0, \alpha d_{2n+2}(z) \leq \beta d_{2n+2}(z) < d_{2n+2}(z).
\]

Hence for every \( n \in \mathbb{N} \), \( d_n(z) \leq \beta d_n(z) < d_n(z) \), which is a contradiction. Therefore \( \{d_n(z)\} \) is a non-increasing sequence in \( \mathbb{R}^+ \).

Now we claim that \( d_n(y_{nm}) = 0 \) for all non-negative integers \( n,m \).

Case 1. \( n > m \). Then we have \( 0 = d_n(y_{nm}) > d_n(y_{nm}) \).

Case 2. \( n < m \). By (M4) we have

\[
d_n(y_{nm}) \leq d_n(y_{nm}) + d_{n-1}(y_{nm}) \leq d_n(y_{nm}) + d_{n-1}(y_{nm}) \leq d_n(y_{nm}).
\]

By using the above inequality repeatedly, we have

\[
d_n(y_{nm}) \leq d_n(y_{nm}) \leq \ldots \leq d_n(y_{nm}) = 0,
\]

which completes the proof of our claim.
Now let $i$, $j$, and $k$ be non-negative integers and suppose $i < j$. By (M4) we have

$$d(y_i, y_j, y_k) = d_i(y_j) + d_j(y_k) + d(y_i, y_j, y_k) = d(y_i, y_j, y_k).$$

Therefore by repetition of the above inequality we have

$$d(y_i, y_j, y_k) \leq d(y_{i+1}, y_j, y_k) \leq \ldots \leq d(y_i, y_j, y_k) = 0.$$ 

This completes the proof.

Lemma 8.3. Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying (8.3.1) and (8.3.2). Then the sequence $(y_n)$ defined by (8.3.3) is a Cauchy sequence in $X$.

Proof. In the proof of Lemma 8.2 since $(d_n(z))$ is a non-increasing sequence in $R^+$, by (8.3.2) we have

$$d_{i+2}(z) = d^i(y_1, y_2, z) = d^i(Bx_1, Ax_1, z)$$

$$\leq \psi(d_{i+2}(z), d_i(z), d_{i+2}(z), d_i(z), 0, d_i(z), d_i(z) + d_i(z), 0, d_n(z), d_n(z))$$

$$\leq \psi(d_{i+2}(z), d_{i+2}(z), d_{i+2}(z), 2d_{i+2}(z), 2d_{i+2}(z), d_{i+2}(z))$$

$$= \gamma(d_{i+2}(z)).$$

In general we have $d_{i+2}(z) \leq \gamma(d_i(z))$, which implies that if $d_i(z) > 0$, by Lemma 8.1 we have $\lim_{n \to \infty} d_n(z) = 0$. Therefore we have $\lim_{n \to \infty} d_n(z) = 0$. For $d_n(z) = 0$, since $(d_n(z))$ is non-increasing, we have clearly $\lim_{n \to \infty} d_n(z) = 0$.

Now we shall prove that $(y_n)$ is a Cauchy sequence in $X$. Since $\lim_{n \to \infty} d_n(z) = 0$ it is sufficient if we show that a subsequence $(y_{2n})$ of $(y_n)$ is a Cauchy sequence in $X$. Suppose that the subsequence $(y_{2n})$ is not a cauchy sequence. Then there exist a point $a$ in $X$, $\varepsilon > 0$ and strictly increasing sequences $(m_k)$ and $(n_k)$ of positive integers such that $k \leq n(k) < m(k)$,

$$(8.3.4) \quad d(y_{2m(k)}, y_{2n(k) + 2}, a) > \varepsilon \quad \text{and} \quad d(y_{2m(k)}, y_{2n(k) + 2}, a) < \varepsilon.$$
\[ d(y_{2nk}, y_{2nk+1}, a) - d(y_{2nk+1}, y_{2nk+2}, a) \leq d(y_{2nk+1}, y_{2nk+2}, a) + d_{2nk+1}(a) + d_{2nk+2} \]

Since \( \{d(y_{2nk}, y_{2nk+1}, a) - \epsilon\} \) and \( \{\epsilon - d(y_{2nk+1}, y_{2nk+2}, a)\} \) are sequences in \( \mathbb{R}^+ \) and \( \lim_{k \to \infty} d(a) = 0 \), we have

\[ (8.3.5) \lim_{k \to \infty} d(y_{2nk}, y_{2nk+1}, a) = \epsilon \text{ and } \lim_{k \to \infty} d(y_{2nk+1}, y_{2nk+2}, a) = \epsilon. \]

Note that by (M4),

\[ (8.3.6) \quad |d(x,y,a) - d(x,y,b)| \leq d(a,b,x) + d(a,b,y) \quad \text{for all } x,y,a,b \in X. \]

Taking \( x = y_{2nk}, y = a, a = y_{2nk+1} \) and \( b = y_{2nk} \) in (8.3.6) and using Lemma 8.2 and (8.3.5) we have,

\[ (8.3.7) \quad \lim_{k \to \infty} d(y_{2nk+1}, y_{2nk+2}, a) = \epsilon. \]

Once again by using lemma 3.2, 3.5 and 3.6 we have,

\[ (8.3.8) \quad \lim_{k \to \infty} d(y_{2nk}, y_{2nk+1}, a) = \epsilon \quad \text{and} \quad \lim_{k \to \infty} d(y_{2nk+1}, y_{2nk+2}, a) = \epsilon. \]

Thus by (8.3.2) we have,

\[ (8.3.9) \quad d^2(y_{2nk}, y_{2nk+1}, a) = d^2(Ax_{2nk}, Bx_{2nk+1}, a) \]

\[ \leq \phi(d^2(y_{2nk+1}, y_{2nk}, a), d(y_{2nk+1}, y_{2nk}, a)d(y_{2nk+1}, y_{2nk+1}, a), \]

\[ d(y_{2nk+1}, y_{2nk}, a)d(y_{2nk+1}, y_{2nk}, a), d(y_{2nk+1}, y_{2nk+1}, a)d(y_{2nk}, y_{2nk+1}, a), \]

\[ d(y_{2nk+1}, y_{2nk+1}, a)d(y_{2nk+1}, y_{2nk+1}, a), d(y_{2nk+1}, y_{2nk+1}, a)d(y_{2nk}, y_{2nk+1}, a), \]

\[ d(y_{2nk+1}, y_{2nk+1}, a)d(y_{2nk+1}, y_{2nk+1}, a), d(y_{2nk+1}, y_{2nk+1}, a)d(y_{2nk}, y_{2nk+1}, a), \]

\[ \}

using (8.3.4), (8.3.5), (8.3.6) and (8.3.7), since \( \phi \in F \) as \( k \to \infty \) we see that

\[ \epsilon^2 \leq \phi(\epsilon^2; 0,0,0,0,0,0,0) \leq \gamma(\epsilon^2) < \epsilon^2 \text{ which is a contradiction. Therefore } \{y_{2n}\} \]
is a Cauchy sequence in $X$.

Now we are ready to prove our main theorem.

**Theorem 8.1.** Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, \sigma)$ into itself satisfying $(8.3.1)$, $(8.3.2)$ and $(8.3.10)$

$$(8.3.10) \quad S(X) \cap T(X) \text{ is a complete subspace of } X. \text{ Then}$$

1. $A$ and $S$ have a coincidence point in $X$ and
2. $B$ and $T$ have a coincidence point in $X$.

**Proof.** By Lemma 8.2 sequence $\{y_n\}$ defined by $(8.3.4)$ is a Cauchy sequence in $S(X) \cap T(X)$. Since $S(X) \cap T(X)$ is a complete subspace of $X$, the sequence $\{y_n\}$ must converge to some point say $w$ in $S(X) \cap T(X)$. On the other hand since subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also Cauchy sequences in $S(X) \cap T(X)$, they should also converge to the same point $w$ in $X$. Hence there should exist two points $u$ and $v$ in $X$ such that $Su = w$ and $Tv = w$. Using $(8.3.2)$ we get,

$$d^2(Au, Bx_{2n+1}, z) \leq \phi(d(Su, Tx_{2n+1}, z), d(Su, Au, z), d(Tx_{2n+1}, Bx_{2n+1}),$$

$$d(Su, Tx_{2n+1}, z), d(Su, Au, z), d(Tx_{2n+1}, Bx_{2n+1}),$$

$$d(Su, Tx_{2n+1}, z), d(Su, Bx_{2n+1}), d(Su, Tx_{2n+1}, z), d(Tx_{2n+1}, Au, z),$$

$$d(Su, Bx_{2n+1}, z), d(Tx_{2n+1}, Au, z), d(Su, Au, z), d(Tx_{2n+1}, Au, z),$$

$$d(Su, Bx_{2n+1}, z), d(Tx_{2n+1}, Bx_{2n+1}, z))$$

i.e.

$$d^2(Au, y_{2n+1}, z) \leq \phi(d(Su, y_{2n}, z), d(Su, Au, z), d(y_{2n}, y_{2n+1}, z), d(Su, y_{2n}, z), d(Su, Au, z),$$

$$d(Su, y_{2n}, z), d(y_{2n}, y_{2n+1}, z), d(Su, y_{2n}, z), d(Su, y_{2n}, z), d(y_{2n}, Au, z),$$

$$d(Su, y_{2n+1}, z), d(y_{2n}, Au, z), d(Su, Au, z), d(y_{2n}, Au, z), d(Su, y_{2n+1}, z), d(y_{2n}, y_{2n+1}, z).$$

Since $\lim_{n \to \infty} d(z) = 0$, as $n \to \infty$ we get, $d(Au, w, z) \leq \phi(0, 0, 0, 0, 0, 0, 0, d^2(w, Au, z), 0)$ which is a contradiction. Hence $Au = w = Su$. Similarly we can show that $v$ is a coincidence point of $B$ and $T$.

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Theorem 8.2. Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying (8.3.1), (8.3.2), (8.3.10) and (8.3.11)

(8.3.11) The pairs $(A, S)$ and $(B, T)$ are compatible of type $(A - 1)$ or type $(A - 2)$.

Then $A, B, S,$ and $T$ have a unique common fixed point.

Proof. By Theorem 8.1, $A u = S u = w$ and $B v = T v = w$. If the pairs $(A, S)$ and $(B, T)$ are compatible of type $(A - 1)$ then by proposition (8.3) we get $A S u = S S u$ and $B T v = T T v$, i.e. $A w = S w$ and $B w = T w$. If the pairs $(A, S)$ and $(B, T)$ are compatible of type $(A - 2)$ then by proposition (8.4) we get $A S u = A A u$ and $T B v = B B v$.

i.e. $S w = A w$ and $T w = B w$. i.e. in both the cases we get $A w = S w$ and $B w = T w$. Using (8.3.2) we get

\[
d^2(A x_{2n}, B w, z) \leq \phi(d^2(S x_{2n}, T w, z), d(S x_{2n}, A x_{2n}, z), d(T w, B w, z),
\]

\[
d(S x_{2n}, T w, z), d(S x_{2n}, A x_{2n}, z), d(T w, B w, z),
\]

\[
d(S x_{2n}, T w, z), d(S x_{2n}, B w, z), d(S x_{2n}, T w, z), d(T w, A x_{2n}, z),
\]

\[
d(S x_{2n}, B w, z), d(T w, A x_{2n}, z), d(S x_{2n}, A x_{2n}, z), d(T w, A x_{2n}, z),
\]

\[
d(S x_{2n}, B w, z), d(T w, A x_{2n}, z),
\]

as $n \to \infty$ we get

\[
d^2(y_{2n}, B w, z) \leq \phi(d^2(y_{2n}, T w, z), d(y_{2n}, y_{2n}, z), d(T w, B w, z), d(y_{2n}, T w, z), d(y_{2n}, y_{2n}, z),
\]

\[
d(y_{2n}, T w, z), d(T w, B w, z), d(y_{2n}, y_{2n}, z), d(y_{2n}, T w, z), d(y_{2n}, B w, z),
\]

\[
d(y_{2n}, T w, z), d(T w, y_{2n}, z), d(y_{2n}, B w, z), d(T w, y_{2n}, z),
\]

\[
d(y_{2n}, y_{2n}, z), d(T w, y_{2n}, z), d(y_{2n}, B w, z), d(T w, B w, z))
\]

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\[ \phi (d'(w, Bw, z), 0, 0, d'(w, Bw, z), d'(w, Bw, z), d'(w, Bw, z), 0, 0) \]

which is a contradiction. Hence \( w = Bw = Tw \). Similarly we can prove that \( w = Aw = Sw \). Therefore \( w \) is a common fixed point of \( A, B, S \) and \( T \). The proof of uniqueness follows easily from (8.3.2).

Remark 1 Theorem 8.2 extends, generalises and improves the results of Murthy et al. [124] and a number of fixed point theorems for commuting mappings, weakly commuting mappings and compatible mappings in 2-metric spaces.
9.1. Goebel's [64] results were extended to L-space, metric space, 2-metric space and multivalued contraction maps on metric spaces by Okada [133], Singh and Virendra [198], Kulshreshtha [109] and Naimpally et al. [129] respectively. In [88], Jungck contraction principle appeared for a pair of continuous and commuting self maps. After Jungck, a spate of research papers appeared using these concepts in various ways with several contractive type mappings by many authors. (See [25]-[27], [35], [41], [44-45], [53-54], [56-57], [69], [91], [97], [99], [102], [104-105], [123], [142], [141], [136], [143], [158], [167], [183], [191], [193], [197], [198]).

Cho and Singh [32] and Murthy and Sharma [126] introduced the concepts of commuting and weakly uniformly contraction maps respectively in saks space and proved several fixed point theorems using these concepts.

In this chapter we introduce the concept of compatible mappings and compatible mappings of type (A-1) and type (A-2) in saks space and give some relationship between these mappings. We have also established a series of coincidence and common fixed point theorems for compatible mappings of type (A-1) and type (A-2) in saks space which extends and generalises many known results in saks space as well as metric spaces. Our theorem extends the results of Diviccaro et al. [45], Fisher and Sessa [58], Gregus [64], Jungck [66], Mukherjee et al. [123] and many others.

9.2. In this section we introduce the concept of compatible mappings as well as compatible mappings of type (A-1) and type (A-2) in saks space and derive some relationship between them.

Definition 9.1 [126]: Let S and T be self maps of a saks space \((X, N_1, N_2)\). The mappings S and T are said to be weakly uniformly contraction pair if

\[ N_1(S(Tx - Tx)) < N_1(Sx - Tx) \quad \text{and} \quad N_1(T(Sx - Sx)) < N_1(Sx - Tx) \]

Definition 9.2: Let S and T be self maps of a saks space \((X, N_1, N_2)\). The mappings S and T are said to be compatible if \(\lim_{n \to \infty} N_1(STx_n - TSx_n) = 0\), whenever \(\{x_n\}\) is a sequence in X such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\) for some \(t\) in X.
**Definition 9.3:** Let $S$ and $T$ be self maps of a saks space $(X, N_1, N_2)$. The pair of mappings $(S, T)$ is said to be compatible of type (A-1) if \( \lim_{n \to \infty} N_1(STx_n - TTx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in $X$ such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some $t$ in $X$.

**Definition 9.4:** Let $S$ and $T$ be self maps of a saks space $(X, N_1, N_2)$. The pair of mappings $(S, T)$ is said to be compatible of type (A-2) if \( \lim_{n \to \infty} N_1(TSx_n - SSx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in $X$ such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some $t$ in $X$.

Clearly if a pair of mappings $(S, T)$ is compatible of type (A-1) then the pair $(T, S)$ is compatible of type (A-2). Further from the definitions it's clear that if $S$ and $T$ weakly uniformly contraction pair of mappings then the pair $(S, T)$ are compatible of type (A-1) as well as type (A-2). The following example illustrates that the implication is not reversible.

**Example 9.1.** Let $X = [0, \infty)$ and $N_2 = d$ be the euclidean metric. Consider the mappings $S$ and $T$ defined by $Sx = 2x^2$ and $Tx = 3x^2$.

By routine check up one can easily verify that the pair $(S, T)$ are compatible of type (A-1) and type (A-2) but $S$ and $T$ are not weakly uniformly contraction maps.

**Example 9.2.** Let $X = [0, \infty)$ and $N_2 = d$ be the euclidean metric.

Consider the mappings $S$ and $T$ defined by $Sx = \begin{cases} 1 & \text{if } x \in [0,1] \\ 1 + x & \text{if } x > 1. \end{cases}$ and $Tx = \begin{cases} 1 + x & \text{if } x \in [0,1) \\ 1 & \text{if } x \in [1, \infty). \end{cases}$

Consider the $\{x_n\}$ defined by $x_n = 1/n$ for all $n$. We see that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = 1$. 

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\[ \lim_{n \to \infty} N_2(STx_n - TTx_n) = 0/2 \quad \text{and} \quad \lim_{n \to \infty} N_2(TSx_n - SSx_n) = 0. \text{ Hence the pair of mappings} (S, T) \text{ is compatible of type (A-2) but not of type (A-1).} \]

We now cite the following propositions which gives the condition under which definitions 9.2, 9.3, and 9.4 becomes equivalent.

**Proposition 9.1:** Let \( S \) and \( T \) be self maps of a saaks space \((X, N_1, N_2)\).

a) If \( T \) is continuous then the pair of mappings \((S, T)\) is compatible of type (A-1) iff \( S \) and \( T \) are compatible.

b) If \( S \) is continuous then the pair of mappings \((S, T)\) is compatible of type (A-2) iff \( S \) and \( T \) are compatible.

c) If \( S \) and \( T \) are continuous then the pair \((S, T)\) is compatible of type (A-1) iff the pair \((S, T)\) is compatible of type (A-2).

**Proof:**

a) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in \( X \), and let the pair \((S, T)\) be compatible of type (A-1). Since \( T \) is continuous we have \( \lim_{n \to \infty} STx_n = Tt \) and \( \lim_{n \to \infty} TTx_n = Tt \).

Hence \( N_2(STx_n - TSx_n) \leq N_2(STx_n - TTx_n) + N_2(TTx_n - TSx_n) \).

Hence \( \lim_{n \to \infty} N_2(STx_n - TSx_n) = 0. \)

Now let \( S \) and \( T \) be compatible. Then we have

\[ N_2(STx_n - TTx_n) \leq N_2(STx_n - TSx_n) + N_2(TSx_n - TTx_n) \]

Hence \( \lim_{n \to \infty} N_2(STx_n - TTx_n) = 0. \)

b) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in \( X \), and let the pair \((S, T)\) be compatible of type (A-2). Since \( S \) is continuous we have \( \lim_{n \to \infty} STx_n = St \) and \( \lim_{n \to \infty} SSx_n = St \). Hence \( N_2(STx_n - TSx_n) \leq N_2(STx_n - SSx_n) + N_2(SSx_n - TSx_n) \).

Hence \( \lim_{n \to \infty} N_2(STx_n - TSx_n) = 0. \)

Now let \( S \) and \( T \) be compatible. Then we have

\[ N_2(TSx_n - SSx_n) \leq N_2(TSx_n - STx_n) + N_2(STx_n - SSx_n) \]

Hence \( \lim_{n \to \infty} N_2(TSx_n - SSx_n) = 0. \)
c) Let \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in \( X \), and let the pair \((S, T)\) be compatible of type \((A-1)\). We have

\[
N_2(\text{TS}_n - \text{SS}_n) \leq N_2(\text{ST}_n - \text{ST}_n) + N_2(\text{ST}_n - \text{TT}_n) + N_2(\text{TT}_n - \text{SS}_n).
\]

Since \( S \) and \( T \) are continuous we see that \( \lim_{n \to \infty} \text{SS}_n = \lim_{n \to \infty} \text{ST}_n \) and \( \lim_{n \to \infty} \text{TS}_n = \lim_{n \to \infty} \text{TT}_n \). Hence \( \lim_{n \to \infty} N_2(\text{TS}_n - \text{SS}_n) = 0 \). Therefore the pair \((S, T)\) is compatible of type \((A-1)\).

Next suppose the pair \((S, T)\) is compatible of type \((A-2)\). We have

\[
N_2(\text{ST}_n - \text{TT}_n) \leq N_2(\text{ST}_n - \text{ST}_n) + N_2(\text{TS}_n - \text{SS}_n) + N_2(\text{SS}_n - \text{TT}_n).
\]

Again using continuity of \( S \) and \( T \) we get \( \lim_{n \to \infty} N_2(\text{ST}_n - \text{TT}_n) = 0 \). Therefore the pair \((S, T)\) is compatible of type \((A-1)\).

As a direct consequence of proposition 9.1 we have the following.

**Proposition 9.2.** Let \( S \) and \( T \) be self maps of a metric space \((X, N_1, N_2)\). If \( S \) and \( T \) are continuous then the following statements are equivalent.

a. The pair \((S, T)\) is compatible of type \((A-1)\).

b. The pair \((S, T)\) is compatible of type \((A-2)\).

c. The mappings \( S \) and \( T \) are compatible.

Next we give some properties of compatible mappings of type \((A-1)\) and type \((A-2)\) which will be used in our main theorem.

**Proposition 9.3:** Let \( S \) and \( T \) be self maps of a metric space \((X, N_1, N_2)\). If the pair \((S, T)\) are compatible of \( (A-1) \) and \( Sz = Tz \) for some \( z \) in \( X \) then \( STz = TTz \).

**Proof:** Let \( \{x_n\} \) be a sequence in \( X \) defined by \( x_n = z \) for \( n = 1, 2, \ldots \) and let \( Tz = Sz \). Then we have \( \lim_{n \to \infty} Sx_n = Sz \) & \( \lim_{n \to \infty} Tx_n = Tz \). Since the pair \((S, T)\) is compatible of type \((A-1)\) we have

\[
N_2(\text{ST}_n - \text{TT}_n) = \lim_{n \to \infty} N_2(\text{ST}_n - \text{TT}_n) = 0.
\]

Hence \( STz = TTz \).

**Proposition 9.4:** Let \( S \) and \( T \) be self maps of a metric space \((X, N_1, N_2)\). If the pair \((S, T)\)
is compatible of type (A-2) and $S_2 = T_2$ for some $z$ in $X$ then $T S_2 = S S_2$.

9.3. In this section we prove some coincidence point theorems and common fixed point theorems in saks space which improves many well known results.

We require the following well known lemma for our main theorem.

**Lemma 9.1.** Let $(X, d) = (X, N_1, N_2)$ be a saks space. Then the following statements are equivalent.

1. $N_1$ is equivalent to $N_2$ on $X$.
2. $(X, N_1)$ is a Banach space and $N_1 \leq N_2$ on $X$.
3. $(X, N_2)$ is a Frechet space and $N_2 \leq N_1$ on $X$.

Let $A, B, S, T$ be mappings from a saks space $(X, N_1, N_2)$ into itself such that

1. $A(X) \cup B(X) \subseteq S(X) \cap T(X)$
2. $N_2(Ax - By) \leq aN_2(Sx - Ty) + b \max(N_2(Ax - Sx), N_2(By - Ty),

\frac{1}{2}(N_2(Ax - Ty) + N_2(By - Sx))$

for all $x, y$ in $X$, $a, b > 0$ and $0 < a + b < 1$.

For some arbitrary $x_0$ in $X$, by (9.3.1) we choose $x_1$ in $X$ such that $Ax_0 = Tx_1$, and for this $x_1$ there exists $x_2$ such that $Sx_1 = Bx_1$. Continuing this process we define the sequence $(y_n)$ in $X$ such that

1. $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$

**Lemma 9.2.** Let $A, B, S$ and $T$ be mappings from saks space $(X, N_1, N_2)$ into itself satisfying (4.1) and (4.2). Then the sequence $(y_n)$ defined by (4.3) is a cauchy sequence.

**Proof.** By (9.3.2) we have

$N_2(Ax_{2n} - Bx_{2n+1}) \leq aN_2(Sx_{2n} - Tx_{2n+1}) + \max[N_2(Ax_{2n} - Sx_{2n}), N_2(Bx_{2n+1} - Tx_{2n+1}),

\frac{1}{2}(N_2(Ax_{2n} - Ty_{2n+1}) + N_2(By_{2n+1} - Sx_{2n}))$

or equivalently
\[ N_2(y_{2n} - y_{2n+1}) \leq a \cdot N_2(y_{2n} - y_{2n}) + b \cdot \max \{ N_1(y_{2n} - y_{2n+1}), N_1(y_{2n+1} - y_{2n}) \} + \frac{1}{2} \cdot (N_2(y_{2n} - y_{2n}) + N_1(y_{2n+1} - y_{2n+1})) \]

If \( N_2(y_{2n} - y_{2n+1}) > N_1(y_{2n} - y_{2n+1}) \) in the above inequality then we get

\[ N_2(y_{2n} - y_{2n+1}) \leq (a + b) \cdot N_2(y_{2n} - y_{2n+1}) \] a contradiction

Hence \( N_1(y_{2n} - y_{2n+1}) \leq N_1(y_{2n} - y_{2n+1}) \)

Hence we get \( N_2(y_{2n} - y_{2n+1}) \leq (a + b) \cdot N_2(y_{2n} - y_{2n+1}) \)

It follows that \( N_2(y_{2n} - y_{2n+1}) \leq (a + b)^n \cdot N_1(y_{2n} - y_{2n+1}) \).

If \( m < n \) then the repeated use of above inequality gives

\[ N_2(y_{m} - y_{n}) \leq N_2(y_{m} - y_{m+1}) + N_2(y_{m+1} - y_{m+2}) + \ldots + N_2(y_{n-1} - y_{n}) \]

\[ \leq \{(a + b)^m + (a + b)^{m+1} + (a + b)^{m+2} + \ldots + (a + b)^{n-1}\} \cdot N_2(y_{1} - y_{n}) \]

\[ = (a + b)^{n-1}(1 - (a + b)) \cdot N_2(y_{1} - y_{n}) \]

Hence since \( 0 < a + b < 1 \) we see that sequence \( \{y_n\} \) is a Cauchy sequence.

**Theorem 9.1.** Let \( A, B, S, T \) be from saks space \( (X, N_1, N_2) \) into itself satisfying

(9.3.1), (9.3.2) and (9.3.4) \( S(X) \cap T(X) \) is a complete subspace of \( X \).

Then

a) \( A \) and \( S \) have a coincidence point in \( X \).

b) \( B \) and \( T \) have a coincidence point in \( X \).

**Proof.** By Lemma 9.2 sequence \( \{y_n\} \) defined by (9.3.3) is a Cauchy sequence in \( S(X) \cap T(X) \). Since \( S(X) \cap T(X) \) is a complete subspace of \( X \), the sequence \( \{y_n\} \) must converge to some point say \( w \) in \( S(X) \cap T(X) \). On the other hand since subsequences \( \{y_{2n}\} \) and \( \{y_{2n+1}\} \) are also Cauchy sequences in \( S(X) \cap T(X) \), they should also converge to the same point \( w \) in \( X \). Hence there should exist two points \( u \) and \( v \) in \( X \) such that \( Su = w \) and \( Tv = w \). Using (9.3.2) we get,

\[ N_2(Au - Bx_{2n}) \leq a \cdot N_2(Su - Tx_{2n}) + b \cdot \max \{ N_1(Au - Su), N_2(Bx_{2n} - Tx_{2n}) \} \]
$1/2 \{ N_2(Au - Tx_{2n}) + N_2(Bx_{2n}, Su) \}$

as $n \to \infty$ we get

$N_2(Au - w) \leq 0 + b \text{max}\{N_2(Au - w), 0\} + 1/2 \{ N_2(Au - w) + 0 \}$

i.e. $N_2(Au - w) \leq b N_2(Au - w)$ a contradiction. Therefore $Au = w$. Hence $Su = Au = w$. Similarly it can be shown that $Bv = Tv = w$.

**Theorem 9.2.** Let $A, B, S$ and $T$ be from saks space $(X, N_1, N_2)$ into itself satisfying (9.3.1), (9.3.2), (9.3.4) and (9.3.5).

(9.3.5) The pairs $(A, S)$ and $(B, T)$ are compatible of type $(A-1)$ or type $(A-2)$.

Then $A, B, S,$ and $T$ have a unique common fixed point.

**Proof.** By Theorem 9.1, $Au = Su = w$ and $Bv = Tv = w$. If the pairs $(A, S)$ and $(B, T)$ are compatible of type $(A-1)$ then by proposition (9.3) we get $ASu = SSu$ and $BTv = TTv$, i.e. $Aw = Sw$ and $Bw = Tw$. If the pairs $(A, S)$ and $(B, T)$ are compatible of type $(A-2)$ then by proposition (9.4) we get $SAu = AAu$ and $TBv = BBv$, i.e. $Sw = Aw$ and $Tw = Bw$, i.e. in both the cases we get, $Aw = Sw$ and $Bw = Tw$. Using (9.3.2) we get

$N_2(Ax_{2n} - Bw) \leq a N_2(Sx_{2n} - Tw) + b \text{max}\{N_2(Ax_{2n} - Sx_{2n}), N_2(Bw - Tw)\}$

$1/2 \{ N_2(Ax_{2n} - Tw) + N_2(Bw - Sx_{2n}) \}$

as $n \to \infty$ we get,

$N_2(w - Bw) \leq (a+b) N_2(w - Bw)$ a contradiction. Hence $Bw = w$. Again by (9.3.2), we have

$N_2(Aw - Bw) \leq a N_2(Sw - Tw) + b \text{max}\{N_2(Aw - Sw), N_2(Bw - Tw)\}$

$1/2 \{ N_2(Aw - Tw) + N_2(Bw - Sw) \}$

i.e. $N_2(Aw - w) \leq (a+b) N_2(Aw - w)$, a contradiction.

Hence $Aw = w$. Therefore $w$ is common fixed point of $A, B, S$ and $T$.

The following results follows immediately upon noting that under
the given hypothesis the sequence given by (9.3.3) is a Cauchy sequence in $S(X) \cap T(X)$.

**Theorem 9.3.** Let $A, B, S, \text{and } T$ be from saks space $(X, N_1, N_2)$ into itself satisfying (9.3.1), (9.3.4), (9.3.5) and

\[(9.3.6) \quad N_2(Ax - By) \leq \psi(N_2(Sx - Ty), N_2(Ax - Sx), N_2(By - Ty), N_2(Ax - Ty), N_2(By - Sx))
\]

for all $x,y$ in $X$ where $\phi : [0, \infty)^4 \rightarrow [0, \infty)$ is

1. nondecreasing and upper semicontinuous in each coordinate variable.
2. for each $t > 0$, $\gamma(t) = \max\{\phi(0,0,t,t), \phi(t,t,t,2t), \phi(t,t,0,2t)\} < t$

Then $A, B, S, \text{and } T$ have a unique common fixed point.

**Remark 1.** For $S = T$ Theorem 9.3 includes the results of Cho and Singh [32].

**Remark 2.** If $X$ is a metric space and $N_2(x-y)$ is replaced by $d(x,y)$ in (9.3.6) we get the result of Kang et al. [93].

**Theorem 9.4.** Let $A, B, S, \text{and } T$ be from saks space $(X, N_1, N_2)$ into itself satisfying (9.3.1), (9.3.4), (9.3.5) and

\[(9.3.7) \quad N_2(Ax - By) \leq k \psi(N_2(Sx - Ty), N_2(Ax - Sx), N_2(By - Ty), \phi(N_2(Ax - Ty) + N_2(By - Sx))
\]

for all $x,y$ in $X$ where $\phi : [0, \infty)^4 \rightarrow [0, \infty)$ is

1. nondecreasing and upper semicontinuous in each coordinate variable.
2. for each $t > 0$, $\gamma(t) = \max\{\phi(0,0,t,t), \phi(t,t,t,2t), \phi(t,t,0,2t), \phi(t,t,0,2t)\} < t$

Then $A, B, S, \text{and } T$ have a unique common fixed point.

**Remark 3.** Theorem 9.4 extends and generalises theorem 3 of Murthy and Sharma [126] by replacing weakly uniformly contraction pair of maps with compatible maps of type (A-1) or type (A-2).
Theorem 9.5. Let $A, B, S,$ and $T$ be from saks space $(X, N_1, N_2)$ into itself satisfying (9.3.1), (9.3.4), (9.3.5) and

\[(9.3.8) \ N_2^2(Ax - By) \leq \phi(N_2^2(Sx - Ty), N_2(Ax - Sx), N_2(By - Ty), N_2(Ax - Ty)) \]

for all $x, y$ in $X$ where $\phi: [0, \infty)^5 \rightarrow [0, \infty)$ is

1. nondecreasing and upper semicontinuous in each coordinate variable.
2. for each $t > 0$, $\gamma(t) = \max\{\phi(0,0,t,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\} < t$

Then $A, B, S,$ and $T$ have a unique common fixed point.

Remark 4. Theorem 9.5 extends and generalises Theorem 4 of Murthy and Sharma [126] by replacing weakly uniformly contraction pair of maps with compatible maps of type (A-1) or type (A-2).

As an immediate consequence of Theorem 9.2 we have the following:

Corollary 9.6. Let $A, B, S,$ and $T$ be from saks space $(X, N_1, N_2)$ into itself satisfying (9.3.1), (9.3.4), (9.3.5) and

\[(9.3.9) \ N_2^2(Ax - By)^p \leq a \cdot (N_2^2(Sx - Ty))^p \]

for all $x, y$ in $X$, $0 < a < 1$ and $p \geq 1$.

Then $A, B, S,$ and $T$ have a unique common fixed point in $X$. 

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