INTRODUCTION
CHAPTER I

INTRODUCTION

Functional analysis is an abstract branch of mathematics that originated from classical analysis. The study of fixed point theorems comes under the domain of functional analysis.

The first result on fixed point was proved by Poincare [148] in 1895. Consequently in 1910 the Brouwer’s fixed point theorem appeared which stated as follows:

Theorem A. Every continuous map f of the closed unit ball \( B = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\} \) in \( \mathbb{R}^n \) to itself has a fixed point.

In 1927 J. Schauder extended Brouwer's fixed point theorem to separable spaces and then to arbitrary spaces in 1930 and proved that every continuous mapping on a convex compact subset of a Banach space has a fixed point.

From the mid sixties the concept of contraction mappings have drawn the attention of many mathematicians in the field of point set topology and non linear functional analysis. The work of S. Banach commonly known as Banach's contraction principle is the foundation stone over which the whole bulk of fixed point theorems rest. Banach's contraction principle states that

Theorem B. Let \((X,d)\) be a complete metric space and \(T\) be a self map of \(X\). If there exists a real number \(q, \ 0 < q < 1\), such that for all \(x,y \in X\), \(d(Tx, Ty) \leq q \cdot d(x,y)\), then \(T\) has a fixed point.

In fact the fixed point theorems have extensive applications in proving the existence and uniqueness of the solutions of Differential equations, integral equations, space mechanics, statistics and certain branch of physiology. These theorems also have great applications on non linear mechanics, fluid mechanics, topological dynamics, theory of games and many other related fields. Along with various applications the fixed point theorems have given many fruitfull applications in initial value problems as well as boundary value problems. In the last five years many mathematicians have applied fixed point theorems in proving the existence of solutions of complimentarity problems and variational inequalities.

1.2 COMMON FIXED POINTS IN METRIC SPACES

In 1976 Jungck [88] initially gave a fixed point theorem for commuting maps, which generalises the well known Banach's fixed point theorem. This result was generalised and extended in various ways by S. S. Chang [26], K.M. Das and K.V. Naik [41], B. Fisher [53], S.L. Singh and S.P. Singh [197] etc.
Recently B.Fisher [54], M.S.Khan and M.Imdad [105] proved some common fixed point theorems for four and three commuting mappings, respectively, which extend the result of G Jungck [88].

Let \( T \) and \( I \) be two mappings of a metric space \((X,d)\) into itself. Sessa [167] defined \( T \) and \( I \) to be weakly commuting if \( d(TTx,ITx) \leq d(Tx,Ix) \) for any \( x \in X \). Clearly two commuting mappings weakly commute, but two weakly commuting mappings in general do not commute. He also proved some common fixed point theorem for weakly commuting mappings which generalises the result of K.M.Das and K.V.Naik [41]. In [89] G.Jungck introduced more generalised commutativity called compatibility, which is more general than weak commutativity.

**Definition A.** Let \( S \) and \( T \) be self maps of a metric space \((X,d)\). The mappings \( S \) and \( T \) are said to be compatible if
\[
\lim_{n \to \infty} d(STx_n,TSx_n) = 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \in X.
\]

By employing compatible mappings instead of commuting mappings and using four maps as opposed to three, G.Jungck [90] extended the result of M.S.Khan and M.Imdad [105], S.L.Singh and S.P.Singh [197].

Recently Jungck et al.[92] introduced the concept of compatible mappings of type (A) which is equivalent to compatible mappings under certain conditions, and proved a common fixed point theorem for compatible mappings of type (A) in a metric space. Since then many fixed point theorems have been proved for compatible mappings of type (A).

**Definition B:** Let \( S \) and \( T \) be self maps of a metric space \((X,d)\). The mappings \( S \) and \( T \) are said to be compatible of type (A) if
\[
\lim_{n \to \infty} d(STx_n,TTx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(TSx_n,SSx_n) = 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \in X.
\]

More recently S.M.Kang and Y.P.Kim [94] extending the results of B.Fisher [54], G.Jungck [90], M.S.Khan and M.Imdad [105] proved the following.

**Theorem C.** Let \( A,B,S \) and \( T \) be mappings from a complete metric space \((X,d)\) into itself satisfying the following:

1. \( A(X) \subseteq T(X) \), \( B(X) \subseteq S(X) \)

2. \( d(Ax,By) \leq h \max\{d(Ax,Sx),d(By,Ty),1/2.(d(Ax,Ty)+d(By,Sx)),d(Sx,Ty)\} \)

for all \( x,y \) in \( X \), and \( 0 \leq h < 1 \).

3. The pairs \((A,S)\) and \((B,T)\) are compatible

4. One of \( A,B,S \) and \( T \) is continuous.

Then \( A,B,S \) and \( T \) have a unique common fixed point.
In chapter II we have introduced the concept of compatible mappings of type (A-1) and type (A-2) and shown that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions. In the sequel we prove some common fixed point theorem for compatible mappings of type (A-1) and type (A-2) which generalises the results of Kang and Kim [94], Fisher [53], Jungck [90], Khan and Imdad [105].

Precisely we have proved the following

**Theorem 1.** Let A,B,S and T be from metric space (X,d) into itself satisfying the following:

1.2.1) \( A(X) \cup B(X) \subseteq S(X) \cap T(X) \)

1.2.2) \( S(X) \cap T(X) \) is a complete subspace of X

1.2.3) \[ 1 + p.\{d(Ax,Sx) + d(By,Ty)\}.d(Ax,By) \leq p.\{d'(Ax,Sx) + d'(By,Ty)\} + \max[d(Ax,Sx), d(By,Ty)], 1/2.\{d(Ax,Ty) + d(By,Sx)\}, d(Sx,Ty)] \]

for all \( x,y \) in X, \( 0 \leq h < 1 \) and \( p > 0 \). Then

a) A and S have a coincidence point in X.

b) B and T have a coincidence point in X.

**Theorem 2.** Let A,B,S and T be from metric space (X,d) into itself satisfying \( 1.2.1, 1.2.2, 1.2.3 \) and \( 1.2.4 \) The pairs \( (A,S) \) and \( (B,T) \) are compatible of type (A-1) or type (A-2). Then A,B,S, and T have a unique common fixed point.

**Theorem 3.** Let A,B,S and T be from metric space (X,d) into itself satisfying \( 1.2.1, 1.2.2, 1.2.4 \) and the following

1.2.5) \[ 1 + p.\{d(Ax,Sx) + d(By,Ty)\}.d(Ax,By) \leq p.\{d'(Ax,Sx) + d'(By,Ty)\} + \max[d(Ax,Sx), d(By,Ty)], 1/2.\{d(Ax,Ty) + d(By,Sx)\}, d(Sx,Ty)] \]

for all \( x,y \) in X, \( p \geq 0 \), and \( \phi : [0,\infty) \rightarrow [0,\infty) \) is a nondecreasing and upper semicontinuous function and \( \phi(t) < t \) for all \( t > 0 \) Then

a) A and S have a coincidence point in X.

b) B and T have a coincidence point in X.

**Theorem 4.** Let A,B,S and T be from metric space (X,d) into itself satisfying \( 1.2.1, 1.2.2, 1.2.4 \) and \( 1.2.5 \). Then A,B,S, and T have a unique common fixed point.

In a recent paper see Chatterjee and Singh [29], the following theorem was proved for four self maps \( T_i, i = 1,2,3,4 \) of a complete metric space X.
**Theorem 6** Let \((X, d)\) be a complete metric space. Let \(T_i : X \to X, i = 1, 2, 3, 4\) satisfy the following inequality

\[
d(T_i T_j x, T_i T_j y) \leq \alpha_1 d(x, y)^2 + \alpha_2 d(x, T_i T_j x) d(y, T_i T_j y) + \alpha_3 d(x, T_i T_j y) d(y, T_i T_j x)
\]

for all \(x, y \in X\), where \(\alpha_1 \geq 0, \alpha_2 + \alpha_3 + \alpha_4 < 1\) and \(\alpha_i + \alpha_i < 1\). Further assume that \(T_i T_j = T_j T_i, T_i T_j = T_j T_i\). Then \(T_i, i = 1, 2, 3, 4\) have a unique common fixed point in \(X\).

In this context we have proved the following which extends the result of Chatterjee and Singh [29], for four maps to five maps.

**Theorem 5.** Let \((X, d)\) be a complete metric space. Let \(T_i : X \to X, i = 1, 2, 3, 4, 5\) satisfy the following conditions

\[
d(T_i T_j x, T_i T_j y) \leq \alpha_1 d(x, y)^2 + \alpha_2 d(x, T_i T_j x) d(y, T_i T_j y) + \alpha_3 d(x, T_i T_j y) d(y, T_i T_j x) + \alpha_4 d(x, T_i T_j y) d(y, T_i T_j x)
\]

for all \(x, y \in X\), where \(\alpha_i \geq 0, \alpha_i + \alpha_2 + \alpha_3 + \alpha_4 < 1\) and \(\alpha_i + \alpha_i < 1\). Further assume that \(T_i T_j = T_j T_i, T_i T_j = T_j T_i, T_i T_j = T_j T_i\). Then \(T_i, i = 1, 2, 3, 4, 5\) have a unique common fixed point in \(X\).

### 1.3 Common Fixed Points in Normed Spaces

Naimpally and Singh [128] extended some fixed point theorems of Rhoades [157A] for a mapping \(T\) satisfying certain contractive conditions, if the sequence of Ishikawa iterates converges to a fixed point of \(T\). We recall that the G-iterative process associated with two self mappings \(T_1\) and \(T_2\) of a normed space \(N\) is defined in the following manner.

Let \(x_0 \in N\) and set

\[
x_{2n+1} = (\mu_{2n} - \lambda_{2n})x_{2n} + \lambda_{2n} T_1 x_{2n} + (1 - \mu_{2n})T_2 x_{2n+1}
\]

and,

\[
x_{2n+2} = (\mu_{2n+1} - \lambda_{2n+1})x_{2n+1} + \lambda_{2n+1} T_2 x_{2n+1} + (1 - \mu_{2n+1})T_1 x_{2n+2}
\]

for \(n = 0, 1, 2, \ldots\), where (a) \(\mu_n = \lambda_n = 0\), (b) \(0 < \lambda_n < 1, 0 \leq \mu_n \leq 1\), such that \(\mu_n \geq \lambda_n, n > 0\) (c) \(\lim \lambda_n = h > 0\), and (d) \(\lim \mu_n = 1\).

When \(<\mu_n> = 1\), the G-iterative process reduces to Mann iteration.
Pathak and Dubey [144] extending the result of Naimpally and Singh [128] proved the following.

**Theorem 6.** Let X be a closed, convex, bounded subset of a normed space N and let $T_1$ and $T_2$ be self mappings of X satisfying

\[
(1.3.1) \quad ||T_1 x - T_2 y|| \leq q \max(c ||x - y||, ||x - T_1 x|| + ||y - T_2 y||, ||x - T_2 y|| + ||y - T_1 x||)
\]

for every $x, y \in X$ where $c \geq 0, 0 < q < 1$. Let the sequence $<x_n>$ be defined in accordance with the $G$-iterates associated with $T_1$ and $T_2$ as defined above:

If $<x_n>$ converges to $z$ in $X$, then $z$ is a common fixed point of $T_1$ and $T_2$.

In this context in chapter III, we have proved the following theorems which extends the results of Pathak and Dubey [144], Naimpally and Singh [128] and many others. In fact we have proved the following:

**Theorem 6.** Let X be a closed, convex, bounded subset of a normed space N and let $T_1$ and $T_2$ be self mappings of X satisfying

\[
(1.3.2) \quad ||T_1 x - T_2 y||^p + a_1(||x - T_1 x||^p + ||x - T_2 y||^p) + a_2(||y - T_2 y||^p + ||y - T_1 x||^p) 
\]

\[
\leq q \max(c ||x - y||^p, ||x - T_1 x||^p + ||y - T_2 y||^p, ||x - T_2 y||^p + ||y - T_1 x||^p)
\]

for every $x, y \in X$ where $c \geq 0, 0 < q - (a_1 + a_2) < 1$. Let the sequence $<x_n>$ be defined in accordance with Mann iteration process associated with two mappings $T_1$ and $T_2$ as follows

\[
(1.3.3) \quad x_{2n+1} = (1 - c_{2n})x_{2n} + c_{2n}T_1 x_{2n}
\]

\[
(1.3.4) \quad x_{2n+2} = (1 - c_{2n+1})x_{2n+1} + c_{2n+1}T_2 x_{2n+1}
\]

for $n \geq 0$ where $c_0 = 1, 0 < c_n < 1$ for $n \geq 0$, and $\lim c_n = h > 0$ If $<x_n>$ converges to $z$ in $X$ then $z$ is a common fixed point of $T_1$ and $T_2$. Moreover if $\max\{cq, 2q\} < 1 + a_1 + a_2$, then $z$ is a unique common fixed point of $T_1$ and $T_2$.

**Theorem 7.** Let X be a closed, convex, bounded subset of a normed space N and let $T_1$ and $T_2$ be self mappings of X satisfying any one of the following.

For all $x$ and $y$ in $X$

(I) $||T_1 x - T_2 y||^p \leq q \max\{c ||x - y||^p, ||x - T_1 x||^p, ||y - T_2 y||^p, ||x - T_2 y||^p + ||y - T_1 x||^p\}$

\[0 < q < 1\]

(II) $||x - T_1 x||^p + ||y - T_2 y||^p \leq a_1 ||x - y||^p, 1 \leq a < 2$
(II) \(|x - T_1x|^p + |y - T_2y|^p \leq b \left( |x - T_1y|^p + |y - T_2x|^p + |x - y|^p \right) \quad \frac{1}{2} \leq b < \frac{2}{3}

(IV) \(|x - T_1x|^p + |y - T_2y|^p + T_1x - T_2y|^p \leq c \left( \|x - T_1y\|^p + \|y - T_2x\|^p \right) \quad 1 \leq c < \frac{3}{2}

(V) \left( \|T_1x - T_2y\|^p \leq k \max \{ c|x - y|^p, \|x - T_1x\|^p, \|y - T_2y\|^p, \|T_1x - T_2y\|^p \} / 2 \right) \quad 0 \leq k < 1.

Let the sequence \(\langle x_n \rangle\) be defined as in (1.3.3) and (1.3.4). If \(\langle x_n \rangle\) converges to \(z\) in \(X\) then \(z\) is a common fixed point of \(T_1\) and \(T_2\).

In this chapter we have also investigated the solvability of certain non-linear functional equations in Banach spaces. In fact we have proved the following.

**Theorem 8.** Let \(\{ f_n \}\) and \(\{ g_n \}\) be sequence of elements in a Banach space \(B\). Let \(u_n\) and \(v_n\) be the unique solution of the equation \(u - T_1u = f_n\) and \(v - T_2v = g_n\) where \(T_1\) and \(T_2\) are mappings of \(B\) into itself satisfying condition (1.3.2) of Theorem 6.

If \(\|f_n\| \to 0\) and \(\|g_n\| \to 0\) as \(n \to \infty\) and \(\max(c, q_1 + q_2) < 1 + a_1 + a_2\), then \(\{ u_n \}\) and \(\{ v_n \}\) converges to a common solution of the equation \(x = T_1x = T_2x\).

### 1.4. FIXED POINTS IN FUZZY METRIC SPACES

A fuzzy metric space (shortly an FM-space) is an ordered triplet \((X, M, *)\) consisting of a nonempty set \(X\), a fuzzy set \(M\) on \(X^2 \times [0, \infty)\) and a continuous \(T\)-norm \(*\) ([65], [107]). The functions \(M(x, y, \cdot): [0, \infty) \to [0, \infty)\) are left continuous and are assumed to satisfy the following conditions:

\((FM-1)\) \(M(x, y, t) = 1\) for all \(t > 0\) if and only if \(x = y\).

\((FM-2)\) \(M(x, y, 0) = 0\).

\((FM-3)\) \(M(x, y, t) = M(y, x, t)\).

\((FM-4)\) \(M(x, y, t)*M(y, z, s) \leq M(x, z, t+s)\) for all \(x, y, z\) in \(X\) and \(t, s \geq 0\).

In 1988, Grabiec [65], extending the well known fixed point theorem of Banach [4] and Edelstein [49] to fuzzy metric spaces, in the sense of Kramosil and Michalek [107] proved the following:

**Theorem 9.** Let \((X, M, *)\) be a complete fuzzy metric space and \(P\) be a self map of \(X\) such that

\((1.4.1)\) \(M(Px, Py, kt) \geq M(x, y, t)\)

for all \(x, y\) in \(X\) and \(k \in (0, 1)\). Then \(P\) has a fixed point.
Following Grabiec [65] and Kramosil-Michalek [107], Mishra-Sharma-Singh [120] obtained common fixed point theorems for compatible maps and asymptotically commuting maps on fuzzy metric spaces, which generalise, extend and fuzzify several fixed point theorems for contractive type maps on metric spaces and other spaces.

In chapter 3 we have introduced the concept of compatible mappings of type (A-1) and type (A-2) in fuzzy metric space and show that they are equivalent to compatible mappings under certain conditions. In the sequel we prove some common fixed point theorem for compatible mappings of type (A-1) and type (A-2) on fuzzy metric spaces which generalise, extend, unify and fuzzify several well known fixed point theorems for contractive type maps on metric spaces, Menger spaces, uniform spaces and fuzzy metric spaces.

Precisely we have proved the following:

**Theorem 9.** Let $(X,M,\ast)$ be an FM-space with $t \ast t \geq t$ for all $t$ in $[0,1]$ and $P,Q,S,T$ be self maps of $X$ such that

\begin{align*}
(1.4.2) & \quad P(X) \subseteq T(X) \quad \text{and} \quad Q(X) \subseteq S(X) \\
(1.4.3) & \quad [1 + p_M(Sx,Ty,kt)]^*M(Px,Qy,kt) \geq p[M(Px,Sx,kt) + M(Qy,Ty,kt) + M(Px,Ty,kt) + M(Qy,Sx,kt)] + M(Px,Sx,kt) + M(Qy,Ty,kt) + M(Px,Ty,kt) + M(Qy,Sx,(2-k)t)] \\
& \quad \text{for all } x,y \in X, \quad p \geq 0, \quad t > 0 \quad \text{and } k \in (0,2),
\end{align*}

(1.4.4) the pairs $(P,S)$ and $(Q,T)$ are compatible of type (A-1) or type (A-2)

(1.4.5) one of $S$ and $T$ is continuous

Then $P,Q,S$ and $T$ have a unique common fixed point in $X$.

**Theorem 10.** Let $(X,M,\ast)$ be a complete FM-space with $t \ast t \geq t$, $t \in [0,1]$, and let $P$ and $Q$ be two maps on the product $X \times X$ with values in $X$. If there exists a constant $k \in (0,1)$ such that

\begin{align*}
(1.4.6) & \quad [1 + p_M(x,u,kt)]^*M(P(x,y),Q(u,v),kt) \geq p[M(P(x,y),x,kt) + M(Q(u,v),u,kt) + M(P(x,y),x,t) + M(Q(u,v),u,t)] + M(P(x,y),x,kt) + M(Q(u,v),u,kt) + M(P(x,y),x,(2-k)t)] \\
& \quad \text{for all } x,y \in X, \quad p \geq 0, \quad t > 0 \quad \text{and } k \in (0,2), \quad \text{then there exists exactly one point } w \in X \quad \text{such that} \quad P(w,w) = w = Q(w,w).
\end{align*}

As a consequence many well known results follows as corollaries of theorem 9.
1.5. COORDINATEWISE COMPATIBILITY OF MAPPINGS

The well known Banach contraction principle has been generalised in two directions in the galaxy of contractive principles by Gerald Jungck [88] and Janusz Matkowski [115]-[116]

In 1991, Singh and Gairola [186] formulated a general fixed point theorem, by introducing a new Matkowski type contractive condition and a new concept of commutativity of maps called coordinatewise weakly commuting maps. Their results apart from generalising diverse fixed point theorems of Matkowski, Jungck and their generalisations, includes abounding contractive principles

**Definition B** Let $X_i$, $i = 1, 2, \ldots, n$ be nonempty arbitrary sets and $X = X_1 \times X_2 \times \ldots \times X_n$. Let $F_i, G_i : X \rightarrow X$. The system of maps $(F_1,F_2,\ldots,F_n)$ and $(G_1,G_2,\ldots,G_n)$ are said to be coordinatewise commuting if and only if

$$F_i(G_i(x_1,\ldots,x_n),\ldots,G_n(x_1,\ldots,x_n)) = G_i(F_i(x_1,\ldots,x_n),\ldots,F_n(x_1,\ldots,x_n)),$$

$i = 1, 2, \ldots, n$, for all $(x_1, \ldots, x_n) \in X$.

**Definition C** Let $(X_i,d_i)$, $i = 1, 2, \ldots, n$ be metric spaces and $X = X_1 \times X_2 \times \ldots \times X_n$. Let $F_i, G_i : X \rightarrow X$. The system of maps $(F_1,F_2,\ldots,F_n)$ and $(G_1,G_2,\ldots,G_n)$ are said to be coordinatewise weakly commuting if and only if

$$d_i(F_i(G_i(x_1,\ldots,x_n),\ldots,G_n(x_1,\ldots,x_n)), G_i(F_i(x_1,\ldots,x_n),\ldots,F_n(x_1,\ldots,x_n)))$$

$$\leq d_i(F_i(x_1,\ldots,x_n), G_i(x_1,\ldots,x_n)),$$

$i = 1, 2, \ldots, n$, for all $(x_1, \ldots, x_n) \in X$.

Singh and Gairola [186] proved the following:

**Theorem E** Let $(X_i,d_i)$ be metric spaces and $P_i, Q_i, S_i, T_i : X \rightarrow X$, $i = 1, \ldots, m$, such that

1.5.1 $P_i(X) \cup Q_i(X) \subseteq S_i(X) \cap T_i(X)$ and $S_i(X) \cap T_i(X)$ is a complete subspace of $X$, $i = 1, 2, \ldots, m$.

1.5.2 $(P_1, \ldots, P_m)$ weakly commutes with $(S_1, \ldots, S_m)$ and $(Q_1, \ldots, Q_m)$ weakly commute with $(T_1, \ldots, T_m)$.

If there exists non negative numbers $b, a_i, k = 1, 2, \ldots, m$ such that

1.5.3 $c^{(i+1)}_k = \begin{cases} c^{(i)}_1 x^{(i)}_{i+1} + c^{(i)}_{i+1} x^{(i+1)}_1 & \text{for } i \neq k \\ c^{(i)}_{i+1} x^{(i)}_{i+1} - c^{(i)}_1 x^{(i+1)}_1 & \text{for } i = k \end{cases}$ for $i = 1,2,\ldots,m$ such that
(1.5.4) $0 \leq b < 1-h$ and $h = \max \left( \frac{1}{r_i} \sum_{j \neq i} a_j, r_i \right)$

(1.5.5) $c_{ik}^{i-1} = \begin{cases} 1 - a_k & \text{for } i = k \\ a_k & \text{for } i \neq k \end{cases}$

$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{m} u_i v_i$

fulfil the conditions

(1.5.6) $c_{ik}^{i-1} > 0$, $i = 1, \ldots, m$, $t = 0, \ldots, m-1$

and

(1.5.7) $d\left( P_i(x(1,n)), Q_i(y(1,n)) \right) \leq \max \left( \frac{1}{r_i} \sum_{j \neq i} d_i \left( S_i(x(1,n)), T_j(y(1,n)) \right), b \max \left( d_i \left( S_i(x(1,n)), P_i(x(1,n)) \right), d_i \left( T_j(y(1,n)), Q_i(y(1,n)) \right) \right) \right)$

for all $x(1,n), y(1,n) \in X$, then the system of equations

$P_i(x(1,m)) = Q_i(x(1,m)) = x_i = S_i(x(1,m)) = T_i(x(1,m))$

has a unique common solution $x_1, \ldots, x_m$ such that $x_i \in X_i$, $i = 1, \ldots, m$.

In this context in chapter V we have introduced the concept of coordinatewise compatibility of mappings and demonstrated the utility of the concept of coordinatewise compatibility of mappings in the context of fixed point theory, by generalising the result of Singh and Gairola [186]. Precisely we have proved the following:

Theorem 11. Let $(X_1, d_1)$ be metric spaces and $P_1, Q_1, S_1, T_1 : X \rightarrow X$, $i = 1, \ldots, m$, such that (1.5.1), (1.5.3), (1.5.4), (1.5.5), (1.5.6), (1.5.7) and the following holds

(1.5.8) $(P_1, \ldots, P_m)$ and $(S_1, \ldots, S_m)$ as well as $(Q_1, \ldots, Q_m)$ and $(T_1, \ldots, T_m)$ are coordinatewise compatible.

Then the system of equations

$P_i(x(1,m)) = Q_i(x(1,m)) = x_i = S_i(x(1,m)) = T_i(x(1,m))$

has a unique common solution $x_1, \ldots, x_m$ such that $x_i \in X_i$, $i = 1, \ldots, m$.

1.6. COMMON FIXED POINTS IN MENGERS

A probabilistic metric space (briefly PM-space) is a pair $(X, F)$ where $X$ is a non empty set and $F$ is a mapping from $X \times X$ to $L$. For any $(u, v) \in X \times X$, the distribution
function $F(u,v)$ is denoted by $F_{uv}$. The functions $F_{uv}$ are assumed to satisfy the following conditions

(P1) $F_{uv}(x) = 1$ for all $x > 0$ if and only if $u = v$

(P2) $F_{uv}(0) = 0$ for all $u,v \in X$

(P3) $F_{uv}(x) = F_{vu}(x)$ for all $u,v \in X$.

(P4) If $F_{uv}(x) = 1$ and $F_{uv}(y) = 1$, then $F_{uv}(x+y) = 1$ for all $u,v,w \in X$.

A function $t : [0,1] \times [0,1] \rightarrow [0,1]$ is called a $T$-norm if it satisfies the following conditions:

(t1) $t(a,1) = a$ for all $a \in [0,1]$ and $t(0,0) = 0$.

(t2) $t(a,b) = t(b,a)$ for all $a,b \in [0,1]$

(t3) If $c \geq a$ and $d \geq b$, then $t(c,d) \geq t(a,b)$.

(t4) $t(t(a,b),c) = t(a,t(b,c))$ for all $a,b,c \in [0,1]$.

A Menger space is a triplet $(X,F,t)$ where $(X,F)$ is a PM-space and $t$ is a $T$-norm with the following condition:

(P5) $F_{uv}(x + y) \geq t(F_{uv}(x) , F_{uv}(y))$ for all $u,v,w \in X$ and $x,y \in \mathbb{R}^+$.

The existence of fixed points for compatible mappings in metric spaces and probabilistic metric spaces was shown by Jungck [89]-[90], Mishra [119] and Sessa et al [170].

Recently Jungck et al [92] introduced the concept of compatible mappings of type (A) in metric spaces and in 1992 Cho et al [31] introduced the concept of compatible mappings of type (A) in Menger spaces and proved some fixed point theorems for these mappings in Menger spaces which extend, generalise and improve many known results.

Cho et al [31] proved the following:

**Theorem**. Let $(X,F,t)$ be a complete Menger space with $t(x,y) = \min(x,y)$ for all $x,y \in (0,1)$ and $A,B,S$ and $T$ be self maps of $X$ such that

(1.6.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(1.6.2) the pairs $(A,S)$ and $(B,T)$ are compatible of type (A)

(1.6.3) $[F_{Ab}(kk)]^2 \geq \min\{[F_{Sa},(x)]^2, F_{Sa}(x), F_{Sb}(y), F_{Sb}(y), F_{Sa}(x), F_{Sb}(y), F_{Sa}(x), F_{Sb}(y), F_{Sb}(y)\}$
for all \( u, v \in X \), \( k \in (0,1) \) and \( x \geq 0 \)

(1.6.4) One of \( A, B, S \) and \( T \) is continuous

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \)

In this context, in chapter VI we introduce the concept of compatible mappings of type (A-1) and type (A-2) in Menger spaces and show that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions. In the sequel we have proved some common fixed point theorems which improves the result of Cho et al [31]. In fact we have proved the following

**Theorem 12.** Let \( (X, F, t) \) be a complete Menger space with \( t(x, y) = \min(x, y) \) for all \( x, y \in (0,1) \) and \( A, B, S \) and \( T \) be self maps of \( X \) such that

(1.6.5) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \)

(1.6.6) the pairs \((A, S)\) and \((B, T)\) are compatible of type (A-1) or type (A-2)

(1.6.7) \[
[F_{Su,Bv}(kx)]^2 \geq \min\{ [F_{Su,Tv}(x)]^2, F_{Su,Au}(x)F_{Tu,Bv}(x), F_{Su,Tv}(x)F_{Tu,Au}(x), \\
F_{Su,Bv}(x)F_{Su,Tv}(2x), F_{Su,Tv}(x)F_{Tu,Au}(x), \\
F_{Su,Bv}(2x)F_{Tu,Au}(x), F_{Su,Tv}(x)F_{Tu,Bv}(x), F_{Su,Au}(x)F_{Tu,Bv}(x) \}
\]

for all \( u, v \in X \), \( k \in (0,1) \) and \( x \geq 0 \)

(1.6.8) One of \( A, B, S \) and \( T \) is continuous

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Theorem 13** Let \( A, B, S \) and \( T \) be mappings from a complete metric space \((X, d)\) into itself such that

(1.6.9) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \),

(1.6.10) The pairs \((A, S)\) and \((B, T)\) are compatible of type (A-1) or type (A-2)

(1.6.11) one of \( A, B, S \) and \( T \) is continuous and

(1.6.12) \[
[d^2(Au,Bv)] \leq k \max\{d^2(Su,Tv), d(Su,Au)d(Tv,Bv), d(Su,Tv)d(Su,Au), \\
d(Su,Tv)d(Tv,Bv), d(Su,Tv)d(Su,Bv), d(Su,Tv)d(Tv,Au), \\
d(Su,Bv)d(Tv,Au), d(Su,Au)d(Tv,Au), d(Su,Bv)d(Tv,Bv) \}
\]

for all \( u, v \in X \), and \( k \in (0,1) \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).
In 1990, J. L. C. Camargo [23], presented the locally convex version of a fixed point theorem of R. Kannan, by proving a fixed point theorem in locally convex Hausdorff topological vector space.

The following result was obtained by him:

**Theorem 6.** Let \( X \) be a locally convex Hausdorff Topological vector space, whose topology is generated by a family \( U \) of continuous seminorms, \( M \) be a nonempty sequentially complete subset of \( X \), \( T_1 \) and \( T_2 \) be two self maps of \( M \) such that for each \( P \in U \) and \( x, y \) in \( M \), there exists a number \( a(p) \), such that \( 0 < a(p) < 1/2 \), and

\[
(1.7.1) \quad P(T_1(x) - T_2(y)) \leq a(p) \left[ P(x - T_1(x)) + P(y - T_2(y)) \right].
\]

Then \( T_1 \) and \( T_2 \) has a unique common fixed point.

In this context, in chapter VII, we have obtained the locally convex version of the results of Naimpally and Singh [128], as in the following:

**Theorem 14.** Let \( X \) be a locally convex Hausdorff Topological vector space, whose topology is generated by a family \( U \) of continuous seminorms. \( M \) be a nonempty sequentially complete subset of \( X \), \( T_1 \) and \( T_2 \) be two self maps of \( M \) such that for each \( P \in U \) and \( x, y \) in \( M \), there exists a number \( a(p) \), satisfying any one of the following conditions:

I. For \( 0 \leq a(p) < 1/2 \) at least one of the following conditions hold.
   
   (A) \( P(T_1 x - T_2 y) \leq a(p) \max \{ P(x - y), P(x - T_1 x), P(y - T_2 y), P(x - T_2 y), P(y - T_1 x) \} \)
   
   (B) \( P(T_1 x - T_2 y) \leq a(p) \max \{ P(x - y), P(x - T_1 x), P(y - T_2 y), \[ P(x - T_2 y) + P(y - T_1 x) \] / 2 \} \)
   
   (C) \( P(T_1 x - T_2 y) \leq a(p) \max \{ P(x - y), P(x - T_1 x), P(y - T_2 y), P(x - T_2 y) + P(y - T_1 x) \} \)
   
   (D) \( P(T_1 x - T_2 y) \leq a(p) \max \{ P(x - y), P(x - T_1 x) + P(y - T_2 y), P(x - T_2 y) + P(y - T_1 x) \} \)

II. \( P(x - T_1 x) + P(y - T_2 y) + P(T_1 x - T_2 y) \leq a(p) \left[ P(x - T_2 y) + P(y - T_1 x) \right] \), \( 1 \leq a(p) < 3/2 \);

III. \( P(x - T_1 x) + P(y - T_2 y) \leq a(p) \left[ P(x - T_2 y) + P(y - T_1 x) + P(x - y) \right] \), \( 1/2 \leq a(p) < 2/3 \);

IV. \( P(x - T_1 x) + P(y - T_2 y) \leq a(p) P(x - y) \), \( 1 \leq a(p) < 2 \).
Then $T_1$ and $T_2$ has a unique common fixed point

1.8. (COMMON FIXED) POINTS IN 2-METRIC SPACES

A 2-metric space is a set $X$ with a real valued function $d$ on $X \times X \times X$ satisfying the following conditions:

(M1) For distinct points $x$ and $y$ in $X$ there exists a point $z$ in $X$ such that $d(x,y,z) = 0$.

(M2) $d(x,y,z) = 0$ if at least two of $x,y,z$ are equal.

(M3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$

(M4) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $u$ in $X$.

The concept of 2-metric spaces was initially investigated by Gahler [61]-[63] and then developed by many others. On the other hand a number of authors ([30], [83]-[85], [98], [100], [101], [103], [104], [106], [108], [110], [111], [124], [127], [152], [159], [171]-[174], [181-182], [194], [196], [198]) have studied the aspects of fixed point theory in the setting of 2-metric spaces.

In 1984 Khan [98] and in 1986 Naidu and Prasad [127] introduced the concepts of weakly commuting mappings in 2-metric spaces and weak continuity of a 2-metric space respectively and proved several fixed point theorems in 2-metric spaces using these concepts. In 1992 Murthy et al [124] introduced the concept of compatible mappings and compatible mappings of type (A) in 2-metric spaces and derived some relations between these mappings.

They also proved a coincidence point theorem and a fixed point theorem for compatible mappings of type (A) in 2-metric space which extends, generalizes and improves a number of fixed point theorems for commuting mappings, weakly commuting mappings and compatible mappings in 2-metric spaces. Murthy et al [124] proved the following:

Theorem H. Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself satisfying the following:

(1.8.1) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$

(1.8.2) $S(X) \cap T(X)$ is a complete subspace of $X$

(1.8.3) $d^2(Ax,By) \leq \phi [d^2(Sx,Ty,z) + d(Sx,Ax,z)d(By,Ty,z) + d(Sx,By,z)d(Ax,Ty,z) + d(Ax,Sx,z)d(Ty,Ax,z) + d(Sx,By,z)d(Ty,By,z)]$

for all $x,y$ in $X$, where $\phi : R^2 \rightarrow R$ is upper semicontinuous.
non decreasing in each coordinate variable, and for any $t > 0$

\[\phi(t, 1, 0, 0) = 0 \text{ and } \phi(t, t, 0, 0) = 0 \text{ for } t > 0, \beta = 1 \text{ for } \alpha \geq 2, \beta = 1 \text{ for } \alpha \geq 2, \]

\[\gamma(t) = (1) \cdot (1, 1, 1, 1) \cdot 1, \text{ where } \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a mapping and } a_1 + a_2 + a_3 + 4.\]

(1.8.4) The pairs $(A, S)$ and $(B, T)$ are compatible of type (A)

Then $A, B, S$ and $T$ have a unique common fixed point

In this context, in chapter VIII, we have introduced the concept of compatible mappings of type (A-1) and type (A-2) in 2-metric spaces and show that they are equivalent to compatible mappings and compatible mappings of type (A) under certain conditions.

We have also proved a coincidence point theorem and a common fixed point theorem for compatible mappings of type (A-1) and type (A-2) in 2-metric spaces which improves and generalises the results of Murthy et al [124]. In fact we proved the following:

**Theorem 15.** Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying (1.8.1), (1.8.2) and the following:

\[(1.8.5) \quad d(Ax, By, z) \leq \phi(d(Sx, Ty, z) + d(Sx, Ax, z) + d(Ty, By, z) + d(Sx, Ty, z) + d(Sx, By, z), d(Sx, Ty, z) + d(Ty, By, z) + d(Sx, Ax, z) + d(Ty, Ax, z) + d(Sx, By, z) + d(Ty, By, z))\]

for all $x, y, z$ in $X$ and $\phi$ is as defined in (1.8.2).

Then (1) $A$ and $S$ have a coincidence point in $X$ and

(2) $B$ and $T$ have a coincidence point in $X$.

**Theorem 16.** Let $A, B, S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself satisfying (1.8.1), (1.8.2), (1.8.5) and the following:

\[(1.8.6) \quad \text{The pairs } (A, S) \text{ and } (B, T) \text{ are compatible of type (A-1) or type (A-2).}\]

Then $A, B, S$, and $T$ have a unique common fixed point.

**1.9. COMMON FIXED POINTS IN SAKS SPACE**

The concept of compatible maps and weakly uniformly contraction maps were introduced in metric spaces by Jungck [90] and Pathak [143] respectively. On the other hand Murthy and Sharma [126] and Cho and Singh [32]-[33], and many others have studied the aspects of coincidence and common fixed point theorems in the setting of Saks space. They have been motivated by various concepts already known in ordinar
metric spaces and have thus introduced analogue of various concepts in the framework of the saks space. Especially Cho and Singh [12] and Murthy and Sharma[126] introduced the concepts of commuting and weakly uniformly contraction maps respectively in saks space and proved several fixed point theorems using these concepts.

Definition D. Let $X$ be a linear space. A real valued function $f$ defined on $X$ will be called a $B$-norm if it satisfies the following conditions:

1. $f(x) = 0$ if and only if $x = 0$.
2. $f(x+y) \leq f(x) + f(y)$.
3. $f(ax) = |a|f(x)$ where $a$ is any real number.

Definition E. Let $X$ be a linear space. A real valued function $f$ defined on $X$ will be called an $F$-norm if it satisfies (1) and (2) of definition D and the following:

3. If the sequence \{a_n\} of real numbers converges to $a$ and $f(a_n - x) \rightarrow 0$ as $n \rightarrow \infty$ then $f(a_n x_n - ax) \rightarrow 0$ as $n \rightarrow \infty$.

A two norm space is a linear space $X$ with two norms, a $B$-norm $N_1$ and $F$-norm $N_2$ and denoted by $(X, N_1, N_2)$.

Definition F. Let $(X, N_1, N_2)$ be a two norm space. A sequence $\{x_n\}$ in $X$ is said to be gamma convergent to a point $x$ in $X$ if $\sup N_1(x_n) < \infty$ and $\lim N_2(x_n - x) = 0$.

Definition G. Let $(X, N_1, N_2)$ be a two norm space. A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence if $N_2(x_n - x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

Definition H. A two norm space $(X, N_1, N_2)$ is said to be $\gamma$-complete if every $\gamma$-Cauchy sequence in $X$ is $\gamma$-convergent in $X$.

Definition I. Let $X$ be a linear set with two norms $B$-norm $N_1$ and $F$-norm $N_2$ on $X$ respectively. Let $X_* = \{ x \in X : N_1(x) < 1 \}$ and $d(x, y) = N_2(x-y)$ for all $x, y$ in $X_*$. Then $d$ is a metric on $X_*$ and the metric space $(X_*, d)$ is called a saks set.

Definition J. A complete saks set is called a saks space and denoted by $(X_*, N_1, N_2)$.

In this context in chapter IX we introduce the concept of compatible mappings and compatible mappings of type (A-1) and type (A-2) in saks space and give some relationship between these mappings. In the sequel we established a series of coincidence and common fixed point theorems for compatible mappings of type (A-1) and type (A-2) in saks space which extends and generalises many known results in saks space as well as metric spaces. Our theorem extends the results of Diviccaro...
et al[45], Fisher and Sessa [58], Gregus [66], Mukherjee et al [123] and many others.

Precisely we have proved the following

**Theorem 17.** Let $A,B,S$ and $T$ be from saks space $(X, N_1')$ into itself satisfying the following:

1. (1.9.1) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$
2. (1.9.2) $N_2(Ax - By) \leq a(N_1(Sx - Ty)) + b \max\{N_2(Ax - Sx), N_2(By - Ty), \n \}
3. (1.9.3) $S(X) \cap T(X)$ is a complete subspace of $X$

Then

a) $A$ and $S$ have a coincidence point in $X$.

b) $B$ and $T$ have a coincidence point in $X$.

**Theorem 18.** Let $A,B,S$ and $T$ be from saks space $(X, N_1')$ into itself satisfying (1.9.1), (1.9.2), (1.9.3) and

(1.9.4) The pairs $(A,S)$ and $(B,T)$ are compatible of type (A-1) or type (A-2).

Then $A,B,S,$ and $T$ have a unique common fixed point.

**Theorem 19.** Let $A,B,S$ and $T$ be from saks space $(X, N_1')$ into itself satisfying (1.9.1), (1.9.3), (1.9.4) and

(1.9.6) $N_2(Ax - By) \leq \psi(N_1(Sx - Ty), N_2(Ax - Sx), N_2(By - Ty), N_2(Ax - Ty), N_2(By - Sx))$

for all $x,y$ in $X$ where $\psi : [0,\infty)^4 \rightarrow [0,\infty)$ is

1. nondecreasing and upper semicontinuous in each coordinate variable.
2. for each $t > 0$, $\psi(t) = \max\{\psi(0,0,t,t), \psi(t,t,0,t), \psi(t,t,0,2t)\} < t$

Then $A,B,S,$ and $T$ have a unique common fixed point.

**Theorem 20.** Let $A,B,S$ and $T$ be from saks space $(X, N_1')$ into itself satisfying (1.9.1), (1.9.3), (1.9.4) and

(1.9.7) $N_2(Ax - By) \leq k \psi(N_1(Sx - Ty), N_2(Ax - Sx), N_2(By - Ty), (1/2) (N_2(Ax - Ty) + N_2(By - Sx)))$
for all \( x, y \in X \) where \( \phi : [0, \infty) \longrightarrow [0, \infty) \) is

1. nondecreasing and upper semicontinuous in each coordinate variable.

2. for each \( t > 0 \),

\[
\max \{ \phi(0,0,1,1,1), \phi(t,1,2,0), \phi(t,0,2,1) \} < t
\]

Then \( A, B, S, \) and \( T \) have a unique common fixed point.

**Theorem 21.** Let \( A, B, S, \) and \( T \) be from saks space \((X, N_1, N_2)\) into itself satisfying

\( (1.9.1), (1.9.3), (1.9.4) \) and

\[
(1.9.8) \ N_2^2(Ax - By) \leq \phi(N_2^2(Sx - Ty), N_2^2(Ax - Sx), N_2^2(By - Ty), N_2^2(Ax - Ty), N_2^2(By - Sx))
\]

for all \( x, y \in X \) where \( \phi : [0, \infty) \longrightarrow [0, \infty) \) is

1. nondecreasing and upper semicontinuous in each coordinate variable.

2. for each \( t > 0 \),

\[
\max \{ \phi(0,0,1,1,1), \phi(t,1,2,0), \phi(t,0,2,1) \} < t
\]

Then \( A, B, S, \) and \( T \) have a unique common fixed point.

**1.10. Variational Inequalities and Their Solutions**

Variational inequalities arise in optimal stochastic control[13], as well as in other problems in mathematical physics, e.g. deformation of elastic bodies stretched over solid obstacles, elasto-plastic torsion, etc.[48]. The iterative methods for solution of discrete V.I.'s are very suitable for implementation on parallel computers with single instruction, multiple-data architecture, particularly on massively parallel processors.

The variational inequality problem is to find a function \( u \) such that

\[
(1.10.1) \ \max \{ Lu - f, u - \phi \} = 0 \ \text{on} \ \Omega: u = 0 \ \text{on} \ \partial \Omega
\]

where \( j \) is a bounded, open subset of \( \mathbb{R}^n \), \( L \) is an elliptic operator defined on \( \Omega \) by

\[
L = -a_j(x) \partial^2 \phi / \partial x_j \partial x_j + b_j(x) \partial \phi / \partial x_j + c(x) \phi
\]

where summation with respect to repeated indices is implied; \( c(x) \geq 0, [a_j(x)] \) is a strictly positive definite matrix, uniformly in \( x \), for \( x \in \Omega \); \( f \) and \( \phi \) are smooth functions defined in \( \Omega \) and \( \phi \) satisfies the condition \( \phi(x) \geq 0 \) for \( x \in \Omega \).

A problem related to \((1.10.1)\) is the two-obstacle variational inequality. Given two functions \( \phi \) and \( \mu \) defined on \( \Omega \), and satisfying \( \phi \leq \mu \) in \( \Omega \), \( \phi \leq 0 \leq \mu \) on \( \Omega \), the corresponding variational inequality is

\[
(1.10.2) \ \max \{ \min \{ Lu - f, u - \phi \}, u - \mu \} = 0 \ \text{in} \ \Omega: u = 0 \ \text{on} \ \partial \Omega
\]
The problem (1.10.2) arises in stochastic game theory. In this situation, two players are trying to control a diffusion process by stopping the process, the first player is trying to maximize a cost functional, and the second player is trying to minimize a similar functional. Here, $\beta$ represents the continuous rate of cost for both players, $\phi$ is the stopping cost for the maximizing player, and $\mu$ is the stopping cost for the minimizing player.

Recently DIVICCARO et al. [45], established the following result.

**Theorem 1.** Let $T$ and $I$ two weakly commuting mappings of a closed, convex subset $C$ of a Banach space $X$ into itself satisfying the inequality

$$
||Tx - Ty||^p \leq a ||x - y||^p + (1-a).\max\{||Tx - lx||^p, ||Ty - ly||^p\}
$$

for all $x, y \in C$, where $0 < a < 1/2^{p+1}$ and $p \geq 1$. If $I$ is linear, nonexpansive in $C$ and such that $I(C)$ contains $T(C)$, then $T$ and $I$ have a unique common fixed point at which $T$ is continuous.

In this context, in chapter $X$, we have introduced the concept of compatible mappings of type (B-1) and type (B-2) and show that they are equivalent to compatible mappings and compatible mappings of type (B) under certain conditions. In the sequel we have proved some fixed point theorems which extends and improves the result of Divicaro et al. [45] and many others. In fact we have proved the following:

**Theorem 22.** Let $T$ and $I$ be two compatible self maps of a closed convex bounded subset $C$ of a Normed space $X$ satisfying the following

(1.10.3) \[ ||Tx - Ty||^p \leq a \cdot ||x - y||^p + (1-a).\max\{||Tx - lx||^p, ||Ty - ly||^p\}. \]

(1.10.4) \[ I(C) \supseteq (1-k).I(C) + k.T(C). \]

for all $x, y \in C$ where $0 < a < 1$, $p \geq 0$ and for some fixed $k$ such that $0 < k < 1$. If for some $x_0 \in C$, the sequence $\langle x_n \rangle$ defined by

(1.10.5) \[ x_{n+1} = (1-k).lx_n + k.Tx_n, \quad \forall \ n \geq 0 \]

converges to a point $z$ of $C$ and if $I$ is continuous at $z$ then $T$ and $I$ have a unique common fixed point. Further if $I$ is continuous at $Tz$ then $T$ and $I$ have a unique common fixed point at which $T$ is continuous.

**Theorem 23.** Let $(S, I)$ and $(T, J)$ be two pairs of compatible mappings of type (B-1) or (B-2) of a normed space $X$ into itself and let $C$ be a closed convex bounded subset of $X$ satisfying the following conditions:

18
\[ (3.1) \|Sx - Ty\|_p \leq \alpha \|x - y\|_p + \beta \max\{\|Sx - 1y\|_p, \|y - Jy\|_p\} + \gamma \max\{\|x - Jy\|_p, \|f - y\|_p\} \]

for all \( x, y \in C \) where \( p > 0 \), \( 0 \leq \alpha, \beta, \gamma \leq 1 \).

\[ (3.2) I(C) \geq (1 - k) I(C) + k S(C) \text{ and } J(C) \geq (1 - k') J(C) + k' T(C) \]

for some fixed \( k, k' \) such that \( 0 < k, k' < 1 \).

If for some \( x_0 \) in \( X \) the sequence \( \{x_n\} \) defined by

\[ (3.3) x_{2n+1} = (1 - k) Ix_{2n} + k Sx_{2n} \text{ and } Jx_{2n+1} = (1 - k') Jx_{2n} + k' Tx_{2n} \]

for all \( n \geq 0 \) converges to a point \( z \) in \( C \), and if \( I \) and \( J \) are continuous at \( z \), then \( S, T, I \) and \( J \) have a unique common fixed point. Further if \( I \) and \( J \) are continuous at \( Tz \), then \( S, T, I \) and \( J \) have a unique common fixed point at which \( S \) and \( T \) are continuous.

We have also applied our results to prove the existence of solutions of certain discrete variational inequalities.

1.11. APPLICATION OF FIXED POINTS IN DYNAMIC PROGRAMMING

The basic form of the functional equation of dynamic programming is as follows:

\[ f(x) = \text{opt} \{ H(x, y, f(T(x, y))) \} \]

where \( x \) and \( y \) denote the state and decision vectors respectively, \( T \) the transformation of process and \( f(x) \) the optimal return with the initial state \( x \), where \( \text{opt} \) denotes max or min.

In chapter XI we have applied the results of chapter II to study the existence and uniqueness of common solution of the following functional equations arising in dynamic programming:

\[ \begin{align*}
(1.11.1) & \quad f(x) = \sup H(x, y, f(T(x, y))), \quad x \in S, \\
(1.11.2) & \quad g(x) = \sup F(x, y, g(T(x, y))), \quad x \in S
\end{align*} \]

where \( T : S \times D \rightarrow S \) and \( H, F : S \times D \times R \rightarrow R, \ i = 1, 2 \), where \( X, Y \) are a Banach space, \( S \subseteq X \) is the state space, \( D \subseteq Y \) is the decision space, and \( R = (-\infty, +\infty) \).

In fact we have proved the following

**Theorem 24.** Suppose that the following conditions are satisfied

\[ (1.11.3) \quad H_i \text{ and } F_i \text{ are bounded for } i = 1, 2 \]

19
\( (1.11.4) \ |H_i(x,y,h(t)) - H_j(x,y,k(t))| \leq \frac{1}{2} \{ |A_i h(t) - d h(t)|^2 + |A_j k(t) - d k(t)|^2 \} \]

where \( \frac{1}{2} \{ |A_i h(t) - d h(t)|^2 + |A_j k(t) - d k(t)|^2 \} \) for all \( (x,y) \in S \times D \), \( h,k \in B(S), t \in S, 0 < d < 1, p \geq 0 \) and the mappings \( A_i \) and \( T_j \) are defined as follows:

\[
A_i h(x) = \sup H_i(x,y,h(T(x,y))), x \in S, h \in B(S), i = 1,2
\]

\[
T_j k(x) = \sup F_j(x,y,k(T(x,y))), x \in S, k \in B(S), i = 1,2
\]

(1.11.5) for any \( h \in B(S) \), there exists \( k_1, k_2 \in B(S) \) such that \( A_1 h(x) = T_2 k_1(x) \) \( A_2 h(x) = T_2 k_2(x) \), \( x \in S \)

(1.11.6) \( T_1(B(S)) \cap T_2(B(S)) \) is closed

(1.11.7) for any \( \{k_n\} \in B(S) \), if there exists \( h \in B(S) \) such that

\[
\lim_{n \to \infty} \sup |A_1 k_n(x) - h(x)| = \lim_{n \to \infty} \sup |T_2 k_n(x) - h(x)| = 0, i = 1,2
\]

then \( \lim_{n \to \infty} \sup |A_1 T_2 k_n(x) - T_2 T_2 k_n(x)| = 0 \) or

\[
\lim_{n \to \infty} \sup |T_2 A_2 k_n(x) - A_2 A_2 k_n(x)| = 0, i = 1,2.
\]

Then the system of functional equations (1.11.1) and (1.11.2) has a unique common solution in \( B(S) \).

**Theorem 25**: Suppose that the following conditions are satisfied

(1.11.8) \( H_i \) is bounded for \( i = 1,2 \)

(1.11.9) \( |H_i(x,y,h(t)) - H_j(x,y,k(t))| \leq \frac{1}{2} \{ |A_i h(t) - d h(t)|^2 + |A_j k(t) - d k(t)|^2 \} \)

where \( \frac{1}{2} \{ |A_i h(t) - d h(t)|^2 + |A_j k(t) - d k(t)|^2 \} \) for all \( (x,y) \in S \times D \), \( h,k \in B(S), t \in S, 0 < d < 1, p \geq 0 \) and the mappings \( A_i \) are defined as follows:

\[
A_i h(x) = \sup H_i(x,y,h(T(x,y))), x \in S, h \in B(S), i = 1,2
\]

Then the system of functional equations (1.11.1) and (1.11.2) has a unique common
Theorem 26. Suppose that the following conditions are satisfied

1. $H_i$ and $F_i$ are bounded for $i = 1, 2$

2. $|H_i(x, y, h(t)) - H_i(x, y, k(t))| \leq (1/M)(|A_i h(t) - T_i h(t)| + |A_j k(t) - T_j k(t)|) + h(|f| h(t) - T_j k(t))$

where $M = [1 + p(|A_i h(t) - T_j h(t)| + |A_j k(t) - T_j k(t)|)]$

for all $(x, y) \in S \times D$, $h, k \in B(S)$, $0 < h < 1$, $p \geq 0$ and the mappings $A_i$ and $T_i$ are defined as follows:

$A_i h(x) = \sup_{x \in S} H_i(x, y, h(T_i(x, y)))$, $x \in S$, $h \in B(S)$, $i = 1, 2$

$T_i k(x) = \sup_{x \in S} F_i(x, y, k(T_i(x, y)))$, $x \in S$, $k \in B(S)$, $i = 1, 2$

1. for any $h \in B(S)$ with $\sup_{x \in S} |h(x)| = (1-h) r$, there exists $k_1, k_2 \in B(S)$ such that $\sup_{x \in S} |h(x)| \leq (1-h) r$ and $T_i k_i(x) = h(x)$, $x \in S$, $i = 1, 2$, $r > 0$.

1. for any $h \in B(S)$ with $\sup_{x \in S} |h(x)| = (1-h) r$, there exists $k \in B(S)$ such that $\sup_{x \in S} |k(x)| \leq (1-h) r$ and $A_i h(x) \cup A_j h(x) = T_i k(x) \cap T_j k(x)$, $x \in S$

1. $T_i(B(S)) \cap T_j(B(S))$ is closed

1. for any $h \in B(S)$ with $\sup_{x \in S} |h(x)| \leq (1-h) r$

$\sup_{x \in S} |T_i h(x)| = (1-h) r \Rightarrow \sup_{x \in S} |A_i h(x)| \leq (1-h) r$ for $i, j = 1, 2$.

1. for any $\{k_n\} \in B(S)$, if there exists $h \in B(S)$ such that

$\lim_{n} \sup_{x \in S} |A_i k_n(x) - h(x)| = \lim_{n} \sup_{x \in S} |T_i k_n(x) - h(x)| = 0$, $i = 1, 2$

then $\lim_{n} \sup_{x \in S} |A_i T_i k_n(x) - T_i T_i k_n(x)| = 0$ or

$\lim_{n} \sup_{x \in S} |T_i A_i k_n(x) - A_i A_i k_n(x)| = 0$, $i = 1, 2$.

Then the system of functional equations (1.11.1) and (1.11.2) has a unique common solution $h^* \in B(S)$ and $\sup_{x \in S} |h^*(x)| \leq (1-h) r$.

Theorem 27. Suppose that the following conditions are satisfied

1. $H_i$ and $F_i$ are bounded for $i = 1, 2$

21
(1.11.18) $|H_1(x,y,h(t)) - H_2(x,y,k(t))|$
\[
\leq \left(\frac{1}{M}\right)p \left[|A_1h(t) - T_1h(t)|^2 + |A_2k(t) - T_2k(t)|^2\right] + \phi\left(\max\{|A_1h(t) - T_1h(t)|, |A_2k(t) - T_2k(t)|\}\right)
\]

where $M = [1 + p \{ |A_1h(t) - T_1h(t)| + |A_2k(t) - T_2k(t)|\}]$

for all $(x,y) \in S \times D$, $h,k \in B(S)$, $t \in S$, $p \geq 0$ and $\phi$ is as in Theorem 3, and the mappings $A_i$ and $T_i$ are defined as follows:

$A_i h(x) = \sup H_i(x,y,h(T(x,y))), x \in S, h \in B(S), i = 1,2$

$T_i k(x) = \sup F_i(x,y,k(T(x,y))), x \in S, k \in B(S), i = 1,2$

(1.11.19) for any $h \in B(S)$, there exists $k_1, k_2 \in B(S)$ such that $A_1 h(x) = T_2 k_1(x)$, $A_2 h(x) = T_1 k_2(x), x \in S$

(1.11.20) $T_1(B(S)) \cap T_2(B(S))$ is closed

(1.11.21) for any $(k_i) \subseteq B(S)$, if there exists $h \in B(S)$ such that

$\lim_{n \to \infty} \sup |A_i k_i(x) - h(x)| = \lim_{n \to \infty} \sup |T_i k_i(x) - h(x)| = 0, i = 1,2$

then $\lim_{n \to \infty} \sup |A_i T_i k_i(x) - T_i T_i k_i(x)| = 0$ or

$\lim_{n \to \infty} \sup |A_i k_i(x) - \Lambda A_i k_i(x)| = 0, i = 1,2$

Then the system of functional equations (1.11.1) and (1.11.2) has a unique common solution in $B(S)$.

Theorem 28: Suppose that the following conditions are satisfied (1.11.22) $H_i$ is bounded for $i = 1,2$

(1.11.23) $|H_1(x,y,h(t)) - H_2(x,y,k(t))|$
\[
\leq \left(\frac{1}{M}\right)p \left[|A_1h(t) - h(t)|^2 + |A_2k(t) - k(t)|^2\right] + \phi\left(\max\{|A_1h(t) - h(t)|, |A_2k(t) - k(t)|\}\right)
\]

where $M = [1 + p \{ |A_1h(t) - h(t)| + |A_2k(t) - k(t)|\}]$

for all $(x,y) \in S \times D$, $h,k \in B(S)$, $t \in S$, $p \geq 0$, $\phi$ is as in Theorem 3 and the mappings $A_i, i = 1,2$ are defined as follows:

$A_i h(x) = \sup H_i(x,y,h(T(x,y))), x \in S, h \in B(S), i = 1,2$

Then the system of functional equations (1.11.1) and (1.11.2) has a unique common solution in $B(S)$.

Theorem 29: Suppose that the following conditions are satisfied (1.11.24) $H_i$ and $F_i$ are bounded for $i = 1,2$
(1.11.25) \[ |H(x,y,h(t)) - H(x,y,k(t))| < (1/M) p (|A_1, h(t) - T_1, h(t)|^2 + |A_2, k(t) - T_2, k(t)|^2) + \phi(|T_1, h(t) - T_2, k(t)|) \]

where \( M = 1 + p (|A_1, h(t) - T_1, h(t)| + |A_2, k(t) - T_2, k(t)|) \)

for all \((x,y) \in S \times D, h, k \in B(S), v \in S, \rho \geq 0, \phi\) is as in Theorem 3 and the mappings \( A, T \) are defined as follows.

\( A_i, h(x) = \sup_{y \in S} H_i(x,y,h(T(x,y))) \), \( x \in S, h \in B(S), i = 1,2 \)

\( T_i, k(x) = \sup_{y \in S} F_i(x,y,k(T(x,y))) \), \( x \in S, k \in B(S), i = 1,2 \)

(1.11.26) for any \( h \in B(S) \) with \( \sup_{x \in S} h(x) = (1-h)r \), there exists \( k_1, k_2 \in B(S) \) such that \( \sup_{x \in S} |h(x)| \leq (1-h)r \) and \( T_i, k(x) - h(x) \), \( x \in S, i = 1,2, r > 0 \).

(1.11.27) for any \( h \in B(S) \) with \( \sup_{x \in S} |h(x)| \leq (1-h)r \), there exists \( k \in B(S) \) such that \( \sup_{x \in S} |k(x)| \leq (1-h)r \) and \( A_i, h(x) \cup A_2, h(x) \cap T_i, k(x), x \in S \)

(1.11.28) \( T_i(B(S)) \cap T_j(B(S)) \) is closed

(1.11.29) for any \( h \in B(S) \) with \( \sup_{x \in S} |h(x)| \leq (1-h)r \),
\[ \sup_{x \in S} |T_i, h(x) - (1-h)r| \Rightarrow \sup_{x \in S} |A_i, h(x)| \leq (1-h)r \]

(1.11.30) for any \( \{k_n\} \subseteq B(S) \), if there exists \( h \in B(S) \) such that
\[ \lim_{n \to \infty} \sup_{x \in S} |A_i, k_n(x) - h(x)| = \lim_{n \to \infty} \sup_{x \in S} |T_i, k_n(x) - h(x)| = 0, i = 1,2 \]
then \[ \lim_{n \to \infty} \sup_{x \in S} |A_i, T_i, k_n(x) - T_i, T_i, k_n(x)| = 0 \] or
\[ \lim_{n \to \infty} \sup_{x \in S} |T_i, A_i, k_n(x) - A_i, A_i, k_n(x)| = 0, i = 1,2. \]

Then the system of functional equations (1.11.1) and (1.11.2) has a unique common solution \( h^* \) in \( B(S) \) and \( \sup_{x \in S} |h^*(x)| \leq (1-h)r \).

1.12. COMPLEMENTARITY PROBLEMS AND THEIR SOLUTIONS

The study of complementarity problems came into existence in the early sixties. Since then a variety of research papers appeared in this field. Particularly the explicit complementarity problems and implicit complementarity problems were discussed and studied by many authors. For details we refer to [2], [6], [8], [18], [59-60], [74], [81], [95-96], [112], [121], [136].

It is a fairly well known fact that the complementarity problems has got a wide range of applications in the areas such as Optimization theory, Engineering, Structural...
In this context in chapter XII we have considered a more general class of complementarity problem called simultaneous complementarity problem and studied the existence and uniqueness of its solution via fixed point.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $K \subset H$ be a closed convex cone. If $D \subset H$ is a subset of $H$ and $f, g : D \rightarrow H$ are two mappings.

The simultaneous implicit complementarity problem is:

\[(S.I.C.P) : \text{find } x \in D \text{ such that } g(x) \in K, f_1(x) \in K^*, f_2(x) \in K^*, <g(x), f_1'(x)> = 0 \text{ and } <g(x), f_2'(x)> = 0.\]

We note that our result includes many known results as special cases.

Precisely we have proved the following:

**Theorem 30.** Let $H$ be a Hilbert space and $K \subset H$ be a closed convex cone. If, for a subset $D \subset H$, the mappings $f, g : D \rightarrow H$ satisfy the following

1) $f_1$ and $f_2$ are pairwise $n$-strongly monotone with respect to $g$
2) $f_1$ and $f_2$ are pairwise $n$-Lipschitz with respect to $g$
3) there exists a real number $\sigma > 0$ such that $\sigma \beta^2 < 2 \alpha < 1/\sigma + \sigma \beta^2$
   where $\alpha$ and $\beta$ are as in definitions 3.1 and 3.2 respectively

4) $K \subset g(D)$

Then the problem $(S.I.C.P)$ is solvable. Moreover if $g$ is one-one then problem $(S.I.C.P)$ has a unique solution.