Chapter 2

Generalized Laguerre Transform and Differential Operators

2.1 Introduction:

The method we shall use is related to Hilbert space techniques, and its prototype in the Fourier series expansion of a periodic distribution. A procedure will be developed for expanding a generalized function \( f \), into a series of the form:

\[
f = \sum_{n=1}^{\infty} F(n) \psi_n
\]

(2.1.1)

Where the \( \psi_n \) constitute a complete system of orthonormal functions and the \( F(n) \) are the corresponding Fourier coefficients of \( f \). This procedure leads to a whole new class of generalized integral transformations. The basic idea is to view the mapping \( f \rightarrow F(n) \) or a transformations \( \mu \) from a certain class or generalized functions \( f \) into the space of functions \( F(n) \) mapping the integers into the complex plane. Then (2.1.1) defines the inverse transformation; of course, the convergence of series (2.1.1) is interpreted in a generalized sense. Moreover, the permissible orthonormal functions \( \psi_n \) will be eigenfunctions of a certain type of self-adjoint differential operator \( \eta \). As a result, the corresponding transformation \( \mu \) will generate an operational calculus for solving differential equations involving the operator \( \eta \). Particular generalized integral transformations that are encompassed by this technique are the finite Fourier transformation (i.e. the transformation corresponding to any Fourier series), the Laguerre transformation, the Hermite transformation, the Jacobi transformation with its special cases such as the Legendre, Chebyshev, and Gegenbauer transformations, and finally the finite Hankel transformations.

In this chapter we Generalized Laguerre transform of functions of two variables with the use of Zemanian’s [99] technique related to the transformation
arising from orthonormal series expansions. Inversion formula is obtained. Also several operations, defined in [72] for a Laguerre transform of functions of two variables are extended to generalized functions in $A(I)$. Finally we discussed the use of differential operator for solving certain type of differential equations.

A.K. Shukla [72] defined Laguerre transform $F_n(\alpha, \beta)$ of functions of two variables $f(x, y)$ as,

$$F_n(\alpha, \beta) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x+y)} x^{\alpha} y^{\beta} K_n^{(\alpha, \beta)}(x, y) f(x, y) dx dy. \quad (2.1.2)$$

where, $f(x, y)$ is a Riemann integrable function [58] defined on set

$$S = \mathbb{R}^+ \times \mathbb{R}^+, \alpha>-1, \beta>-1, \text{n is a non negative integer and}$$

$$K_n^{(\alpha, \beta)}(x, y) = \sum_{r=0}^{n} \frac{(-xy)^r}{r!(-n)_r} L_{n-r}^{(\alpha+r, \beta+r)}(x, y) \quad (2.1.3)$$

The Pochhammer symbol [75], for $(x)_n$ defined as,

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \ (-x)_n = (-1)^n(x-n+1)_n$$

Ragab [70] obtained

$$L_n^\alpha(x)L_n^\beta(y) = \sum_{t=0}^{n} \frac{(-xy)^t}{r!(-n)_r} L_{n-r}^{(\alpha+r, \beta+r)}(x, y) \quad (2.1.4)$$

Using (2.1.3) & (2.1.4) we get,

$$K_n^{(\alpha, \beta)}(x, y) = L_n^\alpha(x)L_n^\beta(y) \quad (2.1.5)$$
Where \( L_n^\alpha(x) \) is a generalized Laguerre polynomial [69], defined by the formula,

\[
L_n^\alpha(x) = \sum_{r=0}^{n} (-1)^r \binom{n+\alpha}{n-r} x^r r!
\]

Therefore equivalent definition of (2.1.2) becomes

\[
F_n(\alpha, \beta) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x+y)} x^\alpha y^\beta L_n^\alpha(x)L_n^\beta(y) f(x, y) dxdy.
\]  \( (2.1.6) \)

Now we shall develop the theory for generalized Laguerre transform for functions of two variables \( f(x, y) \).

2.2. The spaces \( L^2(I) \) and \( \mathcal{A}(I) \):

We consider the space \( L^2(I) \) of quadratically double integrable function on \( I \). That is a family of equivalence classes of functions \( f \), those are locally integrable on \( I \) such that,

\[
\alpha_0(f) \equiv \left[ \int_{0}^{\infty} \int_{0}^{\infty} |f(x, y)|^2 dxdy \right]^{1/2} < \infty
\]  \( (2.2.1) \)

Where, \( I \) is set of all \( (x, y) \) such that \( 0 < x < \infty \) and \( 0 < y < \infty \).

\( L^2(I) \) is linear space; with zero element as the class of all functions that are equal to zero almost everywhere on \( I \). \( \alpha_0 \) is the norm on \( L^2(I) \). Topology for \( L^2(I) \) is generated by \( \alpha_0 \).

**Definition 2.1:** An inner product of two functions \( f, g \in L^2(I) \) is defined by,

\[
(f, g) \equiv \int_{0}^{\infty} \int_{0}^{\infty} f(x, y) \overline{g(x, y)} dxdy
\]  \( (2.2.2) \)

Where, \( \overline{g(x, y)} \) denotes the complex conjugates of \( g(x, y) \).
We list the properties of inner product for $f, g, h \in L^2(I)$. 

1) $(f + h, g) = (f, g) + (h, g)$,

2) $(\beta f, g) = (f, \beta g) + \beta(f, g)$

3) $(f, g) = (g, f)$,

4) $(f, f) = \left[\alpha_0(f)\right]^2 \geq 0$,

5) The inner product is continuous with respect to each of its arguments that is, if $f_m \to f$ in $L^2(I)$ as $m \to \infty$, then $(f_m, g) \to (f, g)$ and $(g, f_m) \to (g, f)$

6) **Schwarz inequality** $| (f, g) | \leq \alpha_0(f) \alpha_0(g)$.

We now fix the differential operator $\eta$ by,

$$\eta = e^{-\frac{x+y}{2}} e^{-\frac{r+y}{2}} \frac{\partial^2}{\partial x \partial y} x^{\alpha+1} y^{\beta+1} e^{-\frac{x+y}{2}} \frac{\partial^2}{\partial x \partial y} x^{-\alpha/2} y^{-\beta/2} e^{-\frac{x+y}{2}}$$  \hspace{1cm} (2.2.3)

And let us define for a nonnegative integer $n$

$$\psi_n(x, y) = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} n^{1/2} x^{\alpha/2} y^{\beta/2} e^{-\frac{x+y}{2}} L_\alpha^n(x) L_\beta^n(y)$$ \hspace{1cm} (2.2.4)

Comparing self adjoint form of the differential equation [99], for function of the one variable, we can have a sequence $\{\lambda_n\}_{n=0}^\infty$ of eigen values where

$$\lambda_n = n^2$$ \hspace{1cm} (2.2.5)

Definitely $|\lambda_n| \to \infty$ as $n \to \infty$ and

$$\eta \psi_n = \lambda_n \psi_n \hspace{1cm} n=0,1,2,3\ldots$$ \hspace{1cm} (2.2.6)
The sequence \( \{\psi_n\}_{n=0}^{\infty} \) of smooth functions in \( L^2(I) \) form a complete orthonormal system in it. Indeed

\[
(\psi_n, \psi_m) = \begin{cases} 
0 & \text{when } n \neq m \\
1 & \text{when } n=m
\end{cases}
\]

This proves orthogonality.

The completeness is established due to the fact that every \( f \in L^2(I) \) can be expanded into the series expansion

\[
f = \sum_{n=0}^{\infty} (f, \psi_n)\psi_n
\]  

Which converges in \( L^2(I) \).

That is \( \alpha_0[f - \sum_{m=0}^{n} (f, \psi_m)\psi_m] \to 0 \) as \( n \to \infty \)

Now we construct the testing function space \( \mathcal{A}(I) \) as the collection of all complex-valued smooth functions \( \phi(x, y) \) defined on \( I \) such that

a) for any non-negative integer \( k \),

\[
\gamma_k(\phi) \triangleq \gamma_0(\eta^k \phi) \triangleq \left[ \int_{0}^{\infty} \int_{0}^{\infty} \left| \eta^k \phi(x, y) \right|^2 \, dx \, dy \right]^{1/2}
\]

exist (i.e. finite) \( (2.2.8) \)

b) For each pair of non-negative integers \( n, k \),

\[
(\eta^k \phi, \psi_n) = (\phi, \eta^k \psi_n)
\]  

\( (2.2.9) \)

Every member of sequence \( \{\psi_n\}_{n=0}^{\infty} \) of eigenfunctions is a member of \( \mathcal{A}(I) \). For,

i) Every \( \psi_n \) is complex-valued smooth function defined on \( I \),

ii) For each non-negative integer \( k \),

\[
\gamma_k(\psi_n) = \gamma_0(\eta^k \psi_n) = \left[ \int_{0}^{\infty} \int_{0}^{\infty} \left| \eta^k \psi_n \right|^2 \, dx \, dy \right]^{1/2} = \left[ \int_{0}^{\infty} \int_{0}^{\infty} \left| \lambda^k \psi_n \right|^2 \, dx \, dy \right]^{1/2}
\]
= \lambda^k_n(\psi_n, \psi_n) = \lambda^k_n \text{ is finite,}

iii) For any non-negative integers \( n, m, k \)

\[
(\eta^k \psi_n, \psi_m) = \lambda^k_n(\psi_n, \psi_m) = 0 = (\psi_n, \lambda^k_m \psi_m) = (\psi_n, \eta^k \psi_m), \quad n \neq m
\]

and for \( n = m \)

\[
(\eta^k \psi_n, \psi_n) = \lambda^k_n(\psi_n, \psi_n) = (\psi_n, \lambda^k_n \psi_n) = (\psi_n, \eta^k \psi_n).
\]

**Theorem 2.1:** The operator \( \eta \) is continuous linear mapping \( \mathcal{A}(I) \) into itself.

Linearity is established from,

\[
(\eta(c_1 \phi_1 + c_2 \phi_2), \psi_n) = (c_1 \phi_1 + c_2 \phi_2, \eta \psi_n) = (c_1 \phi_1 + c_2 \phi_2, \lambda_n \psi_n)
\]

\[
= c_1(\lambda_n \psi_n) + c_2(\lambda_n \psi_n) = c_1(\phi_1, \eta \psi_n) + c_2(\phi_2, \eta \psi_n)
\]

\[
= c_1(\eta \phi_1, \psi_n) + c_2(\eta \phi_2, \psi_n) = (c_1 \eta \phi_1 + c_2 \eta \phi_2, \psi_n)
\]

Continuity is proved from the fact that,

\[
(\eta \psi_n, \psi_n) = (\phi_n, \eta \psi_n) = \lambda_n(\phi_n, \psi_n) \to 0 \quad \text{as} \quad n \to \infty
\]

Whenever \( \{\phi_n\}_{n=0}^\infty \) converges to the zero function in \( \mathcal{A}(I) \).

\( \mathcal{A}(I) \) is linear space under addition and multiplication by complex numbers.

Clearly \( \gamma_0 \) is a norm and \( \{\gamma_k\}_{k=0}^\infty \) is a separating collection of seminorms, hence is a countable multinorm on \( \mathcal{A}(I) \).

We equip with \( \mathcal{A}(I) \) the topology generated by \( \{\gamma_k\}_{k=0}^\infty \).

Thus \( \mathcal{A}(I) \) is a countably multinormed space. Every Cauchy sequence in \( \mathcal{A}(I) \) converges in it. Hence \( \mathcal{A}(I) \) is complete and therefore it is Frechet space. As in the definition \( \mathcal{A}(I) \) consists complex valued smooth functions \( \phi(x, y) \) defined on \( I \). \( \mathcal{A}(I) \) is complete countably multinormed space. Let \( \{\phi_k\}_{k=1}^\infty \) be a sequence in \( \mathcal{A}(I) \) which converges to the zero function in it. Then for every non negative integer \( m \), \( \{D^m \phi_k\}_{k=1}^\infty \) also converges to the zero function on every compact subset \( K \) of \( I \) [2]. Indeed,
Integrating by parts m times and taking limit as $v \to \infty$ we get on every compact subset $K$ of $I$, $(D^m \phi, \psi_n) \to 0$ as $v \to \infty$.

**Lemma 2.1:** If $\phi \in \mathcal{A}(I)$ then $\phi$ can have series expansion

$$\phi = \sum_{n=0}^{\infty} (\phi, \psi_n) \psi_n$$

(2.2.10)

Which converges in $\mathcal{A}(I)$.

**Proof:** Since $\phi \in \mathcal{A}(I)$, for each non-negative integer $k$, $\eta^k \phi \in L^2(I)$

Due to (2.2.8). Hence $\eta^k \phi$ can be expressed as a series

$$\eta^k \phi = \sum_{n=0}^{\infty} (\eta^k \phi, \psi_n) \psi_n$$

Where $\{\psi_n\}_{n=0}^{\infty}$ forms a complete orthonormal system of smooth functions in $L^2(I)$. Now applying (2.2.6) and (2.2.9) we can have

$$\eta^k \phi = \sum_{n=0}^{\infty} (\phi, \eta^k \psi_n) \psi_n = \sum_{n=0}^{\infty} (\phi, \lambda^k_n \psi_n) \psi_n = \sum_{n=0}^{\infty} (\phi, \psi_n) \lambda^k_n \psi_n$$

$$= \sum_{n=0}^{\infty} (\phi, \psi_n) \eta^k \psi_n$$

This series converges in $L^2(I)$. Further for each $k$,

$$\gamma_k [\phi - \sum_{m=0}^{n} (\phi, \psi_m) \psi_m] = \gamma_0 [\eta^k \phi - \sum_{m=0}^{n} (\phi, \psi_m) \eta^k \psi_m] \to 0$$

as $n \to \infty$. This proves our assertion.

**Proposition 2.2:** $\eta$ is self-adjoint differential operator. That is
\((\eta \phi_1, \phi_2) = (\phi_1, \eta \phi_2)\). Where, \(\phi_1, \phi_2 \in \mathcal{A}(I)\) (2.2.11)

**Proof:** It is evident from the fact that when \(\phi_1, \phi_2 \in \mathcal{A}(I)\)

\[
(\eta \phi_1, \phi_2) = \left( \sum_{n=0}^{\infty} (\phi_1, \psi_n) \eta \psi_n, \phi_2 \right)
= \int_0^\infty \int_0^\infty \left( \sum_{n=0}^{\infty} (\phi_1, \psi_n) \eta \psi_n(x,y) \bar{\phi}_2(x,y) \right) dx \, dy.
= \sum_{n=0}^{\infty} \left( \phi_1, \psi_n \right) \int_0^\infty \int_0^\infty \eta \psi_n(x,y) \bar{\phi}_2(x,y) \, dx \, dy.
= \sum_{n=0}^{\infty} \left( \phi_1, \psi_n \right) \int_0^\infty \int_0^\infty \psi_n(x,y) \eta \bar{\phi}_2(x,y) \, dx \, dy.
= \int_0^\infty \int_0^\infty \left[ \sum_{n=0}^{\infty} \left( \phi_1, \psi_n \right) \eta \psi_n \right] \bar{\phi}_2(x,y) \, dx \, dy.
= \int_0^\infty \int_0^\infty \phi_1 \eta \bar{\phi}_2 \, dx \, dy = (\phi_1, \eta \phi_2).

**Lemma 2.3:** For \(b_n\) to be complex number, the series \(\sum_{n=0}^{\infty} b_n \psi_n\) converges in \(\mathcal{A}(I)\) if and only if the series \(\sum_{n=0}^{\infty} |\lambda_n|^{2k} |b_n|^2\) converges for every non-negative integer \(k\)

**Proof:** - we use (2.2.6) and get

\[
\int_0^\infty \int_0^\infty \left| \eta^k \sum_{n=p}^{q} b_n \psi_n \right|^2 \, dx \, dy = \int_0^\infty \int_0^\infty \left| \sum_{n=p}^{q} b_n \eta^k \psi_n \right|^2 \, dx \, dy.
= \int_0^\infty \int_0^\infty \left| \sum_{n=p}^{q} b_n \lambda_n^k \psi_n \right|^2 \, dx \, dy.
= \int_0^\infty \int_0^\infty \sum_{n=p}^{q} \sum_{m=p}^{q} b_n \lambda_m^k \psi_n \bar{\psi}_m \, dx \, dy.

Now due to orthonormality we have
\[
\int_0^\infty \int_0^\infty \eta^k \sum_{n=p}^\infty b_n \psi_n \, dx \, dy = \sum_{n=p}^\infty |\lambda_n|^{2k} |b_n|^2.
\]

From this equation our assertion is established.

### 2.3. The Dual Space \( \mathcal{A}(I) \):

The collection of all linear continuous functional on \( \mathcal{A}(I) \) is the dual space \( \mathcal{A}(I) \). Since \( \mathcal{A}(I) \) is testing function space, \( \mathcal{A}(I) \) is a space of generalized functions. As usual the number \( f \in \mathcal{A}(I) \) assigns to any \( \phi \in \mathcal{A}(I) \) is denoted by

\[ \langle f, \phi \rangle. \]

In order to keep good relation with the definition in (2.2.2) of the scalar product it will be more convenient when \( f \in \mathcal{A}'(I) \) will assign a number with complex conjugate of \( \phi \).

We define, \( (f, \phi) = \langle f, \overline{\phi} \rangle, \quad \phi \in \mathcal{A}(I). \) (2.3.1)

The multiplication by a complex number \( a \) will follow the rule,

\[ (af, \phi) = a(f, \phi) = (f, a\overline{\phi}). \] (2.3.2)

Clearly \( \mathcal{A}(I) \) is a linear space. Since \( \mathcal{A}(I) \) is complete, \( \mathcal{A}(I) \) is also complete, by theorem 5.5 chapter 1.

We now define the generalized differential operator \( \overline{\eta}' \) on \( \mathcal{A}(I) \) through

\[ (f, \eta \phi) = \langle f, \overline{\eta' \phi} \rangle = \langle \overline{\overline{\eta'} f}, \phi \rangle = (\overline{\eta'} f, \phi). \] (2.3.3)

Since \( \eta \) is self-adjoint \( \overline{\eta'} = \eta \).
So, \((\eta f, \phi) = (f, \eta \phi)\) \(f \in \mathcal{A}(I), \phi \in \mathcal{A}(I)\). \hspace{1cm} (2.3.4)

We can easily prove the property that, \(\eta : \mathcal{A}(I) \to \mathcal{A}(I)\) is continuous linear mapping by making use of the fact that \(\eta\) is continuous linear mapping of \(\mathcal{A}(I)\) into itself.

2.4. Differentiation of generalized function in \(\mathcal{A}'(I)\):

For a generalized function \(f(x, y)\) in \(\mathcal{A}'\) we define its derivative w.r.t. \(x\) and \(y\),

\[
\frac{\partial f}{\partial x_j} \in \mathcal{A} \text{ where } x_i \text{ is } x \text{ or } y \text{ as,}
\]

\[
< \frac{\partial f}{\partial x_j}, \phi > = -\langle f, \frac{\partial \phi}{\partial x_j} \rangle \quad \text{for every } \phi \in \mathcal{A}'(I). \hspace{1cm} (2.4.1)
\]

In general for \(f \in \mathcal{A}', \frac{\partial f}{\partial x_j} \in \mathcal{A}'\) is given by

\[
< \frac{\partial^k f}{\partial x_j}, \phi > = (-1)^k \langle f, \frac{\partial^k \phi}{\partial x_j} \rangle \hspace{1cm} (2.4.2)
\]

The partial derivative \(\partial^k : \mathcal{A}'(I) \to \mathcal{A}'(I)\) defined above is continuous in the following sense,

If \(\partial^k : T_j \to T\) in \(\mathcal{A}'(I)\) then \(\partial^k T_j \to \partial^k T\) in \(\mathcal{A}'(I)\).

Theorem 4.1:

a) Derivative of generalized function in \(\mathcal{A}'(I)\) is also a generalized function in \(\mathcal{A}(I)\).
b) The order of differentiation of partial derivatives can be interchanged.

Proof:

a) Clearly $\partial^k T$ defined above is linear functional on $\mathcal{A}(I)$. To show that it is also continuous, let $\{\phi_n\}_{n \in \mathbb{N}}$ be a sequence which converges in $\mathcal{A}'(I)$ to zero. Then $\{\partial^k \phi_n\}_{n \in \mathbb{N}}$ also converges in $\mathcal{A}'(I)$ to zero. Hence,

$$\langle \partial^k T, \phi_n \rangle = (-1)^{kk}(T, \partial^k \phi_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

b) Let $\phi \in \mathcal{A}(I)$. Then by Schwarz’s theorem

$$\langle \frac{\partial^2 T}{\partial x_i \partial x_j}, \phi \rangle = \langle \frac{\partial T}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \rangle \text{ in classical sense.}$$

Therefore,

$$\langle \frac{\partial^2 T}{\partial x_i \partial x_j}, \phi \rangle = \langle \frac{\partial^2 T}{\partial x_i \partial x_j}, \phi \rangle$$

$$= \langle \frac{\partial T}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \rangle$$

$$= (-1)^2 \langle T, \frac{\partial^2 \phi}{\partial x_i \partial x_j} \rangle$$

$$= \langle \frac{\partial T}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \rangle = \langle \frac{\partial^2 T}{\partial x_j \partial x_i}, \phi \rangle$$

Hence theorem.

**Theorem: 4.2** Differentiation is continuous linear operation in $\mathcal{A}'(I)$ in the following sense:

1) Linearity: $\partial^k (aT_1 + bT_2) = a\partial^k T_1 + b\partial^k T_2$, $T_1, T_2 \in \mathcal{A}(I)$ and $a, b \in \mathbb{C}$

2) Continuity: if $T_n \rightarrow T$ in $\mathcal{A}'(I)$ then $\partial^k T_n \rightarrow T$ in $\mathcal{A}'(I)$, $\forall \ k \in \mathbb{N}$. 

Proof: The linearity is trivial. To prove continuity, let $\phi \in \mathcal{A}(I)$, then $\partial^k \phi \in \mathcal{A}(I)$.

Therefore,

$$\langle \partial^k T_n, \phi \rangle = (-1)^{|k|} \langle T_n, \partial^k \phi \rangle \rightarrow (-1)^{|k|} \langle T, \partial^k \phi \rangle = \langle \partial^k T, \phi \rangle$$

Hence proved.

We use the following facts based on Lebedev [47] and Rainville [69] for our latter investigation:

$$x \frac{d}{dx} \{L_n^{\alpha}(x)\} = n L_n^{\alpha}(x) - (n + \alpha)L_{n-1}^{\alpha}(x) \quad (2.4.3)$$

$$L_n^{\alpha}(x) = L_{n-1}^{\alpha}(x) + L_n^{\alpha-1}(x) \quad (2.4.4)$$

$$xL_{n-1}^{\alpha+1}(x) = -x \frac{d}{dx} L_n^{\alpha}(x) \quad (2.4.5)$$

$$xL_{n-1}^{\alpha+1}(x) = -x \frac{d}{dx} L_n^{\alpha}(x) \quad (2.4.6)$$

2.5. Some properties of $\mathcal{A}(I)$ and $\mathcal{A}'(I)$:

1. $\mathcal{A}(I) \subset L^2(I)$ when we identify that each function in $\mathcal{A}(I)$ with corresponding equivalence class in $L^2(I)$. Convergence in $\mathcal{A}(I)$ implies the convergence in $L^2(I)$.

2. $D(I) \subset \mathcal{A}(I)$. Convergence in $D(I)$ implies the convergence in $\mathcal{A}(I)$. The topology of $D(I)$ is stronger than that induced on it by $\mathcal{A}(I)$. The restriction of any member $f$ in $\mathcal{A}(I)$ to $D(I)$ is a member of $\mathcal{A}(I)$. Moreover convergence in $\mathcal{A}(I)$ implies the convergence in $D'(I)$. Hence in Zemanian sense, the member of $\mathcal{A}(I)$ are distributions.
3. \( A(I) \subset E(I) \). Furthermore if \( \{ \phi_v \} \) converges in \( A(I) \) to the limit say \( \phi \) then \( \{ \phi_v \} \) also converges in \( E(I) \) to the same limit \( \phi \).

4. Since \( D(I) \subset A(I) \subset E(I) \) and \( D(I) \) is dense in \( E(I) \), \( A(I) \) is also dense in \( E(I) \). The topology of \( A(I) \) is stronger than the topology induced on \( A(I) \) by \( E(I) \). Hence \( E(I) \) is subspace of \( A(I) \).

5. We make \( L^2(I) \) as a subspace of \( A(I) \) by defining the number that \( f \in L^2(I) \) assigns to any \( \phi \in A(I) \) as
\[
(f,\phi) = \int_0^\infty \int_0^\infty f(x,y)\overline{\phi(x,y)} dxdy
\]

Now since \( A(I) \) is subspace of \( L^2(I) \) it is clear that \( A(I) \) is imbedded in \( A(I) \) also \( f \) is linear and continuous on \( A(I) \). Linearity can be easily verified. For, continuity we consider a sequence \( \{ \phi_v \} \) which converges to the zero function in \( A(I) \).

By Schwarz inequality
\[
|f,\phi_v| \leq \gamma_0(f)\gamma_0(\phi) \rightarrow 0 \quad \text{as} \quad v \rightarrow \infty
\]

Thus imbedding of \( L^2(I) \) in to \( A(I) \) is one – to – one.

6. If \( f(x,y) = \eta^k g(x,y) \) for some \( g \in L^2(I) \) and some \( k \) then \( f \in A'(I) \). Indeed, \( \eta \) is linear and continuous mapping of \( A(I) \) into itself and \( L^2(I) \subset A'(I) \) implies that \( f \in A'(I) \).

7. As a particular case of theorem. (1.8.1)[99], we have following for each \( f \in A^1(I) \) there exists a positive constant \( C \) and a non- negative integer \( r \) such that for every \( \phi \in A(I) \)
\[
|f,\phi| \leq c \rho_r(\phi)
\]

Where \( \rho_r = \max\{\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_r\} \) and \( C, r \) depends or \( f \) but not on \( \phi \)
2.6. Orthonormal Series Expansion of Generalized Function in $\mathcal{A}(I)$:

The following fundamental theorem [99] provides an orthonormal series expansion of any $f \in \mathcal{A}(I)$ with respect to $\psi_n$ which in turn yields an inversion formula for the generalized integral transformation.

**Theorem 6.1:** Every $f \in \mathcal{A}(I)$ has a series expansion

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\psi_n$$  \hspace{1cm} (2.6.1)

Which converges in $\mathcal{A}(I)$

**Proof:** For any $\phi \in \mathcal{A}(I)$ we have from Lemma 2.1

$$(f, \phi) = \left( f, \sum_{n=0}^{\infty} (\overline{\phi}, \psi_n)\psi_n \right)$$

complex $(\phi, \psi_n)$ constant as in (2.3.2) we then have

$$(f, \phi) = \sum_{n=0}^{\infty} (\overline{\phi}, \psi_n)(f, \psi_n)$$

$$= \sum_{n=0}^{\infty} (f, \psi_n)(\psi_n, \phi)$$  \hspace{1cm} (2.6.2)

Now the right hand side converges for every $\phi \in \mathcal{A}(I)$

Thus, $(f, \phi) = \left( \sum_{n=0}^{\infty} (f, \psi_n)\psi_n, \phi \right)$ Proves our assertion.

The orthonormal series expansion gives inversion formula for distributional generalized Laguerre transform $\mu$ of function of two variables defined by
$\mathcal{L}^{(\alpha, \beta)}(f)(n) = \mu f = F(n) = (f, \psi_n) \quad f \in \mathcal{A}(I), \ n = 0, 1, 2, 3\ldots \quad (2.6.3)$

In this way $\mu$ is a mapping of $\mathcal{A}(I)$ in to the space of complex-valued functions $F(n)$ defined on $n$ and

$$(\mathcal{L}^{(\alpha, \beta)})^{-1}(F)(n) = \mu^{-1}F(n) = f = \sum_{n=0}^{\infty} F(n)\psi_n. \quad (2.6.4)$$

**Claim:** $\mu$ is a continuous linear mapping.

For continuity we consider a sequence $\{f_v\}_{v=1}^{\infty}$ which converges in $\mathcal{A}^\prime(I)$ to $f$. Then $(f_v, \psi_n) \to (f, \psi_n)$ as $v \to \infty$. That is, $\{F_v(n)\}_{v=1}^{\infty}$ converges to $F(n)$ for every $n$.

**Theorem 6.2:** (Uniqueness): $f, g \in \mathcal{A}(I)$ and $\mu f = F(n), \mu g = G(n)$ satisfy

$F(n) = G(n)$ for every $n$, then $f = g$ in the sense of equality in $\mathcal{A}^\prime(I)$.

**Proof:** Using theorem (2.6.1), we get

$$f-g = \sum_{v=0}^{n} (f, \psi_n) \psi_n = \sum_{v=0}^{n} [(f, \psi_n) - (g, \psi_n)] \psi_n$$

$$= \sum_{v=0}^{n} [F(n) - G(n)] \psi_n = 0$$

Hence, in the sense of equality in $\mathcal{A}^\prime(I)$, $f = g$.

**2.7. Characterization of Distributional Generalized Laguerre transform in two variables:**

We now characterize the functions $F(n)$ which are Generalized Laguerre transforms of $f$ in $\mathcal{A}(I)$. We shall prove that a sequence a sequence $\{b_n\}_{n=0}^{\infty}$ of complex numbers
is the transform of some member \( f \) of \( A(I) \). (i.e. \( b_n = (f, \psi_n) \)) if and only if the sequence satisfies the condition (2.7.2) stated below.

**Theorem 7.1:** For \( b_n \) to be complex number, the series

\[
\sum_{n=0}^{\infty} b_n \psi_n
\]  
(2.7.1)

Converges in \( A(I) \) if and only if there exist a non-negative integer \( q \) such that

\[
\sum_{\lambda_i=0} \left| \lambda_i \right|^{-2q} \left| b_n \right|^2
\]  
(2.7.2)

Moreover if \( f \) denotes the sum (2.7.1) in \( A(I) \)

\[
b_n = (f, \psi_n).
\]

**Proof: NECESSITY.** We assume the series (2.7.1) converges in \( A(I) \), say to \( f \).

Then, \( b_n = (f, \psi_n) \).

By orthagonality of \( \Psi_n \). This proves the last part of the theorem. Now we prove that series (2.7.2) converges for some \( q \) under the assumption of convergence of series (2.7.1). Let us denote the statement,

‘ For every \( \phi = \sum_{N=0}^{\infty} a_n \psi_n \in A(I) \), the series \( \sum a_n \bar{b}_n \sum \bar{a}_n b_n \) converges.’ By A

Refer equation (2.6.2)

Here we select \( a_n \) such that \( |\bar{a}_n b_n| = |a_n b_n| \).
We first prove that the sequence \( \{ F(n)\lambda_{n}^{-q}\}_{n=1}^{\infty} \) is bounded for some \( q \), say \( q_0 \). If not, the sequence is unbounded for every \( q = 1, 2, 3\ldots \) Hence there is an increasing sequence \( \{ n_q \} \) of positive integers such that

\[
\left| F(n_q)\lambda_{n_q}^{-1}\right| \geq 1, \quad q = 1, 2, 3\ldots
\]

Now for every \( q = 1, 2, 3\ldots \) we set

\[
a_n = \begin{cases} 
q\lambda_{n_q}^{-1} & \text{if } n = n_q \\
0 & \text{if } n \neq n_q
\end{cases}
\]

Then for any fixed non-negative integer \( k \),

\[
\sum_{n=1}^{p} |\lambda_{n_q}^k a_n|^2 = \sum_{q=1}^{\infty} \left| \lambda_{n_q}^k \right| \left| \lambda_{n_q}^{q-q} \right|^{-1} = \sum_{q=1}^{\infty} q^{-2} \left| \lambda_{n_q} \right|^{2k-2q}
\]

Since, \( \left| \lambda_{n_q} \right|^{2k-2q} \) is bounded for sufficiently large \( q \), the series \( \sum_{q=1}^{\infty} q^{-2} \left| \lambda_{n_q} \right|^{2k-2q} \) converges. Hence, \( \sum_{n=1}^{\infty} |\lambda_{n_q}^k a_n|^2 \) converges for every non-negative integer \( k \).

But \( \sum_{n=1}^{p} |\lambda_{n_q}^k a_n|^2 = \int_{0}^{\infty} \int_{0}^{\infty} \left| \lambda_{n_q}^{k} \right| \left| \lambda_{n_q}^{q-q} \right|^{-1} dxdy \) \hspace{1cm} (2.7.3)

Hence the series \( \sum_{n=1}^{\infty} a_n \psi_n \) converge in \( A(I) \), say to \( \phi \). For this \( \phi \in A(I) \)

\[
\sum_{n=1}^{\infty} |a_n b_n| \geq \sum_{q=1}^{\infty} \left| a_{n_q} \lambda_{n_q}^{q} \right| = \sum_{q=1}^{\infty} q^{-1} = \infty.
\]

This contradicts the statement A. Thus sequence \( \{ F(n)\lambda_{n}^{-q}\} \) is bounded for some positive \( q_0 \). Now from the fact that \( |\lambda_{n_q}| \rightarrow \infty \) as \( n \rightarrow \infty \) we can say that \( |\lambda_{n_q}^{-q} F(n)| \rightarrow \infty \) as \( n \rightarrow \infty \) for each \( q > q_0 \).
Secondly, we prove that the series in (2.6.2) converges for some \( q > q_0 \). On the contrary we assume that the series in (2.7.3) diverges for some \( q > q_0 \). Then there will be increasing sequence \( \{m_n\} \) of positive integers such that

\[
1 \leq \sum_{n=m_{q_0}}^{m_{q_0}-1} \left| \lambda^{-q} F(n) \right|^2 < 2 \quad q=q_0+1, q_0+2, q_0+3, \ldots
\]

Here we select \( |a_n| = \left| F(n) \lambda^{-2q} q^{-1} \right| \) when \( m_{q_0}-1 < n < m_q, \ q > q_0 \).

Then for every non-negative integer \( k \),

\[
\sum_{n=m_{q_0}}^{m_{q_0}-1} \left| \lambda^{k} a_n \right|^2 = \sum_{n=q_{k+1}}^{q_k-1} \left| \lambda^{2k-2q} \lambda^{-q} F(n) \right|^2 q^{-2} < 2q^{-2}
\]

For all sufficiently large \( q \).

Hence the series \( \sum_{n=1}^{\infty} \left| \lambda^{k} a_n \right|^2 \) converge for each \( k \).

Then again by (7.2) the series \( \sum_{n=1}^{\infty} a_n \psi_n \) converge in \( \mathcal{A}(I) \), say to \( \phi \). On the other hand

\[
\sum_{n=1}^{\infty} \left| a_n F(n) \right| \text{ diverge because}
\]

\[
\sum_{n=m_{q_0}}^{m_{q_0}-1} \left| a_n F(n) \right| = \sum_{n=m_{q_0}}^{m_{q_0}-1} \left| F(n) \right|^2 \lambda^{-2q} q^{-1} \geq q^{-1}
\]

This again contradicts the statement A. Thus the series (2.7.2) converge for every \( q > q_0 \).

**SUFFICIENCY:** Now we assume that the series (2.7.2) converge for some positive \( q \). Let \( \phi \in L^2(I) \). Then for every \( \phi \in \mathcal{A}(I) \)

\[
\sum_{\lambda_n=0}^{\infty} \left| \lambda_n \psi_n, \phi \right| = \sum_{\lambda_n=0}^{\infty} \left| \lambda_n (\psi_n, \phi) \right| = \sum_{\lambda_n=0}^{\infty} \left| \lambda_n \psi_n (\phi, \psi_n) \right|.
\]
By Schwarz inequality for sum of real numbers we can have

\[
\sum_{\lambda_n=0} \left| (b_n \psi_n, \phi) \right| \leq \left[ \sum_{\lambda_n=0} \left| \lambda_n b_n \right|^2 \right]^{1/2} \cdot \left[ \sum_{\lambda_n=0} \left| \lambda_n (\phi, \psi_n) \right|^2 \right]^{1/2}
\]

By assumption the first series on the right side converges. Now because \( \phi \in A(I) \), from lemma 2.1, the series expansion \( \phi = \sum_{n=1}^{\infty} (\phi, \psi_n) \psi_n \) converge in \( A(I) \). Then using (2.7.2) for \( k=q \) we confirm that the second series on the right side also converges. Thus the series \( \sum_{\lambda_n=0} (b_n \psi_n, \phi) = \left( \sum_{\lambda_n=0} b_n \psi_n, \phi \right) \) converges which further implies that the series (2.7.2) converge in \( A(I) \).

**Theorem: 7.2** \( f \) is a member of \( A(I) \) if and only if there exist some non-negative integer \( m \) and \( g \in L^2(I) \) such that

\[
f = \eta^m g + \sum_{\lambda_n=0} b_n \psi_n
\]

(2.7.4)

Where \( b_n \) are complex numbers.

**Proof:** - First we assure that \( f = \sum_{n=0}^{\infty} F(n) \psi_n \in A'(I) \).

Let \( G(n) = \begin{cases} \lambda_n^m F(n) & \lambda_n \neq 0 \\ 0 & \lambda_n = 0 \end{cases} \)

Here we take \( \eta^m \) as a generalized differential operator and \( m \geq 0 \) is such that

\[
\sum_{\lambda_n=0} \left| \lambda_n^{-m} F(n) \right|^2 \text{ Converges. So that}
\]

\[
\sum_{n=0}^{\infty} \left| G(n) \right|^2 \text{ converges. Then by theorem 9.2.1[99], there exists a function} \ g \in L^2(I) \text{ satisfying} \ G(n) = (g, \psi_n).
\]
Since, \( \psi_n \in A(I) \)

\[ (g, \lambda^m \psi_n) = (g, \eta^m \psi_n) = (\eta^m g, \psi_n). \]

(2.7.5)

Now \( f = \sum_{n=0}^{\infty} F(n) \psi_n = \sum_{\lambda_n=0} F(n) \psi_n + \sum_{\lambda_n=0} F(n) \psi_n \)

\[ = \sum_{\lambda_n=0}^\infty G(n) \psi_n + \sum_{\lambda_n=0} F(n) \psi_n \]

\[ = \sum_{n=0}^{\infty} (g, \lambda^m \psi_n) \psi_n + \sum_{\lambda_n=0} F(n) \psi_n \]

Which proves (2.7.4).

Conversely, Let, (2.7.4) is true. We discussed in property (7) of section 2.5 that \( \eta^m g \in A'(I) \). Now \( \psi_n \in A(I) \) and \( A(I) \subset E(I) \subset A'(I) \) implies that \( \psi_n \in A'(I) \). Hence f defined in (2.7.4) is a member of \( A'(I) \).

2.8. Some Results:

Using (2.4.3), (2.4.4), (2.4.5), (2.4.6) and product rule of differentiation we can obtain the following results easily,

\[ \frac{\partial}{\partial x} (\psi_{n}^{(\alpha,\beta)}(x, y)) = -\psi_{n}^{(\alpha,\beta)}(x, y) + (n + \alpha)\psi_{n}^{(\alpha-1,\beta)}(x, y) \]

(2.8.1)

\[ \frac{\partial}{\partial y} (\psi_{n}^{(\alpha,\beta)}(x, y)) = -\psi_{n}^{(\alpha,\beta)}(x, \beta) + (n + \beta)\psi_{n}^{(\alpha,\beta-1)}(x, y) \]

(2.8.2)

\[ \frac{\partial^2}{\partial x^2} (\psi_{n}^{(\alpha,\beta)}(x, y)) = -\psi_{n}^{(\alpha,\beta)}(x, \beta) + 2(n + \alpha)\psi_{n}^{(\alpha-1,\beta)}(x, y) \]

\[ -(n + \alpha)(n + \alpha - 1)\psi_{n}^{(\alpha-2,\beta)}(x, y) \]

(2.8.3)
\[ \frac{\partial^2}{\partial y^2} (\psi_n^{(\alpha,\beta)}(x,y)) = -\psi_n^{(\alpha,\beta)}(x,\beta) + 2(n + \beta)\psi_n^{(\alpha,\beta-1)}(x,y) + (n + \beta)(n + \beta - 1)\psi_n^{(\alpha,\beta-2)}(x,y) \]  
\[ (2.8.4) \]

\[ x \frac{\partial}{\partial x} (\psi_n^{(\alpha,\beta)}(x,y)) = -\psi_n^{(\alpha+1,\beta)}(x,y) + (\alpha + 1)\psi_n^{(\alpha,\beta)}(x,y) \]  
\[ (2.8.5) \]

\[ y \frac{\partial}{\partial y} (\psi_n^{(\alpha,\beta)}(x,y)) = -\psi_n^{(\alpha,\beta+1)}(x,y) + (\beta + 1)\psi_n^{(\alpha,\beta)}(x,y) \]  
\[ (2.8.6) \]

\[ x \frac{\partial}{\partial x} (\psi_n^{(\alpha,\beta)}(x,y)) + y \frac{\partial}{\partial y} (\psi_n^{(\alpha,\beta)}(x,y)) = -\psi_n^{(\alpha+1,\beta)}(x,y) - \psi_n^{(\alpha,\beta+1)}(x,y) + (\alpha + \beta + 2)\psi_n^{(\alpha,\beta)}(x,y) \]  
\[ (2.8.7) \]

\[ x^2 \frac{\partial^2}{\partial x^2} (\psi_n^{(\alpha,\beta)}(x,y)) + 2xy \frac{\partial^2}{\partial x^2 \partial y^2} (\psi_n^{(\alpha,\beta)}(x,y)) + y^2 \frac{\partial^2}{\partial y^2} (\psi_n^{(\alpha,\beta)}(x,y)) = -\psi_n^{(\alpha,\beta+2)}(x,y) - \psi_n^{(\alpha+2,\beta)}(x,y) - 2(\psi_n^{(\alpha+1,\beta+1)}(x,y)) + (2\alpha + 2\beta + 5)\psi_n^{(\alpha+1,\beta)}(x,y) 
- \psi_n^{(\alpha,\beta+1)}(x,y) + (\alpha + \beta + 2)^2 \psi_n^{(\alpha,\beta)}(x,y) \]  
\[ (2.8.8) \]

### 2.9. Generalized Laguerre transform of derivatives:

A.K. Shukla [72] gave the properties of classical Laguerre transform of functions of two variables. We shall extend these properties for generalized Laguerre transform of functions of two variables.

**Theorem: 9.1** Let \( f \in \mathcal{A} \), and \( n \) is a non-negative integer then,

\[ \mathcal{L}^{(\alpha,\beta)}(\frac{\partial f}{\partial x})(n) = \mathcal{L}^{(\alpha,\beta)}(f)(n) - (n + \alpha)\mathcal{L}^{(\alpha-1,\beta)}(f)(n) \]  
\[ (2.9.1) \]
\[ L^{(\alpha,\beta)} \frac{\partial f}{\partial y}(n) = L^{(\alpha,\beta)}(f)(n) - (n + \beta)L^{(\alpha,\beta-1)}(f)(n) \]  \hspace{1cm} (2.9.2)

\[ L^{(\alpha,\beta)} \frac{\partial^2 f}{\partial x^2}(n) = L^{(\alpha,\beta)}(f)(n) - 2(n + \alpha)L^{(\alpha-1,\beta)}(f)(n) \]
\[ + (n + \alpha)(n + \alpha - 1)L^{(\alpha-2,\beta)}(f)(n) \]  \hspace{1cm} (2.9.3)

\[ L^{(\alpha,\beta)} \frac{\partial^2 f}{\partial y^2}(n) = L^{(\alpha,\beta)}(f)(n) - 2(n + \beta)L^{(\alpha,\beta-1)}(f)(n) \]
\[ + (n + \beta)(n + \beta - 1)L^{(\alpha,\beta-2)}(f)(n) \]  \hspace{1cm} (2.9.4)

\[ L^{(\alpha,\beta)}(x \frac{\partial f}{\partial x})(n) = L^{(\alpha+1,\beta)}(f)(n) - (\alpha + 1)L^{(\alpha,\beta)}(f)(n) \]  \hspace{1cm} (2.9.5)

\[ L^{(\alpha,\beta)}(y \frac{\partial f}{\partial x})(n) = L^{(\alpha,\beta+1)}(f)(n) - (\beta + 1)L^{(\alpha,\beta)}(f)(n) \]  \hspace{1cm} (2.9.6)

\[ L^{(\alpha,\beta)}(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y})(n) = L^{(\alpha+1,\beta)}(f)(n) - (\beta + 1)L^{(\alpha,\beta+1)}(f)(n) \]
\[ - (\alpha + \beta + 2)L^{(\alpha,\beta)}(f)(n) \]  \hspace{1cm} (2.9.7)

\[ L^{(\alpha,\beta)}(x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2})(n) \]
\[ = L^{(\alpha,\beta+2)}(f)(n) + L^{(\alpha+2,\beta)}(f)(n) + 2L^{(\alpha+1,\beta+1)}(f)(n) \]
\[ - (2\alpha + 2\beta + 5)\{L^{(\alpha+1,\beta)}(f)(n) + L^{(\alpha,\beta+1)}(f)(n)\} + (\alpha + \beta + 2)^2 L^{(\alpha,\beta)}(f)(n) \]  \hspace{1cm} (2.9.8)

**Proof:** By (2.6.3) we have,

\[ L^{(\alpha,\beta)}(\frac{\partial f}{\partial x})(n) = \langle \frac{\partial f}{\partial x}, \psi^{(\alpha,\beta)}_n(x, y) \rangle = -\langle f, \frac{\partial}{\partial x} \psi^{(\alpha,\beta)}_n(x, y) \rangle \]

Now using (2.8.1)

\[ = \langle f, \psi^{(\alpha,\beta)}_n(x, y) \rangle - (n + \alpha)\langle f, \psi^{(\alpha-1,\beta)}_n(x, y) \rangle \]
\[ = L^{(\alpha,\beta)}(f)(n) - (n + \alpha)L^{(\alpha-1,\beta)}(f)(n) \]

Hence proved.
Similarly we can prove all other properties.

2.10. Application of differential operator $\eta$:

Using series expansion of $f \in \mathcal{A}^\prime(I)$, we can write for non negative integer $k$,

$$\eta^k f = \sum_{n=0}^\infty (f, \psi_n) \eta^k \psi_n = \sum_{n=0}^\infty (f, \psi_n) \lambda_n^k \psi_n$$  \hspace{1cm} (2.10.1)

Now consider the differential equation

$$P(\eta) f = g$$  \hspace{1cm} (2.10.2)

Where $P$ is a polynomial, and the given $g$ and unknown $f$ are required to be in $\mathcal{A}^\prime$.

Since $f \in \mathcal{A}^\prime(I)$,

$$P(\eta) f = P(\eta) \sum_{n=0}^\infty (f, \psi_n) \psi_n = \sum_{n=0}^\infty (f, \psi_n) p(\eta) \psi_n$$

$$= \sum_{n=0}^\infty (f, \psi_n) \lambda_n \psi_n$$

Now applying distributional generalized Laguerre transform $\ell^{(\alpha, \beta)}$ to both the sides of (2.10.2) we get,

$$P(\lambda_n) F(n) = G(n) \text{ where } F = \ell^{(\alpha, \beta)} f \text{ and } G = \ell^{(\alpha, \beta)} g$$

i.e. $P(\lambda_n)(f, \psi_n) = (g, \psi_n)$

Case (i) $p(\lambda_n) \neq 0$ then $(f, \psi_n) = [p(\lambda_n)]^{-1} (g, \psi_n)$.

We now apply $(\ell^{(\alpha, \beta)})^{-1}$ to both side we get

$$f = \sum_{n=0}^\infty (g, \psi_n) [P(\lambda_n)]^{-1} \psi_n$$  \hspace{1cm} (2.10.3)
By characterization theorem 7.1 and Uniqueness theorem 6.2 solution in (2.10.3) exist and is unique.

Case (ii) $p(\lambda_n) = 0$ for some $\lambda_n$. Let $(p(\lambda_{n_k}) = 0$ for $k=1,2\ldots m$.

Then in this case solution will be exist in $\mathcal{A}'(I)$ if and only if $G(n_k)=0$, for $k=1,2,3\ldots m$. The solution will be then

$$f = \sum_{p(\lambda_n) = 0}^{\infty} (g_n \psi_n) [p(\lambda_n)]^{-1} \psi_n$$

(2.10.4)

Which is not unique in $\mathcal{A}'(I)$. We may add to (2.10.4) any complementary solution $f_c = \sum_{k=1}^{m} a_k \psi_{n_k}$, where $a_k$ are arbitrary numbers.