Chapter 3

STUDY OF SUB CLASSES OF
MULTI VALENT FUNCTIONS AND SOME APPLICATIONS ASSOCIATED WITH
DIFFERENTIAL SUB ORDINATION

3.0 Introduction

In this chapter we have investigated and derived two Sub classes \( U_q^p (\zeta) \) and \( \lambda_q^p (\zeta) \) of Meromorphic (analytic except for isolated singularities i.e. poles) multivalent functions in 
\[ D = \{ z : 0 < |z| < 1 \} \]. The properties like Distort\(^n\) Thms, convolution of functions, the radii of star likeness, coefficient inequalities, and convexity closure theorems etc. which are in these classes has been obtained. In this chapter we investigated few more new Sub classes \( D_q^p (\alpha, \beta) \), \( f_{q,k}^m (\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi) \), \( g_q^m (\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi) \), and \( h_q^m (\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi) \), as well as \( \Phi_{q,k}^m (\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi) \), \( G_q^m (\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi) \), and \( H_q^m (\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi) \) of Meromorphic (analytic except for isolated singularities i.e. poles) multivalent functions in 
\[ D^* = \{ z : 0 < |z| < 1 \} = D \setminus \{ 0 \} \]. Applying differential subordination methods, we have derived some particular properties like distortion theorems, convolution of functions etc. of Meromorphic (analytic except for isolated singularities i.e. poles) multivalent functions.

3.1 Sub classes of Meromorphic Multivalent Functions

Let us assume that \( \sum_q \) represents as given below
\[
f (z) = \frac{1}{z^q} + \sum_{n=1}^{\infty} a_{q+n-1} z^{q+n-1} \quad (q \in N).
\] (3.1.1)

These functions are p-valently and Holomorphic (an analytic) in D 
\[ D = \{ z : 0 < |z| < 1 \} \]
f \( \in \sum_q \) Contained in the class \( U_p (\zeta) \) which is p-valently and Meromorphic (analytic except for isolated singularities i.e. poles) star like function iff
\[ \text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in D; 0 \leq \zeta, \alpha < q; q \in N). \quad (3.1.2) \]

Moreover \( f \in \sum_p \) contained in \( \lambda q(\zeta) \) which is \( p \)-valently and Meromorphic (analytic except for isolated singularities i.e. poles) star like function of order \( \zeta \) in unit disk \( D \) if & only if

\[ \text{Re}\left\{-1 - \frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in D; 0 \leq \zeta, \alpha < q; q \in N). \quad (3.1.3) \]

The classes \( U(\zeta), \lambda(\zeta) \) and other such functions \( \Sigma_1 \) are studied more extensively by [50], [5], [19], [71], [60], [61], [18], [67], [32], [62], [59], [71], [26] and [27]. Now we obtain sufficient conditions for \( f(z) \) so that \( f(z) \) will be in \( U(\zeta) \) and \( \lambda(\zeta) \) which we can derive making use of Coefficient inequalities.

3.2 Coefficient Inequalities

**Theorem 3.2.1**

Let \( \sigma_n(q,k,\alpha,\beta) = (q+n+k-1) + (2\beta - 1)(q+n-1) + 2\beta\alpha - k \).

If \( f(z) \in H_q \) which satisfies

\[ \sum_{n=1}^{\infty} \sigma_n(q,k,\alpha,\beta)a_{p+n-1} < 2\beta(q-\alpha) \quad (3.2.1) \]

For some \( \alpha(0 \leq \alpha < q), \beta(\frac{1}{2} < \beta \leq 1) \) and \( k(k \geq q) \), it is \( f(z) \in Uq(\alpha, \beta) \).

**Proof** Let (2.2.1) holds true, i.e. here after it is to be taken as for \( \alpha(0 \leq \alpha < 1), \beta(\beta < \beta \leq 1) \) and \( k(k \geq 1) \). For \( f(z) \in H_q \) it suffices to show that

\[ \left| \frac{zf'(z)}{f(z)} + k \right| < 1 \quad (z \in D). \]

We note that
\[
\begin{align*}
&z f'(z) + k \\
&2\beta \left( z f'(z) + \alpha \right) - \left( z f'(z) + k \right)
\end{align*}
\]

\[
\begin{align*}
k - q + \sum_{n=1}^{\infty} (q + n + k - 1) a_{q+n-1} z^{2q+n-1} \\
q(1 - 2\beta) + 2\beta \alpha - k + \sum_{n=1}^{\infty} [(2\beta - 1)(q + n - 1) + 2\beta \alpha - k] a_{q+n-1} z^{2q+n-1}
\end{align*}
\]

\[
\begin{align*}
\leq \frac{k - q + \sum_{n=1}^{\infty} (q + n + k - 1) a_{q+n-1}}{(2\beta - 1)q + k - 2\beta \alpha - \sum_{n=1}^{\infty} [(2\beta - 1)(q + n - 1) + 2\beta \alpha - k] a_{q+n-1}}
\end{align*}
\]

Above expansion which is bounded above

\[
\begin{align*}
n - q + \sum_{m=1}^{\infty} (q + m + n - 1) |a_{q+m-1}| \\
< (2\beta - 1)q + n = 2\beta \alpha - \sum_{m=1}^{\infty} [(2\beta - 1)(q + m - 1)2\beta \alpha - n] |a_{q+m-1}|
\end{align*}
\]

This is similar to the condition (3.2.1).

**Example 3.2.1** The function \( f(z) \) defined below

\[
f(z) = \frac{1}{z^3} + \sum_{n=1}^{\infty} \frac{4\beta(q - \alpha)}{n(n + 1)\sigma_n(q, k, \alpha, \beta)} z^{p+n-1} \quad (q \in N) \tag{3.2.2}
\]

It contained in the class \( U_q(\zeta) \). Since \( f(z) \in U_q(\zeta) \) if & only if \( z f'(z) \in \lambda_q(\zeta) \), we can prove:

**Theorem 3.2.2** If \( f(z) \in H_q \) satisfies

\[
\sum_{n=1}^{\infty} (q + n - 1) \sigma_n(q, k, \alpha, \beta) |a_{q+n-1}| < 2\beta(q - \alpha). \tag{3.2.3}
\]

For some
\[ \alpha(0 \leq \alpha < q), \beta(\frac{1}{2} < \beta \leq 1) \quad \text{And} \]
\[ k(k \geq q) \quad \text{I. e. here after it is to be taken as} \quad f(z) \in \lambda_q(\alpha, \beta) \]

**Example 3.2.2** Here the function \( f(z) \) is taken as
\[ f(z) = \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{4\beta(q-\alpha)}{n(n+1)(q+n-1)\sigma_n(q,k,\alpha,\beta)} z^{q+n-1} \].

Above function \( f(z) \) contained in the class \( \lambda_q(\zeta) \). In agreement with theorem 3.2.1 & theorem 3.2.2, we are defining the sub classes \( U_q^*(\zeta) \subset U_q(\zeta) \) & \( \lambda_q^*(\zeta) \subset \lambda_q(\zeta) \). The above Sub classes containing the functions like \( f(z) \in H_q \) which satisfies the conditions (3.2.1) and (3.2.3). Assuming \( q = 1 \) and \( 1 \leq k \leq (2\beta - 1)n + 2\alpha\beta \), here \( 0 \leq \alpha < 1 \) and \( \frac{1}{2} < \beta \leq 1 \) in theorems 3.2.1 & 3.2.2, we get the corollaries as taken as follows.

**Corollary 3.2.1** If the function \( f \) is s. t. \( f(z) \in H_1 \) which satisfies the condition
\[ \sum_{n=1}^{\infty} \beta(n+\alpha)|a_n| < \beta(1-\alpha). \]
I. e. here after it is to be taken as
\[ f(z) \in U_1(\zeta) = H^*(\zeta) \]
This is the class of Meromorphic (analytic except for isolated singularities i. e. poles) star like functions with order \( \delta(0 \leq \delta < 1) \) in\( D \).

**Corollary 3.2.2** If the function \( f \) is s. t. \( f(\square) \in H_p \) which satisfies the condition
\[ \sum_{n=1}^{\infty} \beta n(n+\alpha)|a_n| < \beta(1-\alpha). \]
I. e. here after it is to be taken as
\[ f(z) \in \lambda_k(\zeta) = H_k^*(\zeta) \]
This is the class of Meromorphic (analytic except for isolated singularities i. e. poles) star like functions with order \( \delta(0 \leq \delta < 1) \) in\( D \).
3.3 Distortion Theorems

**Theorem 3.3.1** If function \( f \) is s. t. \( f(z) \) defined in (3.1.1) contained in the class \( U_p^\ast(\zeta) \), i. e. hereafter it is to be taken as for all \( 0 < |z| = r < 1 \), we have obtained

\[
\frac{1}{r^q} - \frac{2\beta(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} r^q \leq |f(z)| \leq \frac{1}{r^q} + \frac{2\beta(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} r^q
\]

\[
\frac{q}{r^{q+1}} - \frac{2\beta q(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} r^{q+1} \leq |f'(z)| \leq \frac{q}{r^{q+1}} + \frac{2\beta q(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} r^{q+1}
\]

The bounds in (3.3.1) and (3.3.2) \( \forall f(z) \) are s. t.

\[
f(z) = \frac{1}{z^q} + \frac{2\beta(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} z^q \quad (q \in N; z \in D)
\]

**Proof** Since \( f \in U_q^\ast(\zeta) \), from the inequality (3.2.1), we have

\[
\sum_{n=1}^{\infty} |a_{q+n-1}| \leq \frac{2\beta(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} \quad (3.3.1)
\]

Thus, for \( 0 < |z| = r < 1 \), and using (3.3.4) we get

\[
|f(z)| \leq \left| \frac{1}{z^q} \right| + \sum_{n=1}^{\infty} |a_{q+n-1}| |z|^{q+n-1} \quad (3.3.2)
\]

\[
\leq \frac{1}{r^q} + r^q \sum_{n=1}^{\infty} |a_{q+n-1}| \leq \frac{1}{r^q} + \frac{2\beta(q - \alpha)}{q + k + \left|(2\beta - 1)q + 2\beta\alpha - k\right|} r^q
\]
Here over it is

\[ |f(z)| \geq \frac{1}{z^q} - \sum_{n=1}^{\infty} |a_{q+n-1}| \left| \frac{\beta}{z^{q+n}} \right| \]

\[ \geq \frac{1}{r^q} - r^q \sum_{n=1}^{\infty} |a_{q+n-1}| \geq \frac{1}{r^q} - \frac{2\beta(q - \alpha)}{q + k + |(2\beta - 1)q + 2\beta\alpha - k|} r^q \]

We can verify that

\[ \frac{q + k + |(2\beta - 1)q + 2\beta\alpha - k|}{q} \sum_{n=1}^{\infty} (q + n - 1) |a_{q+n-1}| \leq 2\beta(q - \alpha) \quad (3.3.3) \]

Which gives the following Distortion inequality

\[ \left| f'(z) \right| \leq \frac{q}{|z|^{q+1}} + \sum_{n=1}^{\infty} (q + n - 1) |a_{q+n-1}| \left| \frac{1}{z^{q+n-2}} \right| \]

\[ \leq \frac{q}{r^{q+1}} + r^{q-1} \sum_{n=1}^{\infty} (q + n - 1) |a_{q+n-1}| \leq \frac{q}{r^{q+1}} + \frac{2\beta q(q - \alpha)}{q + k + |(2\beta - 1)q + 2\beta\alpha - k|} r^{q-1} \]

And

\[ \left| f'(z) \right| \geq \frac{q}{|z|^{q+1}} - \sum_{n=1}^{\infty} (q + n - 1) |a_{q+n-1}| \left| \frac{1}{z^{q+n-2}} \right| \]

\[ \geq \frac{q}{r^{q+1}} - r^{q-1} \sum_{n=1}^{\infty} (q + n - 1) |a_{q+n-1}| \]

\[ \geq \frac{q}{r^{q+1}} - \frac{2\beta q(q - \alpha)}{q + k + |(2\beta - 1)q + 2\beta\alpha - k|} r^{q-1} \]

Hence theorem proved for \( f(z) \in \lambda_q^*(\zeta) \), using (3.2.3), the next theorem.

**Theorem 3.3.2** If the function \( f \) is s. t. \( f(z) \) defined in (3.1.1) contained in the class \( \lambda_q^*(\zeta) \) and i. e. here after it is to be taken as \( \forall \ 0 < |z| = r < 1 \) we obtained

\[ \frac{1}{r^q} - \frac{2\beta(q - \alpha)}{q + k + |(2\beta - 1)q + 2\beta\alpha - k|} r^q \leq |f(z)| \quad (3.3.5) \]

\[ \leq \frac{1}{r^q} + \frac{2\beta(q - \alpha)}{q + k + |(2\beta - 1)q + 2\beta\alpha - k|} r^q. \]

Where it is obviously for all,
\[
\frac{q}{r^{q+1}} - \frac{2\beta(q-\alpha)}{q+k + |(2\beta-1)q + 2\beta\alpha - k|} r^{q-1} \leq |f'(z)| \tag{3.3.6}
\]

\[
\leq \frac{q}{r^{q+1}} + \frac{2\beta(q-\alpha)}{q+k + |(2\beta-1)q + 2\beta\alpha - k|} r^{q-1}
\]

This bound in (3.3.10) and (3.3.11) for \((q \in N; \ z \in D) f(z)\) as given below

\[
g(z) = \frac{1}{z^{q+1}} + \frac{2\beta(q-\alpha)}{q+k + |(2\beta-1)q + 2\beta\alpha - k|} z^q \tag{3.3.7}
\]

### 3.4 Radii of Star likeness and Convexity

**Theorem 3.4.1** If function \(f\) is s. t. \(f(z)\) defined in (3.1.1) contained in the class \(U_q^+(\xi)\) i. e. here after it is to be taken as \(f(z)\) which is called as Meromorphic (analytic except for isolated singularities i. e. poles) \(p\)-valently convex of order \(\delta(0 \leq \delta < q)\) in \(|z| < r_1\).

Where it is obviously for all,

\[
r_1 = \inf_{n \in \mathbb{N}} \left[ \frac{(q-\delta)\sigma_n(q,k,\alpha,\beta)}{2\beta(3q+n-1-\delta)(q-\alpha)} \right] \frac{1}{2^{q+n-1}} (q \in N) \tag{3.4.1}
\]

Furthermore, \(f(z)\) is \(p\)-valently Meromorphic (analytic except for isolated singularities i. e. poles) convex of order \(\delta(0 \leq \delta < q)\) in \(|z| < r_2\).

Where it is obviously for all,

\[
r_2 = \left[ \frac{q(\delta-\delta)\sigma_n(q,k,\alpha,\beta)}{2\beta(q+n-1)(3q+n-1-\delta)(q-\alpha)} \right] \frac{1}{2^{q+n-1}} \tag{3.4.2}
\]

Thus the results (3.4.1) & (3.4.2) \(\forall f(z)\) are sharp. Where \((q \in N; z \in D)\)

\[
f(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{\sigma_n(q,k,\alpha,\beta)} z^{q+n-1} \tag{3.4.3}
\]

**Proof** We prove
\[ \left| z \frac{f'(z)}{f(z)} + q \right| \leq q - \delta \]  

(3.4.4)

\[ \forall \ |z| \leq r_1, \ \text{get} \]

\[ \left| z \frac{f'(z)}{f(z)} + q \right| = \frac{\sum_{n=1}^{\infty} (2q + n - 1)a_{q+n}z^{q+n-1}}{1 + \sum_{n=1}^{\infty} a_{q+n}z^{q+n-1}} \]

\[ \leq \frac{\sum_{n=1}^{\infty} (2q + n - 1)\|a_{q+n}\|z^{2q+n-1}}{1 - \sum_{n=1}^{\infty} \|a_{q+n}\|z^{2q+n-1}} \]  

(3.4.5)

Hence (3.4.5) true iff

\[ \sum_{n=1}^{\infty} (2q + n - 1)\|a_{q+n}\|z^{2q+n-1} \leq (q - \delta) \left( 1 - \sum_{n=1}^{\infty} \|a_{q+n}\|z^{2q+n-1} \right). \]  

(3.4.6)

Or

\[ \sum_{n=1}^{\infty} 3q + n - 1 - \delta \|a_{q+n}\|z^{2q+n-1} \leq 1 \]  

(3.4.7)

With the aid of (3.2.1), (3.4.7) is true if

\[ \frac{3q + n - 1 - \delta}{q - \delta} \|z^{2q+n-1} \leq \sigma_n(q, k, \alpha, \beta) \]  

(3.4.8)

Solving (3.4.8) for \(|z|\)

\[ |z| < \left[ \frac{(q - \delta)\sigma_n(q, k, \alpha, \beta)}{2\beta(3q + n - 1 - \delta)(q - \alpha)} \right]^{\frac{1}{2q+n-1}} \]  

(3.4.9)

Similar to this we can obtain radius of convexity given by (3.4.2), by using

\[ \left| z \frac{f''(z)}{f'(z)} + q + 1 \right| \leq q - \delta. \]  

(3.4.10)

In the same manner we will find convexity and radii of star likeness \( \forall \) functions in the class \( \lambda_q^*(\zeta) \).

**Theorem 3.4.2** If function \( f \) is s. t. \( f(z) \) defined in (3.1.1) contained in the class \( \lambda_q^*(\zeta) \) i. e.
here after it is to be taken as we have \( f(z) \) which is said to be Meromorphic (analytic except for isolated singularities i.e. poles) \( p \)-valently star like

\[
r_i = \inf_{n \in \mathbb{N}} \left\{ \frac{(q-\delta)(q+n-1)\sigma_n(q,k,\alpha,\beta)}{2\beta(3q+n-1-\delta)(q-\alpha)} \right\}^{\frac{1}{2q+n-1}} \quad (3.4.11)
\]

Further moreover, function \( f(z) \) is Meromorphic (analytic except for isolated singularities i.e. poles) univalent (or schlicht i.e. a single valued function) convex of order \( \delta(0 \leq \delta < p) \) in \( |z| < r_d \).

Where it is obviously for all \( (q \in \mathbb{N}) \) and

\[
r_d = \inf_{n \in \mathbb{N}} \left\{ \frac{q(q-\delta)(q+n-1)\sigma_n(q,k,\alpha,\beta)}{2\beta(q+n-1)(3q+n-1-\delta)(q-\alpha)} \right\} \quad (3.4.12)
\]

The results (3.4.11) and (3.4.12) for \( g(z) \) are as defined below

\[
g(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{(q+n-1)\sigma_n(q,k,\alpha,\beta)}z^{q+n-1} \quad (q \in \mathbb{N}; z \in D), \quad (3.4.13)
\]

### 3.5 Closure Theorems

**Theorem 3.5.1** Let

\[
f_j(z) = \frac{1}{z^q} + \sum_{n=1}^{\infty} a_{q+n-1,j}z^{q+n-1} \quad (3.5.1)
\]

\( j \in \{1,2,\ldots,m\} \), \( z \in D \) contained in the class \( U_q^*(\zeta) \) i.e. here after it is to be taken as the function given

\[h(z) \in U_q^*(\zeta).\]

Where it is obviously for all,

\[
h(z) = \sum_{j=1}^{m} b_j f_j(z), \quad b_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{m} b_j = 1 \quad (3.5.2)
\]

**Proof** Using (3.5.2)

\[
h(z) = \frac{1}{z^q} + \sum_{n=1}^{\infty} c_{q+n-1}z^{q+n-1} \quad (3.5.3)
\]
Where it is obviously for all,
\[ c_{q+n-1} = \sum_{j=1}^{m} b_j a_{q+n-1,j}, \quad j \in \{1, 2, \ldots, m\} \]  \hspace{1cm} (3.5.4)

Since \( f_j(z) \in U_q^\ast(\zeta) \), from 3.2.1 we have,
\[ \sum_{m=1}^{\infty} \left[ \frac{\sigma_n(q,k,\alpha,\beta)}{2\beta(q-\alpha)} \right] \sum_{j=1}^{m} b_j |a_{q+n-1,j}| = \sum_{j=1}^{m} \left( \frac{\sigma_n(q,k,\alpha,\beta)}{2\beta(q-\alpha)} \right) |a_{q+n-1,j}| \leq \sum_{j=1}^{m} b_j = 1 \]

i. e. \( h(z) \in U_q^\ast(\zeta) \). Using the same procedure as above in thm (3.5.1) we get the next theorem.

**Theorem 3.5.2** Let us assume that \( f_j(z), \ j \in \{1, 2, \ldots, m\} \) as (3.5.1) belonging to class \( \lambda_q^\ast(\zeta) \)
where \( h(z) \in \lambda_q^\ast(\zeta) \), here \( h(z) \) represented by (3.5.2).

**Theorem 3.5.3** Let us assume
\[ f_{q-1}(z) = \frac{1}{z^q} \quad (z \in D) \]  \hspace{1cm} (3.5.5)
\&
\[ f_{q+n-1}(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{\sigma_n(q,k,\alpha,\beta)} \ z^{p+n-1} \]  \hspace{1cm} (3.5.6)
n \( \in N_0 \); I. e. here after it is to be taken as \( f(z) \in U_q^\ast(\alpha) \) if & only if it can be expressed in the form of
\[ f(z) = \sum_{n=0}^{m} \gamma_{q+n-1} f_{q+n-1}(z), \]  \hspace{1cm} (3.5.7)
Where it is obviously for all,
\[ \gamma_{q+n-1} \geq 0, \quad \text{and} \quad \sum_{n=0}^{m} \gamma_{q+n-1} = 1. \]

**Proof** From (3.5.5), (3.5.7) and (3.5.6) it shows that
\[ f(z) = \sum_{n=0}^{m} \gamma_{q+n-1} f_{q+n-1}(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{\sigma_n(q,k,\alpha,\beta)} \gamma_{q+n-1} \ z^{p+n-1} \]  \hspace{1cm} (3.5.8)
Since \[
\sum_{n=1}^{\infty} \sigma_n(q,k,\alpha,\beta) \frac{2\beta(q-\alpha)}{\sigma_n(q,k,\alpha,\beta)} \gamma_{q+n-1} = 1 - \gamma_{q-1} \leq 1
\] (3.2.1) gives \( \forall f(z) \) defined as (3.5.6) contained in the class \( U_q^*(\zeta) \), conversely, let us assume that \( f(z) \in U_q^*(\zeta) \) since
\[
|a_{q+n-1}| \leq \frac{2\beta(q-\alpha)}{\sigma_n(q,k,\alpha,\beta)} \quad (n \geq 1).
\]
Setting
\[
\gamma_{q+n-1} = \frac{\sigma_n(q,k,\alpha,\beta)}{2\beta(q-\alpha)} |a_{q+n-1}| \quad \text{and} \quad \gamma_{q-1} = 1 - \sum_{n=1}^{\infty} \gamma_{q+n-1}
\]
It follows that
\[
f(z) = \sum_{n=0}^{\infty} \gamma_{q+n-1} f_{q+n-1}(z).
\]
Easily it is outcome for \( \lambda_q^*(\zeta) \).

**Theorem 3.5.4** Let us assume
\[
g_{q-1}(z) = \frac{1}{z^q} \quad (z \in D) \quad (3.5.9)
\]
And
\[
g_{q+n-1}(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{(q+n-1)\sigma_n(q,k,\alpha,\beta)} z^{q+n-1}, \quad (3.5.10)
\]
\( n \in N_0, \ (z \in D) \) I.e. here after it is to be taken as \( g(z) \in \lambda_q^*(\zeta) \) if & only if it can be expressed as
\[
g(z) = \sum_{n=0}^{\infty} \gamma_{q+n-1} g_{q+n-1}(z) \quad (3.5.11)
\]
Here it is obviously for all \( \gamma_{q+n-1} \geq 0, \ (n \in N_0) \) and \( \sum_{n=0}^{\infty} \gamma_{q+n-1} = 1. \)
Theorem 3.5.5 Let us assume \( f \) & \( g \) contained in the classes \( U_q^+(\zeta) \) and \( \gamma_q^+(\zeta) \). I. e. here after it is to be taken as the function \( h(z) = tf(z) + (1 - t)g(z) \) \( (0 \leq t \leq 1) \) are in \( U_q^+(\zeta) \) or in \( \gamma_q^+(\zeta) \).

3.6 Convolution Properties

Theorem 3.6.1 Let each of the \( f_j(z) \) defined as \( f_j(z) = \frac{1}{z^q} + \sum_{n=1}^{\infty} a_{q+n-1,j} z^{q+n-1}, \quad j = 1, 2 \) (3.6.1)

\( f_j(z) \) contained in \( U^+(\zeta) \) i. e. here after it is to be taken as \( (f_1 \ast f_2)(z) \in U^+(\eta) \) where it is obviously

\[ (f_1 \ast f_2)(z) = \frac{1}{z^q} + \sum_{n=1}^{\infty} a_{q+n-1,1} a_{q+n-1,2} z^{q+n-1} \] (3.6.2)

Proof For \( f_j(z) \in U^+(\zeta), \quad (j = 1, 2) \) we find largest \( \eta \) s. t.

\[ \sum_{n=1}^{\infty} \frac{\sigma_n(q,k,\eta)}{2\beta(q-\eta)} \left| a_{q+n-1,1} \right| \left| a_{q+n-1,2} \right| \leq 1 \] (3.6.3)

From (3.2.1) we have

\[ \sum_{n=1}^{\infty} \frac{\sigma_n(q,k,\alpha,\beta)}{2\beta(q-\alpha)} \left| a_{q+n-1,1} \right| \leq 1 \] (3.6.4)

\[ \sum_{n=1}^{\infty} \frac{\sigma_n(q,k,\alpha,\beta)}{2\beta(q-\alpha)} \left| a_{q+n-1,2} \right| \leq 1 \] (3.6.5)

Hence using Cauchy- Schwarz’s inequality, obtained

\[ \sum_{n=1}^{\infty} \frac{\sigma_n(q,k,\alpha,\beta)}{2\beta(q-\alpha)} \sqrt{\left| a_{q+n-1,1} \right| \left| a_{q+n-1,2} \right|} \leq 1 \] (3.6.6)

Thus it is sufficient to show that

\[ \frac{\sigma_n(q,k,\eta)}{2\beta(q-\eta)} \left| a_{q+n-1,1} \right| \left| a_{q+n-1,2} \right| \leq \frac{\sigma_n(1,k,\alpha,\beta)}{2\beta(q-\alpha)} \sqrt{\left| a_{q+n-1,1} \right| \left| a_{q+n-1,2} \right|} \] (3.6.7)

From (3.6.7)
\[
\sqrt{|a_{q+n-1}|^4 |a_{q+n-1,2}|^2} \leq \frac{2\beta(q-\alpha)}{\sigma_n(q, k, \alpha, \beta)}. \quad (3.6.8)
\]

consequently, we need only to show

\[
\frac{2\beta(q-\alpha)}{\sigma_n(q, k, \alpha, \beta)} \leq \frac{(q-\eta)\sigma_n(q, k, \alpha, \beta)}{(q-\alpha)\sigma_n(q, k, \eta)} \quad (n \geq 1) \quad (3.6.9)
\]

Let \( \eta \geq \frac{1}{2} (k + 1 - p - n) \).

Obviously for all, \( k \geq p \);

\[
\eta \leq \frac{p[\sigma_n(p, k, \alpha, \beta)]^2 - 4[\beta(p-\alpha)]^2(p + n - 1)}{4[\beta(p-\alpha)]^2 + [\sigma_n(p, k, \alpha, \beta)]^2} = \psi(n) \quad (3.6.10)
\]

Since \( \psi(n) \) increasing functn of \( n \) (\( n \geq 1 \)), letting \( n=1 \) in (3.6.11), we obtain

\[
\eta \leq \psi(1) = q \left( q + k + |(2\beta - 1)q + 2\beta\alpha - k| \right)^2 - \left[ 2\beta(q-\alpha) \right]^2 \left[ 2\beta(q-\alpha) \right]^2 + \left[ q + k + |(2\beta - 1)q + 2\beta\alpha - k| \right]^2 . \quad (3.6.11)
\]

Hence theorem is proved. Similar to this it is easy to prove next theorem.

**Theorem 3.6.2** Let us assume \( f_j(z), j = (1,2) \) represented as (3.6.1) belongs to \( \lambda_q^{*}(\zeta) \) i. e.

hereafter it is to be taken as \( (f_1 \ast f_2)(z) \in \lambda_q^{*}(\zeta) \), where it is,

\[
\frac{1}{2} (k + 1 - q - n) \leq \xi = q \left[ q + k + |(2\beta - 1)q + 2\beta\alpha - k| \right]^2 - \left[ 2\beta(q-\alpha) \right] \left[ 2\beta(q-\alpha) \right] + \left[ q + k + |(2\beta - 1)q + 2\beta\alpha - k| \right]^2
\]

\((k \geq q; q, n \in \mathbb{N})\). The above obtained result is true for the functions given below

\[
f_j(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{(q + n - 1)\sigma_n(q, k, \alpha, \beta)} z^{q+n-1} \quad (j = 1,2) \quad (3.6.12)
\]

**3.7 Applications of Meromorphic Multi valent Functions Associated with Differential Subordination**

Let \( S \) represents the class of functions as given below

\[
g(z) = z^{-q} + \sum_{n=0}^{\infty} b_{q+n-1} z^{q+n-1}, \text{ For all } (q \in \mathbb{N}) \quad (3.7.1)
\]

these functions are multivalent and Holomorphic (an analytic) in the punctured unit disk \( D^* \) given as follows
\( D^* = \{ z : 0 < |z| < 1 \} = D \setminus \{0\}. \)

Let \( g(z) \) and \( f(z) \) be Holomorphic (an analytic) in unit disk \( D \), i.e., hereafter it is to be taken as we can say that \( g(z) \) is subordinate to \( f(z) \) in \( D \), where \( g(z) < f(z) \), if \( \exists \) a Holomorphic (an analytic) function \( h(z) \) in \( D \), so that \( \text{mod } h(z) \leq \text{mod } z \) \& \( g(z) = f[h(z)] \), (\( \forall z \) in \( D \)). If \( g(z) \) is multivalent in \( D \) i.e. hereafter it is to be taken as the subordination \( g(z) < f(z) \) \( (D) \Leftrightarrow \) \( g \) at \( z = 0 \) is equal to \( f \) at \( z = 0 \) \& \( g(D) \) is contained in \( f(D) \).

Let \( t(z) = 1 + t_1 z + t_2 z + \cdots \) be Holomorphic (an analytic) in \( D \).

S.t.

\[
 t(z) < \frac{1+2az}{1+2\beta z} \quad (z \in D). 
\]

\[
\left| t(z) - \frac{1-4a\beta}{1-4\beta^2} \right| < \frac{2(\alpha-\beta)}{1-4\beta^2}, \quad \left( -\frac{1}{2} \leq \beta < \frac{1}{2} \right). 
\]

\[
\text{Re } t(z) > \frac{1-2\alpha}{2}, \quad (2\beta = -1, \ z \in D). 
\]

Recently, several numbers of authors proved and studied some interesting properties of Meromorphic (analytic except for isolated singularities i.e. poles) multivalent functions. In this chapter we want to prove some of subordination properties for the class \( S \).

When \( g(z) = f(z) = z^{-q} + \sum_{n=0}^{\infty} b_{q+n-1} z^{q+n-1}, \text{For all } (q \in N) \).

We obtained convolution of \( g(z) \& f(z) \) as given below

\[
(g * f)(z) = g(z) = z^{-q} + \sum_{n=0}^{\infty} b_{q+n-1} a_{q+n-1} z^{q+n-1}, \text{For all } (q \in N) 
\]

Where \( (\forall q \in N, \forall m \in N_0 = N \cup \{0\}, z \in D) \). We define a linear Operator by

\[
l_q^n(\sigma, \eta, \xi, \varepsilon, \delta, \nu) g(z) = z^{-q} + \sum_{m=0}^{\infty} \left[ \frac{\sigma \varepsilon (\xi+n) (q+m)}{(\delta-\nu)} + 1 \right]^k b_{q+m-1} z^{q+m-1} = (\psi_q^k_{\sigma, \xi, n, \varepsilon, \delta, \nu} * g)(z). 
\]

Here over it is,

\[
\psi_q^k_{\sigma, \xi, n, \varepsilon, \delta, \nu} g(z) = \frac{1}{z^q} + \sum_{m=0}^{\infty} \left[ \frac{\sigma \varepsilon (\xi+n) (q+m)}{(\delta-\nu)} + 1 \right]^k z^{q+m-1}. 
\]

Throughout this paper for our convenience we are taking

\[
 \frac{(\xi+\eta)}{\sigma \varepsilon (\delta-\nu)} = \exists \text{ and } l_q^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu) g(z) = \psi_q^k g(z) 
\]

\[
(q \in N, k \in N_0 = N \cup \{0\}, z \in D, \xi \geq 0, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}). 
\]

It is simple to check and verify the following

\[
\frac{\sigma \varepsilon (\xi+\eta)}{(\delta-\nu)} z^q \left[ l_q^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu) g(z) \right]. 
\]
\[ l^k_q(\sigma, \eta, \xi, \epsilon, \delta, \nu)g(z) - \left[ \frac{\sigma(\ell + \eta)}{(\delta - \nu)} q + 1 \right] l^{k+1}_q(\sigma, \eta, \xi, \epsilon, \delta, \nu)g(z). \quad (3.7.5) \]
\[ \therefore \exists z[\psi^k_q g(z)]' = \psi^{k+1}_q g(z) - (\zeta + 1)\psi^k_q g(z). \]

We note that,
\[ \psi^0_q(\sigma, \eta, \xi, \epsilon, \delta, \nu)g(z) = g(z) \text{ and} \]
\[ l^1_q \left( \frac{1}{2}, 1, 1, 1, \frac{1}{2}, 1 \right) g(z) = \left[ \frac{z^{q+3}g(z)}{z^q} \right]' = (q + 1)g(z) + zg'(z). \]

Above operators satisfies the conditions given below and are Holomorphic (an analytic) in the unit disk \( D \).

\[ \Re \{p(z)\} > 0, \quad (\forall z \in D). \text{ for } n \in N, \epsilon_n = \exp \left( \frac{2\pi i}{n} \right), \]
\[ f_{q,n}^k[\sigma, \eta, \xi, \epsilon, \delta, \nu; z] = \frac{1}{n} \sum_{j=0}^{n-1} \epsilon_j^k \left[ \psi^k_q f(\epsilon_j^k z) \right] = \frac{1}{z^q} + \cdots \quad (f \in S) \quad (3.7.6) \]
\[ g_{q,n}^k[\sigma, \eta, \xi, \epsilon, \delta, \nu; z] = \frac{1}{z^q} \left( \psi^k_q f(z) + \psi^k_q f(\bar{z}) \right) = \frac{1}{z^q} + \cdots \quad (g \in S). \]
\[ h_{q,n}^k[\sigma, \eta, \xi, \epsilon, \delta, \nu; z] = \frac{1}{z^q} \left( \psi^k_q f(z) + \psi^k_q f(\bar{z}) \right) = \frac{1}{z^q} + \cdots \quad (h \in S). \]

For \( n = 1 \) we have been obtained that
\[ f_{q,1}^k[\sigma, \eta, \xi, \epsilon, \delta, \nu; z] = \psi^k_q f(z). \]
\[ \text{Where,} \quad \left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, \right) \]
\[ \left( \alpha > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \right). \]

We now investigate and discuss the Sub classes of Meromorphic (analytic except for isolated singularities i.e. poles) multivalent functions.

**Definitions**

**Definition 3.7.1** A function \( f \in S \) contained in the class \( F_{q,n}^k(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; z) \) if \( f \) will satisfy the subordination condition given below

\[ - \frac{z^q(1 + \infty)(\psi^k_q f)'(z) + \alpha(\psi^{k+1}_q f)'(z)}{q[(1 + \infty)f_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \nu; z) + \alpha f_{q,n}^{k+1}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)]} < \varphi(z). \]

\( (\alpha \geq 0, z \in D^*, \varphi \in P), f \in F_{q,1}^k[\sigma, \xi, \eta, \epsilon, \delta, \nu; z] \]
\[ f_{q,1}^{k+1}[\sigma, \xi, \eta, \epsilon, \delta, \nu; z] \neq 0. \]
For our convenience & simplicity we can write

\[ P_{q,n}^{k}(0; \sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) = P_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \square). \]

Where it is obviously for all,

\[
\begin{pmatrix}
0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \\
\alpha > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D
\end{pmatrix}.
\]

**Definition 3.7.2** Function \( f \in S \) contained in \( G_{q}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \) if \( f \) will satisfy the subordination condition given below

\[
- \frac{z[(1 + \alpha)(\psi_{q}^{k}f)'(z) + \alpha(\psi_{q}^{k+1}f)'(z)]}{q[(1 + \alpha)g_{q}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; z) + \alpha g_{q}^{k+1}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; z)]} < \varphi(z).
\]

\((z \in D, \alpha \geq 0), g \in G_{q}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \nu; z), \ \& \ g_{q}^{k+1}[\sigma, \xi, \eta, \varepsilon, \delta, \nu; z] \neq 0.\)

For our convenience & simplicity we can write

\[ G_{q}^{k}(0; \sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) = G_{q}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi). \]

Where it is obviously for all,

\[
\begin{pmatrix}
0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \\
\alpha > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D
\end{pmatrix}.
\]

**Definition 3.7.3** Function \( f \in S \) contained in the class \( H_{q}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \nu; z) \), if it satisfies the subordination condition given below

\[
- \frac{z[(1 + \alpha)(\psi_{q}^{k}f)'(z) + \alpha(\psi_{q}^{k+1}f)'(z)]}{q[(1 + \alpha)h_{q}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; z) + \alpha h_{q}^{k+1}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; z)]} < \varphi(z).
\]

\(z \in D, \alpha \geq 0, h \in H_{q}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \nu; z), \ \& \ h_{q}^{k+1}[\sigma, \xi, \eta, \varepsilon, \delta, \nu; z] \neq 0.\)

For our convenience & simplicity we can write

\[ H_{q}^{k}(0; \sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) = H_{q}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi). \]

Where it is obviously for all,

\[
\begin{pmatrix}
0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \\
\alpha > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D
\end{pmatrix}.
\]

**Definition 3.7.4** Function \( f \in S \) contained in the class \( G_{q,n}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \nu; z) \) if it satisfies the
subordination condition given below
\[
-\frac{z[(1+\infty)(\psi_{q}^k f)'(z)+\alpha(\psi_{q}^{k+1} f)'(z)]}{q[(1+\infty)\tilde{\gamma}_{3,1}^k(\sigma,\xi,\eta,\epsilon,\delta,\varphi;\varpi; z) + \alpha k+1(\sigma,\xi,\eta,\epsilon,\delta,\varphi; z)]} < \phi(z).
\]
\[
z \in D, \alpha \geq 0, \ell \in \Gamma_{q,n}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \varphi; z) \& \ell^{k+1}[\sigma, \varpi, \eta, \epsilon, \delta, \varphi; z] \neq 0.
\]
For our convenience & simplicity we can write
\[
\tilde{\gamma}_{q,n}^k(0; \sigma, \xi, \eta, \epsilon, \delta, \varphi) = \tilde{\gamma}_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \varphi; \varphi).
\]
Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, \right).
\]

**Definition 3.7.5** Function $f \in S$ contained in the class $\mathcal{U}_{q,n}^k(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \varphi; z)$, if it satisfies the subordination condition given below
\[
-\frac{z[(1+\infty)(\psi_{q}^k f)'(z)+\alpha(\psi_{q}^{k+1} f)'(z)]}{q[(1+\infty)\tilde{\gamma}_{3,1}^k(\sigma,\xi,\eta,\epsilon,\delta,\varphi;\varpi; z) + \alpha k+1(\sigma,\xi,\eta,\epsilon,\delta,\varphi; z)]} < \phi(z).
\]
\[
z \in D, \alpha \geq 0, u \in \mathcal{U}_{q,n}^k(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \varphi; z) \& u_{k+1}[\sigma, \xi, \eta, \epsilon, \delta, \varphi; z] \neq 0.
\]
For our convenience & simplicity we can write $\mathcal{U}_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \varphi) = \mathcal{U}_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \varphi; \varphi)$.
Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, \right).
\]

**Definition 3.7.6** Function $f \in S$ contained in the class $\mathcal{X}_{q,n}^k(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \varphi; z)$, if it satisfies the subordination condition given below
\[
-\frac{z[(1+\infty)(\psi_{q}^k f)'(z)+\alpha(\psi_{q}^{k+1} f)'(z)]}{q[(1+\infty)\tilde{\gamma}_{3,1}^k(\sigma,\xi,\eta,\epsilon,\delta,\varphi;\varpi; z) + \alpha k+1(\sigma,\xi,\eta,\epsilon,\delta,\varphi; z)]} < \phi(z).
\]
\[
z \in D, \alpha \geq 0, x \in \mathcal{X}_{q,n}^k(\alpha; \sigma, \varpi, \eta, \epsilon, \delta, \varphi; z) \& x_{k+1}[\sigma, \xi, \eta, \epsilon, \delta, \varphi; z] \neq 0.
\]
For our convenience & simplicity we can write $\mathcal{X}_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \varphi) = \mathcal{X}_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \varphi; \varphi)$. 
Where it is obviously for all,
\[
\left(0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}\right).
\]

Remarks

**Remark 3.7.1** put \( q = 0, \xi = \eta = \delta = 1, \varepsilon = \varphi = \sigma = \frac{1}{2} \) \& \( \varphi(z) = \frac{14 + z}{1 - z} \) in definition 3.7.1, we have the class
\[
\operatorname{Re} \left\{ -\frac{z[(1+3\xi)f'(z) + \alpha(zg'(z))']}{(1+3\xi)T_{x}g(z) + \alpha z[T_{x}g(z)]'} \right\} > 0,
\]
\[
T_{x}g(z) = \frac{1}{2}[g(z) - g(-z)].
\]
Where it is obviously for all,
\[
\left(0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}\right).
\]

**Remark 3.7.2** For \( \alpha = 0, \xi = \eta = \delta = 1, \varepsilon = \varphi = \sigma = \frac{1}{2} \) we have the class
\[
G^{k}_{q,n} \left( 0; \frac{1}{2}, 1, 1, \varepsilon, \delta, \varphi; \varphi \right) = G^{k}_{q,n} (\varepsilon, \delta, \varphi; \varphi).
\]
Where \( G^{k}_{q,n} (\varepsilon, \delta, \varphi; \varphi) \) contains the \( g(z) \) in \( S \), satisfy the condition given below
\[
-\frac{z[\psi^{k}_{q}g]'}{qg^{k}_{q,n}(\varepsilon, \delta, \varphi; z)} < \varphi(z). \quad \text{where}
\]
\[
g^{k}_{q,n}[\varepsilon, \delta, \varphi; z] = \frac{1}{n} \sum_{j=0}^{n-1} c_{n}^{j}[\psi^{k}_{q}g](c_{n}^{j}z) \neq 0.
\]
S.t.
\[
\left(0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}\right).
\]

**Remark 3.7.3** For \( \alpha = 0, \xi = \eta = 1, \sigma = \frac{1}{2} \) we have the class
\[
G^{k}_{q,n} \left( 0; \frac{1}{2}, \frac{1}{2}, 1, 1; \varepsilon, \delta, \varphi \right) = G^{k}_{q,n} (\varepsilon, \delta, \varphi; \varphi),
\]
Where \( G^{k}_{q,n} (\varepsilon, \delta, \varphi; \varphi) \) contains the \( g(z) \) in \( S \) if it satisfy the condition given below
\[
-\frac{z[l^{k}(\varepsilon, \delta, \varphi; g)]'}{qg^{k}_{q,n}(z)} < \varphi(z).
\]
\[ g_{q,n}^k [\varepsilon, \delta, \varphi; z] = \frac{1}{n} \sum_{j=0}^{n-1} e_j \left[ \phi_j^k g \right] (e_j^i z) \neq 0, \]

Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right) \]
\[
(\sigma > 0, \delta > 0, \varphi \geq 0, \delta > \varphi, z \in D \right). \]

**Remark 3.7.4** If we put \( q = 1, k = 0, n = 2, \xi = \eta = \delta = 1, \varepsilon = \varphi = \sigma = \frac{1}{2} \) & \( \varphi(z) = \frac{1+z}{1-z} \)

in 3.1.3, we have the class
\[
\text{Rep} \left\{ \frac{-z^r (1+3\alpha) z^s (g(z))'}{(1+3\alpha) z^{s+1} g(z)} \right\} > 0, \]

Here it is,
\[
T_{\infty} f(z) = \frac{1}{2} [f(z) - \overline{f(-z)}] \\
\text{and} \left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right) \]
\[
(\sigma > 0, \delta > 0, \varphi \geq 0, \delta > \varphi, z \in D \right). \]

**Preliminary Lemmas**

**Lemma 3.7.1** Let \( \ell (\Box) \) star like univalent (or schlicht) and Holomorphic (an analytic) in unit disk \( D \) with \( h(0) = 0 \). Iff \( g(z) \) is Holomorphic (an analytic) indisk \( D \) & \( zg'(z) < \ell(z) \), i.e. here after it is to be taken as
\[
g(z) < g(0) + \int_0^z \frac{h(t)}{t} dt. \]

**Lemma 3.7.2** Let \( q(z) \) Holomorphic (an analytic) and other than constant in disk \( D \) with \( t(0) = 1 \). If \( 0 < |z_0| < 1 \) and \( \text{Re} \left[ t(z_0) \right] = \min_{|z|<|z_0|} \text{Re} t(z) \).
I.e. here after it is to be taken as
\[
z_0 t'(z_0) \leq \frac{-|1-t(z_0)|^2}{2(1-\text{Re} t(z_0))}. \]

**Lemma 3.7.3** Let \( d, b \in \mathbb{C} \); and \( \Phi(z) \) is convex and univalent (or schlicht i.e. a single valued function) in \( D \) where \( \Phi(0) = 1 \) & \( \text{Re} (d, \Phi(z) + b) > 0 \). If \( t(z) \) is Holomorphic (an analytic) in \( D \) where \( t(0) = 1 \), i.e. here after it is to be taken as the condition
\[ t(z) + \frac{zt'(z)}{d.t(z)+b} < \emptyset(z) = t(z) < \emptyset(z). \]

**Lemma 3.7.4** Let \( d, b \in \mathbb{C}; \) and \( \emptyset(z) \) is Holomorphic (an analytic), convex and univalent (or schlicht) in disk \( D \) with \( \emptyset(0) = 1 \& \text{Re} \left[ d.\emptyset(z) + b \right] > 0. \) Also let \( t(z) < \emptyset(z) \). If \( q(z) \) is Holomorphic (an analytic) in unit disk \( D \)

With \( t(0) = 1 \). I.e. here after it is to be taken as the subordination given as follows

\[ t(z) + \frac{zt'(z)}{d.t(z)+b} < \emptyset(z) \Rightarrow t(z) < \emptyset(z) \quad (\forall \ z \in D). \]

**Lemma 3.7.5** Let \( f \in P^k_{qn}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \varphi) \) i.e. here after it is to be taken as

\[
\frac{-z[(1+\alpha)(f^k_{\xi}(\sigma,\xi,\eta,\varepsilon,\delta,\varphi,f) + \alpha(f^{k+1}_{\xi}(\sigma,\xi,\eta,\varepsilon,\delta,\varphi,f) + \alpha f^{k+1}_{\xi}(\sigma,\xi,\eta,\varepsilon,\delta,\varphi) \emptyset)]}{q[(1+\alpha)f^k_{\xi}(\sigma,\xi,\eta,\varepsilon,\delta,\varphi) + \alpha f^{k+1}_{\xi}(\sigma,\xi,\eta,\varepsilon,\delta,\varphi) \emptyset]} < \varphi(z), \quad (z \in D).
\]

\( \therefore \) if \( \emptyset \in P \) then \( \text{Re} \left[ \frac{1}{\alpha} \left( 2 + \frac{1}{\alpha} \right) + q(1 - \varphi(z)) \right] > 0. \)

Where it is obviously for all,

\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right) \]

\( \square > 0, \delta > 0, \varphi \geq 0, \delta > \varphi, z \in D. \]

**Proof** For \( j \in \{0,1,2,\ldots, n-1\} \) we obtained

\[
f^{k}_{qn}[\sigma, \xi, \eta, \varepsilon, \delta, \varphi; z] = \frac{1}{n} \sum_{j=0}^{n-1} \epsilon^{jq}_{n} \left[\psi_{\xi}^{k}_{q}f\right]\left(\epsilon^{j}_{n}z\right).
\]

\( \therefore \)

\[
f^{k}_{qn}[\sigma, \xi, \eta, \varepsilon, \delta, \varphi; \epsilon^{j}_{n}z] = \frac{1}{n} \sum_{m=0}^{m-1} \epsilon^{mj}_{n} \left[\psi_{\xi}^{k}_{q}f\right]\left(\epsilon^{m+j}_{n}z\right)
\]

\[
= \epsilon^{jq}_{n} \frac{1}{n} \sum_{m=0}^{m-1} \epsilon^{mj}_{n} \left[\psi_{\xi}^{k}_{q}f\right]\left(\epsilon^{m+j}_{n}z\right) = \epsilon^{jq}_{n} f^{k}_{qn} [\sigma, \xi, \eta, \varepsilon, \delta, \varphi; z] \quad (3.7.7)
\]

And

\[
\left[ f^{k}_{qn}[\sigma, \xi, \eta, \varepsilon, \delta, \varphi; z] \right]' = \frac{1}{n} \sum_{m=0}^{m-1} \epsilon^{jq}_{n} f^{k+1}_{qn} [\sigma, \xi, \eta, \varepsilon, \delta, \varphi; f] \left(\epsilon^{j}_{n}z\right). \quad (3.7.8)
\]

Replacing \( k \) by \( k + 1 \) in (3.7.7) we have obtained

\[
f^{k+1}_{qn} [\sigma, \xi, \eta, \varepsilon, \delta, \varphi; \epsilon^{j}_{n}z] = \epsilon^{jq}_{n} f^{k+1}_{qn} [\sigma, \xi, \eta, \varepsilon, \delta, \varphi; z] \quad (3.7.9)
\]

Replacing \( k \) by \( k + 1 \) in (3.7.8) we have obtained

\[
\left[ f^{k+1}_{qn} [\sigma, \xi, \eta, \varepsilon, \delta, \varphi; z] \right]' = \frac{1}{n} \sum_{m=0}^{m-1} \epsilon^{jq}_{n} f^{k+1}_{qn+1} [\sigma, \xi, \eta, \varepsilon, \delta, \varphi; f] \left(\epsilon^{j}_{n}z\right). \quad (3.7.10)
\]
From (3.7.7) and (3.7.10) we have obtained

\[
- \frac{z(1+\infty)(f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)f)'(z) + \alpha(f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)f)'(z)}{q(1+\infty)f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z) + \alpha f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)}
\]

\[
= -\frac{1}{n} \sum_{j=0}^{n-1} \frac{e_{j}^{(q+1)}z(1+\infty)(\psi_{q}^{k}f)'(\epsilon_{j}^{(q)}z) + \alpha(\psi_{q}^{k+1}f)'(\epsilon_{j}^{(q)}z)}{q(1+\infty)f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z) + \alpha f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)}
\]

\[
= -\frac{1}{n} \sum_{j=0}^{n-1} \frac{e_{j}^{(q)}z(1+\infty)(\psi_{q}^{k}f)'(\epsilon_{j}^{(q)}z) + \alpha(\psi_{q}^{k+1}f)'(\epsilon_{j}^{(q)}z)}{q(1+\infty)f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z) + \alpha f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)}
\]

It is clear that

\[
- \frac{e_{j}^{(q)}z(1+\infty)(\psi_{q}^{k}f)'(\epsilon_{j}^{(q)}z) + \alpha(\psi_{q}^{k+1}f)'(\epsilon_{j}^{(q)}z)}{q(1+\infty)f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z) + \alpha f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)} < 0(z).
\]

Considering that \(0(z)\) is univalent (or schlicht) & convex in disk \(D\), we get lemma 3.7.5. From (3.7.5) & (3.7.6) we have obtained

\[
\frac{z(f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)f)' + (q + \frac{1}{3})f_{q,n}^{k}(\sigma,\xi,\eta,\varepsilon,\delta,\varpi; z)}{f_{q,n}^{k}(\sigma,\xi,\eta,\varepsilon,\delta,\varpi; z)}
\]

\[
= \frac{1}{3k} \sum_{j=0}^{n-1} e_{j}^{(q)} [\psi_{q}^{k+1}(f)'(\epsilon_{j}^{(q)}z) = \frac{1}{3k} f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z) \quad (\forall f \text{ in } S).
\]

Let \(f \in f_{q,n}^{k}(\alpha; \sigma,\eta,\xi,\varepsilon,\delta,\varpi; \emptyset)\) and suppose

\[
\Omega(z) = -\frac{z[f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)]'}{f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)}
\]

I. e. here after it is to be taken as \(\Omega(z)\) is Holomorphic (an analytic) in unit disk \(D\) and \(\Omega(0) = 1\)

\[
\therefore q + \frac{1}{3} - q\Omega(z) = \frac{1}{3} f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)
\]

\[
\frac{z[f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)]'}{f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)}
\]

\[
= -q\left[z\Omega'(z) + q + \frac{1}{3} - q\Omega(z)\right] f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z), \quad (\forall z \text{ in } D')
\]

From above relations we obtained

\[
- \frac{z[(1+\infty)(f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)f)'(z) + \alpha(f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)f)'(z)]}{q(1+\infty)f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z) + \alpha f_{q,n}^{k+1}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi; z)}
\]

\[
= \frac{q(1+\infty)f_{q,n}^{k}(\sigma,\xi,\eta,\varepsilon,\delta,\varpi; z) + \alpha q[z + \frac{1}{3} - q\Omega(z)]f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)}{q(1+\infty)f_{q,n}^{k}(\sigma,\xi,\eta,\varepsilon,\delta,\varpi; z) + \alpha q[z + \frac{1}{3} - q\Omega(z)]f_{q,n}^{k}(\sigma,\eta,\xi,\varepsilon,\delta,\varpi)}
\]
\[
\begin{align*}
\Omega(z) &= \frac{(1+\alpha)\Omega(z) + \alpha\left\{z\Omega'(z) + \left[q + \frac{1}{3} - p\Omega(z)\right]\Omega(z)\right\}}{(1+\alpha) + \alpha\left[q + \frac{1}{3} - p\Omega(z)\right]} \\
&= \frac{\alpha\left\{z\Omega'(z) + \left[(1+\alpha) + \alpha\left[q + \frac{1}{3} - p\Omega(z)\right]\right]\Omega(z)\right\}}{(1+\alpha) + \alpha\left[q + \frac{1}{3} - p\Omega(z)\right]} \\
&= \Omega(z) + \frac{z\Omega'(z)}{\frac{1}{3} + \frac{1}{3} - p - p\Omega(z)} < \varphi(z), \quad (z \in D).
\end{align*}
\]

\[
\therefore \quad \text{Re} \left(\frac{\frac{1}{3}(2 + \frac{1}{a}) \cdot q[1 - \varphi(z)]\right) > 0
\]

and by Lemma 3.7.3 we get

\[
\Omega(z) = -\frac{z\left(f_{q,n}^k(\sigma,\eta,\xi,\epsilon,\delta,\nu; z)\right)'}{qf_{q,n}^k(\sigma,\eta,\xi,\epsilon,\delta,\nu; z)} < \varphi(z), \quad (\forall \ z \text{ in } D).
\]

**Lemma 3.7.6** Let \( f \in f_{q,n}^k(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \) i.e. here after it is to be taken as \((\forall \ z \text{ in } D)\).

\[
\frac{z\left[(1+\alpha)(g_{q}^k(\sigma,\eta,\xi,\epsilon,\delta,\nu) f)'(z) + \alpha(g_{q}^{k+1}(\sigma,\eta,\xi,\epsilon,\delta,\nu) f)'(z)\right]}{q\left[(1+\alpha)g_{q}^{k}(\sigma,\eta,\xi,\epsilon,\delta,\nu; z) + \alpha g_{q}^{k+1}(\sigma,\eta,\xi,\epsilon,\delta,\nu; z)\right]} < \varphi(z).
\]

Therefore,

if \( \emptyset \in P, \text{Re} \left[\frac{\frac{1}{3}(2 + \frac{1}{a}) \cdot q(1 - \varphi(z))]\right] > 0 \)

And \( \left(0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, \right)\)

\( a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \).

I.e. here after it is to be taken as, \( \forall (z \in D). \)

\[
\frac{-z\left(g_{q}^k(\sigma,\eta,\xi,\epsilon,\delta,\nu; z)\right)'}{qg_{q}^{k}(\sigma,\eta,\xi,\epsilon,\delta,\nu; z)} < \varphi(z)
\]

**Lemma 3.7.7** Let \( f \in f_{q,n}^k(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \) i.e. here after it is to be taken as

\[
\frac{z\left[(1+\alpha)(h_{q}^k(\sigma,\eta,\xi,\epsilon,\delta,\nu) f)'(z) + \alpha(h_{q}^{k+1}(\sigma,\eta,\xi,\epsilon,\delta,\nu) f)'(z)\right]}{q\left[(1+\alpha)h_{q}^{k}(\sigma,\eta,\xi,\epsilon,\delta,\nu; z) + \alpha h_{q}^{k+1}(\sigma,\eta,\xi,\epsilon,\delta,\nu; z)\right]} < \varphi(z).
\]

Therefore, if \( \emptyset \in P \) with

\[
\text{Re} \left[\frac{\frac{1}{3}(2 + \frac{1}{a}) \cdot q(1 - \varphi(z))]\right] > 0.
\]

Where it is obviously for all,
\[
(0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \\
\alpha > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D).
\]

\[
\frac{z(n^k_h(\sigma, \eta, \xi, \varepsilon, \alpha, \delta; z))^t}{p_h^k(\sigma, \eta, \xi, \varepsilon, \alpha, \delta; z)^t} < \phi(z) \quad (z \in D).
\]

**Theorem 3.7.1** Let \(0 < a \leq 1\) and \(0 < b < 1\). If \(g(z) \in S \Rightarrow g(z) \neq 0, \forall z \in D^* \) &

\[
\left| \frac{1}{z^q f(z)} \left( \frac{zf'(z)}{f(z)} \right) + q \right| < \delta \quad (z \in D)
\] (3.7.11)

Where it is obviously for all \(\delta\) is minimum positive root of the equation

\[
\frac{a}{2} \sin \left( \pi \frac{b}{2} \right) x^2 - x + \left( 1 - \frac{a}{2} \right) \sin \left( \pi \frac{b}{2} \right) = 0
\] (3.7.12)

I. e. here after it is to be taken as

\[
\left| \arg \left( z^a f(z) - \frac{a^2}{2} \right) \right| < \pi \frac{b}{2} \quad (z \in D)
\] (3.713)

The bound \(b\) is the best possible for each \(0 < a \leq 1\)

**Proof** Let

\[
g(x) = \frac{a}{2} \sin \left( \pi \frac{b}{2} \right) x^2 - x + \left( 1 - \frac{a}{2} \right) \sin \left( \pi \frac{b}{2} \right)
\] (3.7.14)

There exist two roots for the Equation given below

\[
\frac{a}{2} \sin \left( \pi \frac{b}{2} \right) x^2 - x + \left( 1 - \frac{a}{2} \right) \sin \left( \pi \frac{b}{2} \right) = 0
\]

\[
\therefore g(0) = \frac{a}{2} \sin \left( \pi \frac{b}{2} \right) \cdot (0)^2 - 0 + \left( 1 - \frac{a}{2} \right) \sin \left( \pi \frac{b}{2} \right) = \left( 1 - \frac{a}{2} \right) \sin \left( \pi \frac{b}{2} \right) > 0
\]

\[
g(1) = \frac{a}{2} \sin \left( \pi \frac{b}{2} \right) \cdot (1)^2 - 1 + \left( 1 - \frac{a}{2} \right) \sin \left( \pi \frac{b}{2} \right) = \frac{a}{2} \sin \left( \pi \frac{b}{2} \right) + \frac{a}{2} \sin \left( \pi \frac{b}{2} \right) = 2 \frac{a}{2} \sin \left( \pi \frac{b}{2} \right) < 0
\]

Hence we have obtained

\[
0 < \frac{a}{2 - \alpha} \delta \leq \delta < 1
\] (3.7.15)

Put

\[
z^p f(z) = \frac{a}{2} + \left( 1 - \frac{a}{2} \right) q(z)
\] (3.7.16)

I. e. here after it is to be taken as \(q(z)\) Holomorphic in the unit disk \(D\) where
\( q(0) = 1 \& \frac{a}{2} + \left(1 - \frac{a}{2}\right)q(z) \neq 0 \) for all \((z \in D)\).

Taking the logarithmic differentiation on both sides of
\[
z^q f(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right)q(z),
\]
We get
\[
\frac{zf'(z)}{f(z)} + q = \frac{(2-a)zq'(z)}{a+(2-a)q(z)}.
\]
(3.7.17)
\[
z^q f(z) \left[ \frac{zf'(z)}{f(z)} + q \right] = \frac{(2-a)zq'(z)}{[a+(2-a)q(z)]^2} \forall (z \in D).
\]
(3.7.18)

Thus the inequality
\[
\left| \frac{1}{z^q f(z)} \left( \frac{zf'(z)}{f(z)} + q \right) \right| < \delta \quad (\forall z \text{ in } D).
\]
It is equivalent to
\[
\frac{(2-a)zq'(z)}{[a+(2-a)q(z)]^2} < \delta z.
\]
(3.7.19)

By applying Lemma 3.7.1, above inequality gives the following
\[
\int_0^z \frac{(2-a)q'(t)}{[a+(2-a)q(t)]^2} dt < \delta z.
\]
Or
\[
1 - \frac{2}{a+(2-a)q(z)} < \delta z.
\]
(3.7.20)

From above eqn
\[
q(z) < \frac{1+\frac{a}{2-a}\delta z}{1-\delta z}.
\]
(3.7.21)

Now by taking \(\alpha = \frac{a}{2-a}\), \(\beta \) and \(\gamma = -\frac{\delta}{z}\) in (1.2), we get
\[
\left| \arg \left( z^q f(z) - \frac{a}{2} \right) \right| = \left| \arg q(z) \right| < \arcsin \left( \frac{2\delta}{2-\alpha + \alpha \delta z} \right) = \frac{\pi b}{2} \forall (z \in D).
\]

Because of \(g(\delta) = 0\), This proves the statement. Next, we are assuming that \(f\) as defined as follows
\[
f(z) = \frac{z^{-a}}{1-\delta z} \quad (\forall z \text{ in } D^*).
\]

It is simple to prove that
\[
\left| z^q f(z) \left[ \frac{zf'(z)}{f(z)} + q \right] \right| = |\delta z| < \delta \quad (\forall z \text{ in } D).
\]

\[
\therefore \quad z^q f(z) = \frac{a}{2} = \frac{1+\frac{a}{2-a}\delta z}{1-\delta z}.
\]

Hence from the result (3.7.3)
\[
\sup_{z \in D} \left| \arg \left( z^a f(z) - \frac{a}{z} \right) \right| = \arcsin \left( \frac{2\delta}{2 - a + a^2} \right) = \frac{\pi b}{2}.
\]

Hence, finally the conclusion is that the bound \( b \) is better \( \forall \ a \in (0,1] \). Next, we derive the following.

**Theorem 3.7.2** If function \( g \in S \Rightarrow g(z) \neq 0, (\forall z \in D') \) &

\[
\text{Re} \left[ z^a g(z) \left( \frac{zg'(z)}{g(z)} + t \right) \right] < \varepsilon \quad (\forall z \in D). \tag{3.7.22}
\]

\[
0 < \varepsilon < \frac{1}{2\log 2}. \tag{3.7.23}
\]

\[
\therefore \frac{1}{z^a g(z)} > 1 - 2\varepsilon \cdot \log 2 \quad (z \in D). \tag{3.7.24}
\]

The above inequality holds good.

**Proof** Let

\[
t(z) = z^a g(z) \tag{3.7.25}
\]

I.e. here after it is to be taken as \( t(z) \) Holomorphic in \( D \) where \( t(0) = 1 \) and \( t(z) \neq 0, \forall z \in D \).

In accordance with (3.7.17) and (3.7.10), we have obtained

\[
1 - zt'(z) < \frac{1 + z}{1 - z}
\]

Let

\[
z \left[ \frac{1}{t(z)} \right]' < \frac{2\varepsilon z}{1 - z} \tag{3.7.26}
\]

Now by the lemma 3.7.1 we get the following subordination

\[
\frac{1}{t(z)} < 1 - 2\varepsilon \cdot \log (1 - z).
\]

\[
\therefore \text{the function } 1 - 2\varepsilon \cdot \log (1 - z) \text{ is multivalent convex in } D \text{ and}
\]

\[
\text{Re} \left[ 1 - 2\varepsilon \cdot \log (1 - z) \right] > 1 - 2\varepsilon \cdot \log 2 \quad (z \in D).
\]

From

\[
z \left[ \frac{1}{q(z)} \right]' < \frac{2\varepsilon z}{1 - z}
\]

We obtained the inequality defined below

\[
\text{Re} \left( \frac{1}{z^a g(z)} \right) > 1 - 2\varepsilon \cdot \log 2.
\]

To show that the bound
\[
\text{Rep } \frac{1}{z^n f(z)} > 1 - 2e \cdot \log 2 \quad (z \in D),
\]

Cannot be increased, we consider
\[
g(z) = \frac{1}{z^n [1 - 2e \cdot \log (1 - z)]} \quad (z \in D^*).
\]

We can easily check and verify that the function \( g(z) \) satisfying the following
\[
\text{Rep } \left[ z^n g(z) \left( \frac{z^n g'(z)}{g(z)} + t \right) \right] < \varepsilon \quad (\forall \ z \in D).
\]
\[\therefore \ \text{Rep } z^n g(z) \rightarrow 1 - 2e \cdot \log 2 \text{ as } z \rightarrow -1.\]

Hence the Theorem holds good.

**Theorem 3.7.3** Let \( g(z) \in S \Rightarrow g(z) \neq 0, \ (z \in D^*). \)
\[
\text{If } \left| \text{Im} \left( \frac{z^n g(z)}{g(z)} \right) \right| < \sqrt{\tau (\tau + \tau t)} \quad (3.7.27)
\]
\( (\forall \ z \in D) \) & \( \tau > 0 \). I.e. here after it is to be taken as
\[
\text{Rep } z^n g(z) > 0 \quad (3.7.28)
\]

**Proof** Let \( q(z) \) be defined as
\[
\text{Rep } z^n g(z) = q(z) \implies t(0) = 1, t(z) \neq 0,
\]
And
\[
\frac{z^n g'(z)}{g(z)} \left( z^n g(z) - \tau \right) = [t(z) - \tau] \cdot \left[ \frac{z^n g'(z)}{t(z)} - t \right] \quad (3.7.29)
\]
\[
\text{Rep } t(z) > 0, \ |z| < |z_0| \text{ & } t(z_0) = ib \quad (3.7.30)
\]
\( (\forall \ z \in D) \), Where \( b \) is real and \( b \neq 0 \), using lemma 3.7.2 we obtained
\[
z_0 \cdot t'(z_0) \leq \frac{-(1 - b^2)}{2}. \quad (3.7.31)
\]
Thus it follows from above obtained results that
\[
J_0 = \text{Im} \left[ \frac{z^n g'(z_0)}{g(z_0)} \left( z^n g(z) - \tau \right) \right] = -tb + \frac{\tau}{b} z_0 \cdot t'(z_0). \quad (3.7.32)
\]

In accordance with \( \tau > 0 \). From (3.7.1) & (3.7.2) we get
\[
J_0 \geq \frac{-(\tau + \tau^2 + \tau^2 b^2)}{2b} \geq \sqrt{\tau (\tau + 2t)} \quad (b > 0). \quad (3.7.33)
\]
\[
J_0 \leq \frac{[\tau + \tau^2 + \tau^2 b^2]}{2b} \leq -\sqrt{\tau (\tau + 2t)} \quad (b > 0) \quad (3.7.34)
\]
\[\therefore \ \left| \text{Im} \left( \frac{z^n g'(z)}{g(z)} \left( z^n g(z) - \tau \right) \right) \right| < \sqrt{\tau (\tau + \tau t)} \quad (\forall \ z \in D) \text{ and } \tau > 0.
\]

Therefore we have,
Re $q(z) > 0$ for all $z \in \mathbb{D}$ $\Rightarrow$ Rep $z^q g(z) > 0$, $(\forall z \in D)$.

Theorem holds true.

### 3.8 Inclusion Relationships

**Theorem**

3.8.1 Let $\varphi \in P$ with

$$\text{Rep} \left[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + q \left( 1 - \varphi(z) \right) \right] > 0.$$ 

Where,

$$\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, \right)$$

$$a > 0, \delta > 0, \nu \geq 0, \delta > \nu, (\forall z \in \mathbb{D}).$$

I. e. here after it is to be taken as

$$F^k_{q,n}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \subset F^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi).$$

**Proof** Let $f \in F^k_{q,n}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \square)$ and

$$t(z) = -\frac{z(\psi^k_q(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi)f')'}{\psi^k_q(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)} (\forall z \in \mathbb{D}).$$

$\therefore$ $t(z)$ is Holomorphic (an analytic) in unit disk $\mathbb{D}$ and $t(0) = 1$ hence

$$t(z)f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z) = -\frac{1}{3q} \psi^{k+1}_q f(z) + \frac{1 + zq}{3q} \psi^k_q f(z)$$

Differentiating both sides we get

$$zt'(z) + \left( \frac{1}{3} + t + \frac{z(f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z))'}{f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)} \right) t(z) = -\frac{1}{3q}, \frac{z(\psi^{k+1}_q f(z))'}{\psi^k_q(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)}.$$  

$$\therefore \frac{-z[(1+\alpha)(\psi^k_q f)^'(z) + \alpha (\psi^{k+1}_q f)^'(z)]}{q[(1+\alpha)f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z) + \alpha f^{m+1}_{p,k}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)]} = \frac{q(1+\alpha)q(z)f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)}{q(1+\alpha)f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z) + \alpha 3q + \frac{1}{3} - q\Omega(z) f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)}$$

$$+ \frac{\alpha q [zq'(z) + q + \frac{1}{3} - q\Omega(z)] q(z)f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)}{q(1+\alpha)f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z) + \alpha 3q + \frac{1}{3} - q\Omega(z) f^k_{q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; z)}.$$
\[
\frac{(1+\infty)q(z) + \alpha \gamma \left[ zq'(z) + \left[ q + \frac{1}{3} - \varphi z \right] q(z) \right]}{(1+\infty) + \alpha \gamma \left[ q + \frac{1}{3} - \varphi z \right]}
\]
\[
\frac{\alpha \gamma zq'(z) + \left[ (1+\infty) + \alpha \gamma \left[ q + \frac{1}{3} - \varphi z \right] \right] \cdot q(z)}{(1+\infty) + \alpha \gamma \left[ q + \frac{1}{3} - \varphi z \right]}
\]
\[
= t(z) + \frac{zq(z)}{3a} < \varphi(z) \quad (\forall z \in D).
\]
\[
\therefore \text{Rep} \left( \frac{1}{3a} + \frac{2}{3} + q - \varphi z \right) > 0
\]

and by lemma 3.2.3 we have obtained
\[
\Omega(z) = -\frac{z\left( f_{q,n}^{k}(\sigma, \eta, \xi, \varepsilon, \delta, \varphi; z) \right)}{q_{k}^{\gamma}(\sigma, \eta, \xi, \varepsilon, \delta, \varphi; z)} < \varphi(z) \quad (\forall z \in D).
\]

By lemma 3.2.2 we have got \( q(z) < \varphi(z) \) \quad (\forall z \in D).
\[
\therefore f \in F_{q,n}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \varphi) \Rightarrow F_{q,n}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \varphi) \subset F_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \varphi).
\]

**Corollary 3.8.1** Let \( \varphi \in P \) with
\[
\text{Rep} \left[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0.
\]

Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right.
\left. a > 0, \delta > 0, \varphi \geq 0, \delta > \varphi, z \in D \right).
\]
I. e. here after it is to be taken as \( G_{q}^{k}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \varphi) \subset G_{q}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \varphi). \)

**Corollary 3.8.2** Let \( \varphi \in P \) with
\[
\text{Rep} \left[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0.
\]

Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right.
\left. a > 0, \delta > 0, \varphi \geq 0, \delta > \varphi, z \in D \right).
\]
I. e. here after it is to be taken as
\[ H^m_p(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \subset H^m_p(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \]

**Theorem 3.8.2** Let \( \varphi \in P \) with
\[ \text{Re} \left[ \frac{1}{3} \left( 2 \cdot \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0. \]

Where it is obviously for all,
\[ \left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, \right) \]
\[ a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D. \]

I. e. here after it is to be taken as
\[ \mathcal{K}^k_{\alpha,n}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \subset \mathcal{K}^k_{\alpha,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi). \]

**Proof** Let \( f \in \mathcal{K}^k_{\alpha,n}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \) and consider
\[ t(z) = \frac{z(\ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z))'}{q \ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z)} \quad (z \in D). \]

Thus \( t(z) \) is Holomorphic (an analytic) in unit disk \( D \) and \( t(0) = 1 \).
\[ \therefore t(z) = \ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z) \]
\[ = \frac{-1}{3q} l^{k+1}_q(\alpha, \sigma, \xi, \eta, \epsilon, \delta, \nu; f(z)) + \frac{(1+3q)}{3q} l^k_q(\alpha, \sigma, \xi, \eta, \epsilon, \delta, \nu; f(z)) \]

Differentiating both sides we get
\[ zt'(z) + \left( \frac{1}{3} + q + \frac{z(\ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z))'}{\ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z)} \right) q(z) = \frac{-1}{3q} \cdot \frac{z(\psi^{k+1}_q(f(z))'}{\ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z)} \]
\[ \varphi(z) = \frac{-z(\ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z))'}{q \ell^k_{\alpha,n}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z)}, \quad (z \in D). \]
\[ \therefore q(z) < \varphi(z) \quad (z \in D). \]
\[ \Rightarrow f \in \mathcal{K}^k_{\alpha,n}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi). \]

**Corollary 3.8.3** Let \( \varphi \) contained in \( P \) with
\[ \text{Re} \left[ \frac{1}{3} \left( 2 \cdot \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0. \]

Where it is obviously for all,
Corollary 3.8.4 Let $\varphi \in P$ with

$$\text{Re} \left[ \frac{1}{\sqrt{3}} \left( 2 + \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0.$$ 

Where it is obvious for all,

$$\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right)$$

$$\left( a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \right).$$

I. e. here after it is to be taken as

$$\mathcal{X}_q^k(\infty; \sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset \mathcal{X}_q^k(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi).$$

Corollary 3.8.5 Let $\varphi \in P$ with

$$\text{Re} \left[ \frac{1}{\sqrt{3}} \left( 2 + \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0.$$ 

Where it is obviously for all,

$$\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right)$$

$$\left( a > 0, \delta > 0, \nu \geq 0, \delta > \nu, \forall z \in D \right).$$

I. e. here after it is to be taken as

$$\mathcal{F}_{q,n}^{k+1}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset \mathcal{F}_{q,n}^{k+1}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi).$$

Corollary 3.8.6 Let $\varphi \in P$ with

$$\text{Re} \left[ \frac{1}{\sqrt{3}} \left( 2 + \frac{1}{a} \right) + q(1 - \varphi(z)) \right] > 0.$$ 

Where it is obviously for all,

$$\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right)$$

$$\left( a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \right).$$

I. e. here after it is to be taken as

$$\mathcal{G}_q^{k+1}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset \mathcal{G}_q^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi).$$
Corollary 3.8.7 Let $\varphi \in P$ with
\[
\text{Re} \left[ \frac{1}{2} \left( 2 + \frac{1}{z} \right) + q(1 - \varphi(z)) \right] > 0.
\]
Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right.
\]
\[
\left. a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \right).
\]
I. e. here after it is to be taken as
\[
H^{k+1}_q(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset H^k_q(\sigma, \xi, \eta, \delta, \nu; \varphi).
\]

Corollary 3.8.8 Let $\varphi \in P$ with
\[
\text{Re} \left[ \frac{1}{2} \left( 2 + \frac{1}{z} \right) + q(1 - \varphi(z)) \right] > 0.
\]
Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right.
\]
\[
\left. a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \right).
\]
I. e. here after it is to be taken as
\[
\mathcal{K}^{k+1}_q(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset \mathcal{K}^k_q(\sigma, \xi, \eta, \delta, \nu; \varphi).
\]

Corollary 3.8.9 Let $\varphi \in P$ with
\[
\text{Re} \left[ \frac{1}{2} \left( 2 + \frac{1}{z} \right) + q(1 - \varphi(z)) \right] > 0.
\]
Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, \right.
\]
\[
\left. a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D \right).
\]
I. e. here after it is to be taken as
\[
\mathcal{Q}^{k+1}_q(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset \mathcal{Q}^k_q(\sigma, \xi, \eta, \delta, \nu; \varphi).
\]

Corollary 3.8.10 Let $\varphi \in P$ with
\[
\text{Re} \left[ \frac{1}{2} \left( 2 + \frac{1}{z} \right) + q(1 - \varphi(z)) \right] > 0.
\]
Where it is obviously for all,
\[
\left( 0 < \sigma \leq \frac{1}{z}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{z}, \right)
\]
\[
(a > 0, \delta > 0, \nu \geq 0, \delta > \nu, z \in D) .
\]

I. e. here after it is to be taken as
\[
\mathfrak{X}_{q}^{k+1}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \subset \mathfrak{X}_{q}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi).
\]

3.9 Integral Representation

In this part of the chapter we want to prove integral representations with the classes
\[
F_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi), Q_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \text{ & } H_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi).
\]

**Theorem 3.9.1** Let \( f \in F_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \) i. e. here after it is to be taken as
\[
F_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; z) = z^{-q} \exp \left( -\frac{q}{n} \sum_{j=0}^{n-1} \int_{0}^{\infty} \varphi^{w(\epsilon_{j}z)^{-1}} d\xi \right). \]
Where,
\[
f_{q,n}^{k} [\sigma, \xi, \eta, \square, \delta, \nu; z] = \frac{1}{n} \sum_{j=0}^{n-1} \epsilon_{n}^{j} \left[ \Omega_{q}^{k}(\sigma, \eta, \xi, \varepsilon, \delta, \nu; f)(\epsilon_{n}^{j}z) \right]
\]
\[
= \frac{1}{z^{q}} + \ldots \quad (f \in S).
\]

Where \( w(z) \) which is Holomorphic (an analytic) in unit disk \( D \) with
at \( z = 0 \) is 0 & \( mod \ w(z) < 1, (\forall z \in D) \).

**Proof** Let us consider \( f \in F_{q,n}^{k}(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \).
\[
\frac{-z(\psi_{q}^{k} f)'(\epsilon_{n}^{j}z)}{q f_{q,n}^{k}(\sigma, \eta, \xi, \varepsilon, \delta, \nu; z)} = \varphi[w(\epsilon_{n}^{j}z)], (\forall z \in D).
\]

Here \( w(z) \) is Holomorphic in \( D \) with \( w \) at \( z = 0 \) is 0 and \( mod \ w(z) \) less than 1. Replacing
\( z \) by \( \epsilon_{n}^{j}z \) \( (j = 0,1,2,\ldots) \) above Equation holds.
\[
\frac{-\epsilon_{n}^{j}z(\psi_{q}^{k} f)'(\epsilon_{n}^{j}z)}{q f_{q,n}^{k}(\sigma, \eta, \xi, \varepsilon, \delta, \nu; \epsilon_{n}^{j}z)} = \varphi[w(\epsilon_{n}^{j}z)], (\forall z \in D).
\]

We note that,
\[
f_{q,n}^{k}(\sigma, \eta, \xi, \varepsilon, \delta, \nu; \epsilon_{n}^{j}z) = \epsilon_{n}^{-j} f_{q,n}^{k}(\sigma, \eta, \xi, \varepsilon, \delta, \nu; z).
\]

considering \( (j = 0,1,2,\ldots) \), successively summing the resultant equations we have obtained
\[-z\frac{f_k^q(\sigma, \eta, \xi, \epsilon, \delta, \sigma; z)^'}{q f_k^q(\sigma, \eta, \xi, \epsilon, \delta, \sigma; z)} = \frac{1}{n}\sum_{j=0}^{n-1} \varphi(\epsilon_j^q z) \quad (\forall \ z \text{ in } D).\]

\[-z\frac{f_k^q(\sigma, \eta, \xi, \epsilon, \delta, \sigma; z)^'}{f_k^q(\sigma, \eta, \xi, \epsilon, \delta, \sigma; z)} + \frac{q}{z} = \frac{-q}{n}\sum_{j=0}^{k-1} \frac{\varphi(\epsilon_j^q z)^{1-1}}{z} \quad (\forall \ z \text{ in } D^*).\]

Upon integration which yields,

\[\log(z^q f_k^q(\sigma, \eta, \xi, \epsilon, \delta, \sigma; z)) = -\frac{q}{n}\sum_{j=0}^{n-1} \int_0^z \frac{\varphi(\epsilon_j^q z)^{1-1}}{\zeta} d\zeta.\]

Taking exponential theorem holds true.

**Theorem 3.9.2** Let \( f \in F_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \) i. e. here after it is to be taken as

\[\psi_{k,f}(z) = -q \int_0^z \varsigma^{-q-1} \varphi(\zeta) \exp \left(-\frac{q}{n}\sum_{j=0}^{n-1} \int_0^\varsigma \frac{\varphi(\epsilon_j^q \zeta)^{1-1}}{\zeta} d\varsigma \right) d\varsigma,\]

Where \( w(z) \) is Holomorphic (an analytic) in unit disk \( D \) with \( w \) at \( z = 0 \) is \( 0 \) & mod \( w(z) < 1 \), \( (\forall \ z \text{ in } D) \).

**Proof** Let us assume \( f \in F_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \)

\[\psi_k^q f(z) = -q \int_0^z \varsigma^{-q-1} \varphi(\zeta) \exp \left(-\frac{q}{n}\sum_{j=0}^{n-1} \int_0^\varsigma \frac{\varphi(\epsilon_j^q \zeta)^{1-1}}{\zeta} d\varsigma \right) d\varsigma,\]

Integrating above Equation, theorem holds true.

**Theorem 3.9.3** Let \( g \in F_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \) i. e. here after it is to be taken as

\[\psi_k^q g(z) = -q \int_0^z \varsigma^{-q-1} \varphi(\zeta) \exp \left(-\frac{q}{n}\sum_{j=0}^{n-1} \int_0^\varsigma \frac{\varphi(\epsilon_j^q \zeta)^{1-1}}{\zeta} d\varsigma \right) d\varsigma,\]

Where \( w_j(z) \) is Holomorphic (an analytic) in unit disk \( D \) with \( w_j(0) = 0 \) & \( |w_j(z)| < 1 \).

\((z \in D, j = 1,2).\)

**Proof** Let us assume that a function \( g \in F_{q,n}^k(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \).

I. e. here after it is to be taken as,
Here \( w_1(z) \) is Holomorphic (an analytic) in the unit disk \( D \) with \( w_1(0) = 0 \) \& \(|w_1(z)| < 1\). (\( z \in D \)).

Thus by using the procedure similar to the thm 2.2.3

\[
g_{n,\alpha}^k (\sigma, \eta, \xi, \epsilon, \delta, \nu; z) = z^{-q} \exp \left( -q \int_0^z \frac{\phi[w_1(\zeta)]}{\zeta} d\zeta \right).
\]

From above Equations we get

\[
\left( \psi_{\alpha}^k g \right) (z) = \frac{-q g_{n,\alpha}^k (\sigma, \eta, \xi, \epsilon, \delta, \nu; z)}{z} \cdot \phi[w_2(z)]
\]

\[
= -q z^{-q-1} \cdot \phi[w_2(z)] \cdot \exp \left( -q \int_0^z \frac{\phi[w_1(\zeta)]}{\zeta} d\zeta \right)
\]

Where \( w_j(z) \) is Holomorphic (an analytic) in unit disk \( D \) with \( w_j(0) = 0 \) \& \(|w_j(z)| < 1\), (\( \Box \ in D \)),

\( (j = 1,2) \).

**Corollary 3.9.1** Let \( f \in C^k(\sigma, \eta, \xi, \epsilon, \delta, \nu; \phi) \) i.e. here after it is to be taken as

\[
g_{n,\alpha}^k (\sigma, \eta, \xi, \epsilon, \delta, \nu; z) = z^{-q} \exp \left( -q \int_0^z \frac{\phi[w(\zeta)] + \phi[w(\zeta)] - 2}{\zeta} d\zeta \right)
\]

Where,

\[
g_{n,\alpha}^k (\sigma, \eta, \xi, \epsilon, \delta, \nu; z) = \frac{1}{2} \left[ \psi_{\alpha}^k (\sigma, \eta, \xi, \epsilon, \delta, \nu; f(z)) + \bar{\psi_{\alpha}^k (\bar{f}(z))} \right]
\]

\[
= \frac{1}{z^q} + \ldots
\]

Here \( w(z) \) is Holomorphic (an analytic) in the unit disk \( D \) with \( w \) at \( z = 0 \) is \( 0 \) \& \( \mod w(z) < 1 \), (\( \forall z \ in D \)).

**Corollary 3.9.2** Let \( f \in C^k(\sigma, \eta, \xi, \epsilon, \delta, \nu; \phi) \) i.e. here after it is to be taken as

\[
\psi_{\alpha}^k f(z) = -q \int_0^z \zeta^{-q-1} \phi[w(\zeta)] \exp \left( -q \int_0^z \frac{\phi[w(\zeta)] + \phi[w(\zeta)] - 2}{\zeta} d\zeta \right) dz.
\]

Here \( w(z) \) is Holomorphic (an analytic) in unit disk \( D \) with \( w \) at \( z = 0 \) is \( 0 \) \& \( \mod w(z) < 1 \), (\( \forall z \ in D \)).

**Corollary 3.9.3** Let \( f \in \bar{H}_q^k(\sigma, \eta, \xi, \epsilon, \delta, \nu; \phi) \) i.e. here after it is to be taken as
\[ h^k_{Q}(\sigma, \eta, \xi, \epsilon, \delta, \nu; z) = z^{-q} \exp \left( \frac{-q}{z} \int_0^z \frac{\varphi[w(\zeta)] - \varphi[w(\zeta)] - 2}{\zeta} \, d\zeta \right). \]

Here over it is,
\[ h^k_{Q}[\sigma, \xi, \eta, \epsilon, \delta, \nu; z] = \frac{1}{z^q} \left[ \psi^k_{Q}(z) \right] + \overline{\psi^k_{Q}(z)} \]
\[ = \frac{1}{z^q} + \ldots \quad (\forall \, h \in \mathcal{S}). \]

Here \( w(z) \) is Holomorphic (an analytic) in the unit disk \( D \) with \( w \) at \( z = 0 \) is \( 0 \) & \( \text{mod} \, w(z) < 1, (\forall \, z \in D). \)

**Corollary 3.9.4** Let \( f \in H^k_{Q}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \) i.e. here after it is to be taken as
\[ \psi^k_{Q}(z) = -q \int_0^z \zeta^{-q-1} \varphi(w(\zeta)) \exp \left( \frac{-q}{z} \int_0^z \frac{\varphi[w(\zeta)] - \varphi[w(\zeta)] - 2}{\zeta} \, d\zeta \right) \, d\zeta. \]

Here \( w(z) \) is Holomorphic (an analytic) in unit disk \( D \) with \( w \) at \( z = 0 \) is \( 0 \) & \( \text{mod} \, w(z) < 1, (\forall \, z \in D). \)

### 3.10 Convolution Properties Using Differential Subordination

In this part we are going to derive and discuss various familiar convex properties for the functional classes \( D^k_{Q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \), \( G^k_{Q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \) and \( H^k_{Q,n}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \).

**Theorem 3.10.1** Let \( f \in F^k_{Q}(\sigma, \xi, \eta, \epsilon, \delta, \nu; \varphi) \)
\[ \therefore f(z) = \left[ -q \int_0^z \zeta^{-q-1} \varphi(w(\zeta)) \exp \left( \frac{-q}{n} \sum_{m=0}^{\infty} \int_0^z \frac{\varphi[w(\zeta)] - 1}{\zeta} \, d\zeta \right) \, d\zeta \right] \ast \sum_{m=0}^{\infty} \left[ \frac{1}{3m+1} \right]^k z^{m-q}. \]

Here we are assuming a function \( w(z) \) which is Holomorphic (an analytic) in the unit disk \( D \) with \( w \) at \( z = 0 \) is \( 0 \) & \( \text{mod} \, w(z) < 1, (\forall \, z \in D) \).

**Proof** Since we know a linear operator defined below
\[ \psi^k_{Q}(z) = \frac{1}{z^q} + \sum_{m=0}^{\infty} [3(q + m) + 1]^k a_{q+m-1} z^{q+m-1} = (\psi^k_{Q}(\sigma, \xi, \eta, \epsilon, \delta, \nu \ast f)(z). \]

Here over it is,
\[ \psi^k_{Q}(\sigma, \xi, \eta, \epsilon, \delta, \nu)(z) = \frac{1}{z^q} + \sum_{m=0}^{\infty} [3(q + m) + 1]^m z^{q+m-1}. \]
And
\[
\psi_q^k f(z) = -q \int_0^Z \zeta^{-q-1} \varphi[w(\zeta)] \exp \left( -\frac{q}{n} \sum_{i=0}^{n-1} \int_0^\zeta \frac{\varphi[w(e_i\zeta)]}{\zeta} \frac{d\zeta}{\zeta} \right) d\zeta
\]
\[= \left[ \sum_{n=0}^{\infty} \left[ \frac{1}{3n+1} \right]^k \zeta^{m-q} \right] * f(z).
\]

Hence the theorem.

\[
f(z) = \left[ -q \int_0^Z \zeta^{-q-1} \varphi[w(\zeta)] \exp \left( -\frac{q}{n} \sum_{i=0}^{n-1} \int_0^\zeta \frac{\varphi[w(e_i\zeta)]}{\zeta} \frac{d\zeta}{\zeta} \right) d\zeta \right] * \sum_{n=0}^{\infty} \left[ \frac{1}{3n+1} \right]^k \zeta^{m-q}.
\]

**Theorem 3.10.2** Let \( f \in F_q^k(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \).

I. e. here after it is to be taken as

\[
f(z) = \left[ -q \int_0^Z \zeta^{-q-1} \varphi[w_2(\zeta)] \exp \left( -\frac{q}{n} \sum_{i=0}^{n-1} \int_0^\zeta \frac{\varphi[w_1(\zeta)]}{\zeta} \frac{d\zeta}{\zeta} \right) d\zeta \right] * \sum_{n=0}^{\infty} \left[ \frac{1}{3n+1} \right]^k \zeta^{m-q}.
\]

Where \( w_i(z) \) is Holomorphic (an analytic) in unit disk \( D \) with \( w_i \) at \( z = 0 \) is \( 0 \) & mod \( w_i(z) < 1 \) (\( \forall z \) in \( D \), \( i = 1, 2 \)).

**Proof** Since we know a linear operator defined below,

\[
\psi_q^k f(z) = \frac{1}{z^q} + \sum_{m=0}^{\infty} \left[ \zeta(q + m) + 1 \right] a_{q+m-1} z^{q+m-1}
\]

\[= \left( \psi_{\sigma, \xi, \eta, \varepsilon, \delta, \nu}^q * f \right)(z).
\]

\[
\psi_{\sigma, \xi, \eta, \varepsilon, \delta, \nu}^q(z) = \frac{1}{z^q} + \sum_{m=0}^{\infty} \left[ \zeta(q + m) + 1 \right] z^{q+m-1}
\]

\[
\psi_q^k f(z) = -q \int_0^Z \zeta^{-q-1} \varphi[w_2(\zeta)] \exp \left( -\frac{q}{n} \sum_{i=0}^{n-1} \int_0^\zeta \frac{\varphi[w_1(\zeta)]}{\zeta} \frac{d\zeta}{\zeta} \right) d\zeta.
\]

We have been obtained

\[
-\sum_{n=0}^{\infty} \left[ \zeta + 1 \right] \zeta^{m-q} \right] * f(z).
\]

Hence the theorem holds true.
3.10.3 Let us assume \( f \in S \) and \( \varphi \in P \). I. e. here after it is to be taken as \( f \in \mathcal{L}_q^k(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \) if & only if
\[
f(z) * \left\{ (-qz^{-q} + \sum_{m=1}^{\infty}[3m + 1]^k(n - p)z^{m-q}) + \right.
q\varphi(e^{i\theta})(z^{-a} + \sum_{m=1}^{\infty}[3m + 1]^k \varphi_{m-q} \right\} * \left( \frac{1}{n} \sum_{m=0}^{\infty} \frac{1}{z^{q(1-e^{\nu}z)}} \right) \neq 0.
\]
\((\forall z \in \mathcal{D}^*; 0 \leq \theta < 2\pi).\)

**Proof** Let us assume that a function \( f \) is in \( \mathcal{L}_q^k(\sigma, \xi, \eta, \varepsilon, \delta, \nu; \varphi) \). I. e. here after it is to be taken as the Subordination condition given below
\[
\psi_{q,f}^{-1}(z) < \varphi(z) \quad (z \in \mathcal{D}).
\]
\[
\therefore \psi_{q,f}^{-1}(z) \neq \varphi(e^{i\theta}), \quad (z \in \mathcal{D}; 0 \leq \theta < 2\pi).
\]

It’s simple to check and verify the above subordination condition we can express as
\[
z(\psi_{q,f}^{-1}(z) + p f^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu; z) \varphi(e^{i\theta}) = 0. \quad (3.10.1)
\]
that is we can obtain from the Equation (3.9.4) that
\[
z(\psi_{q,f}^{-1}(z) = (-qz^{-q} + \sum_{m=1}^{\infty}[3m + 1]^k(m - q)z^{m-q}) * f(z) \quad (3.10.2)
\]
Moreover, from the definition, we obtained
\[
f^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu; z) = \psi_{q,f}^{-1}(z) * \left( \frac{1}{n} \sum_{m=0}^{\infty} \frac{1}{z^{q(1-e^{\nu}z)}} \right)
= (z^{-a} + \sum_{m=1}^{\infty}[3m + 1]^kz^{m-q}) * \left( \frac{1}{n} \sum_{m=0}^{\infty} \frac{1}{z^{q(1-e^{\nu}z)}} \right) * f(z) \quad (3.10.3)
\]
Substituting (3.10.1) and (3.10.2) in (3.10.3) in a simple manner we can reach to the Convolution property (an essential quality) assigned using them. From Corollaries 3.7.2 & 3.7.4 as well as using same procedure like im thm 3.7.1 it is to prove the properties for \( G_q^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu) \) & \( H_q^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu). \)

**Corollary 3.10.1** Let \( g \in G_q^k(\sigma, \eta, \xi, \varepsilon, \delta, \nu, \varphi) \), i. e. here after it is to be taken as
\[
g(z) = \left[ -q f^q_0 r^{q-1} \varphi_{w_2(\zeta)} \exp \left( -\frac{q}{2} \int_0^\zeta \frac{\varphi_{w_1(\zeta)} + \varphi_{w_2(\zeta)}}{\zeta} \right) d\zeta \right] * \left( \sum_{m=0}^{\infty} \frac{1}{[3m+1]^k} z^{m-q} \right).
\]
Here $w(z)$ is Holomorphic in $D$ where $w$ at $z = 0$ is $0 \mod w(z) < 1, (\forall z \in D)$.

**Corollary 3.10.2** Let $g \in H^k_{\xi,\eta}(\sigma, \xi, \eta, \varepsilon, \delta, \nu, \varphi)$, I. e. here after it is to be taken as

\[
g(z) = \left[ -q \int_0^z \zeta^{-q-1} \varphi[w_2(\zeta)] \exp \left( \frac{-q}{2} \int_0^\zeta \frac{\varphi[w(\zeta)]}{\zeta} - 2 d\zeta \right) d\zeta \right] * \left( \sum_{m=0}^{\infty} \left( \frac{1}{3m+1} \right)^k z^{m-q} \right).
\]

Here $w(z)$ is Holomorphic in $D$ where $w$ at $z = 0$ is $0 \mod w(z) < 1, (\forall z \in D)$.

**Corollary 3.10.3** Let $g \in S$ and $\varphi \in P$, I. e. here after it is to be taken as $g \in C^k_{\xi,\eta}(\sigma, \xi, \eta, \varepsilon, \delta, \nu, \varphi)$ if & only if

\[
g \left\{ \frac{-qz^{-q} + \sum_{m=1}^{\infty} [\Im m + 1]^k (m-q)z^{m-q} + \frac{q\varphi(e^{i\theta})}{2} l}{z} \right\} + \frac{q\varphi(e^{i\theta})}{2} (l * g)(z) = 0 \quad (z \in D^*; 0 \leq \theta < 2\pi).
\]

Here $f(z)$ is given by above Equation.

\[
\therefore f(z) = z^{-q} + \sum_{m=1}^{\infty} [\Im m + 1]^k z^{m-q}.
\]

**Corollary 3.10.4** Let $g \in S$ and $\varphi \in P$, I. e. here after it is to be taken as $g \in H^k_{\xi,\eta}(\sigma, \xi, \eta, \varepsilon, \delta, \nu, \varphi)$, if & only if

\[
f \left\{ \frac{-qz^{-q} + \sum_{m=1}^{\infty} [k(n-p)z^{n-p} + \frac{q\varphi(e^{i\theta})}{2} l]}{z} \right\} = \frac{q\varphi(e^{i\theta})}{2} (l * g)(z) = 0. \quad (z \in D^*; 0 \leq \theta < 2\pi).
\]

Here $g(z)$ is given by above Equation.

**Note:** By using substitutions for the parameters $\sigma, \xi, \eta, \varepsilon, \delta, \nu, m, A, B$ and $q$ in our theorems and corollaries, we can obtain various corresponding results which are investigated by several numbers of researchers.