Chapter 8

PROPERTIES OF MULTI VALENT ANALYTIC
FUNCTIONS AND SOME RESULTS FOR UNI VALENT FUNCTIONS
DEFINED BY A GENERALIZED SALAGEAN OPERATOR

7.0 Introduction

Here in this chapter we verified properties like Diff. Subordination, Convolution and Quasi-Convolution of an analytic schlicht i.e. a single valued function & Multi valent functns with +ive, -ve Taylor series expansion.

8.1 Preliminary Results

Here the term “Convolution” arises from the formula

\[ h(r^2 e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i(\theta-t)}) g(re^{it}) dt, \quad r < 1. \]

\[ \ell(Z) = \sum_{m=1}^{\infty} z^m = \frac{z}{1-z}. \]

Acts as the identity element under Convolution \( u \ast \ell = u = u. \)

Literature on Diff. Subordination is available in nature for example [31], [74], [34], [90] [121], [131], [13], [16], [127] [76] etc.

Here in this topic throughout we are assuming

\[ a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+r)+q} = \frac{ru}{\lambda}, \zeta \geq 0, \tau \geq 0, \]

\[ \delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}. \]

8.2 Preliminary Lemma

Lemma 8.2.1 let \( f(z) \) is given as follows

\[ u(z) = z - \sum_{k=2}^{\infty} a_{2k}z^{2k}, \]

I. e. here after it is to be taken as function \( u(z) \) contained in \( T^*_2(n, m, \gamma, \sigma, \eta, \zeta, A, B, \alpha) \) if &

\[ \sum_{k=2}^{\infty} \left( \frac{(2k-1)+2(2k-\infty)B-2(1-\infty)A(1+2k-1)(1+\epsilon(\delta+r)+q)n(1-\gamma+\eta[1+(2k-1)(1+\epsilon(\delta+r)+q)])}{{B-2A}(1-\infty)} \right) \]
\[ x \cdot a_{2k} \leq 1 \]

**Proof** Since \( u(z) \in T^*_2(n, m, \gamma, \sigma, \eta, \varsigma, A, B, \alpha) \), and

\[
T^* (n, m, \gamma, \sigma, \eta, \varsigma, A, B, \alpha) = \left\{ u : u \in T^* \mid \frac{(1-\gamma)z[D^n u(z)]'' + \gamma z[D^m u(z)]''}{(1-\gamma)D^n u(z) + \gamma D^m u(z)} \in P(A, B, \alpha) \right\}
\]

I.e. here, after it is to be taken

\[
\frac{(1-\gamma)z[D^n u(z)]'' + \gamma z[D^m u(z)]''}{(1-\gamma)D^n u(z) + \gamma D^m u(z)} \leq \frac{1 + 2(1-\alpha)A + 2\alpha B}{1 + 2Bz}.
\]

Where, \( X = 1 + (2k - 1)\sigma(\eta + \varsigma) \).

By definition of Subordination, there exists \( w(z) \) Holomorphic (an analytic) and contained in \( U \).

\[
\frac{x - \Sigma_{k=2}^{\infty} 2kX^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{y - \Sigma_{k=2}^{\infty} 2[(2k-\alpha)B - (1-\alpha)A]X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}} < \frac{1 + 2(1-\alpha)A + 2\alpha B}{1 + 2Bz}.
\]

I.e. here, after it is to be taken as by simple calculations, we obtain

\[
w(z) = \frac{\Sigma_{k=2}^{\infty} (2k-1)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{2(B-A)(1-\alpha) - \Sigma_{k=2}^{\infty} 2[(2k-\alpha)B - (1-\alpha)A]X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}} < 1
\]

Thus by noting \(|w(z)| < 1\), we get

\[
w(z) = \frac{\Sigma_{k=2}^{\infty} (2k-1)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{2(B-A)(1-\alpha) - \Sigma_{k=2}^{\infty} 2[(2k-\alpha)B - (1-\alpha)A]X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}} < 1
\]

Letting \( z \to 1^{-} \), we have obtained

\[
\Sigma_{k=2}^{\infty} (2k-1)X^n(1-\gamma+\gamma X^m)a_{2k} < 1
\]

\[
\therefore \Sigma_{k=2}^{\infty} (2k-1)X^n(1-\gamma+\gamma X^m)a_{2k} < 1
\]

And where,

\[
\begin{aligned}
a_{2k} &\geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(n+\varsigma)}{\epsilon(\delta+t)+\eta} = \frac{a\mu}{\lambda}, \varsigma \geq 0, \tau \geq 0, \\
\delta &\geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, n \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{aligned}
\]

### 8.3 Applications of Differential Subordination

Let us assume that \( A(p, 1) \) represents the class of functions as given below
\[ u(z) = z^q + \sum_{n=1}^{\infty} a_{q+n} z^{q+n} \quad (a_n \geq 0; \ q \in \mathbb{N}). \] (8.3.1)

These functions are Holomorphic (an analytic) in the open unit disk \( U \) defined as
\[ U = \{ z : |z| < 1 \} \]
Let \( u(z), v(z) \in A(q, 1) \), where
\[ u(z) = z^q + \sum_{n=1}^{\infty} a_{q+n} z^{q+n} \]
And
\[ v(z) = z^q + \sum_{n=1}^{\infty} b_{q+n} z^{q+n}. \]
I.e. here after it is to be taken as the Convolution
\[ (u * v)(z) = z^q + \sum_{n=1}^{\infty} a_{q+n} b_{q+n} z^{q+n} \] (8.3.2)
Let \( A, B, \sigma, \eta, \zeta, \epsilon, \delta, \tau \) be fixed real numbers. \( u(z) \in A(q, 1) \) Contained in \( I_{\sigma, \eta, \zeta, \epsilon, \delta, \tau}(q; A, B) \)
gives
\[ \ell_{\epsilon, \delta, \tau, \eta, \zeta}(u) < \]
\[ \frac{1+2Ax}{1+2Bz} \quad (z \in U) \] (8.3.3)
\[ \ell_{\epsilon, \delta, \tau, \eta, \zeta}(u) = [1 - \sigma(\eta + \zeta)] \frac{H_{q, \lambda}^{\lambda-1} f(z)}{z^q} + \sigma(\eta + \zeta) \frac{\partial [\delta(\epsilon + \tau) + q u(z)]}{z^q} \]
Where,
\[ H_{q, \lambda}^{\lambda-1} u(z) = z^q + \sum_{n=1}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda n!)} a_{q+n} z^{q+n}. \] (8.3.4)
Hence from above relation we have been obtained
\[ z[H_{q, \lambda}^{\lambda-1} u(z)]' = \lambda H_{q, \lambda}^{\lambda-1} u(z) - (\lambda - q) H_{q, \lambda}^{\lambda-1} u(z). \] (8.3.5)
This work is due to the motivation of [59] & Raut [49].
\[ s.t. \quad Re \left[ \frac{u(z)}{e^{i\theta H_{q, \lambda}^{\lambda-1}(z)}} \right] > \alpha \quad \text{for} \quad z \in U. \]

**Theorem 8.3.1** If a function \( u(z) \in A(q, 1) \) i.e. here after it is to be taken as
\[ z[z^{1-q} H_{q, \lambda}^{\lambda-1} u(z)]' = \lambda[z^{1-q} H_{q, \lambda}^{\lambda-1} u(z)]' - (\epsilon \delta + \tau + q) [z^{1-q} H_{q, \lambda}^{\lambda-1} u(z)]'. \] (8.3.6)

**Proof** we know that
\[ z[H_{q, \lambda}^{\lambda-1} u(z)]' = \lambda H_{q, \lambda}^{\lambda-1} u(z) - (\lambda - q) H_{q, \lambda}^{\lambda-1} u(z). \]
\[ \therefore z[H_{q, \lambda}^{\lambda-1} u(z)]' + 1 - q H_{q, \lambda}^{\lambda-1} u(z) \]
\[ = \lambda H_{q, \lambda}^{\lambda-1} u(z) + (1 - \lambda) H_{q, \lambda}^{\lambda-1} u(z). \]
But owing to
\[ z[H_{q, \lambda}^{\lambda-1} u(z)]' + (1 - q) H_{q, \lambda}^{\lambda-1} u(z) = z^{q} [z^{1-q} H_{q, \lambda}^{\lambda-1} u(z)]'. \]
We obtain $z^{1-q}H_q^λ \lambda u(z)^\prime = \lambda [z^{1-q}H_q^{λ-1} u(z)] + (1-\lambda) [z^{1-q}H_q^{λ} u(z)]$. 

Differentiating both sides of above equation we get

$$z^{1-q}H_q^{λ-1} u(z)^{\prime\prime} = \lambda [z^{1-q}H_q^{λ} u(z)]^\prime - \lambda [z^{1-q}H_q^{λ-1} u(z)]^\prime.$$ 

And where,

$$\left( \begin{array}{c} a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+\tau)+q} = \frac{\alpha}{\lambda}, \zeta \geq 0, \tau \geq 0, \\
\delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \end{array} \right)$$

Thus theorem holds true.

**Corollary 8.3.1** Let $u(z)$ belongs to $A(q, 1)$ and $z^{1-q}D^{\epsilon(\delta+\tau)+q-1} u(z)$ is convex univalent (or schlicht i.e. a single valued function) function. I.e. here after it is to be taken as $z^{1-q}H_q^{λ-1} u(z)$ is close-to-convex of order $\frac{\lambda-1}{|\lambda|}$ With respect to $z^{1-p}H_p^{λ-1} u(z)$.

**Proof** Since

$$z^{1-q}H_q^{λ-1} u(z)^{\prime\prime} = \lambda [z^{1-q}H_q^{λ} u(z)]^\prime - \lambda [z^{1-q}H_q^{λ-1} u(z)]^\prime.$$ 

we obtain

$$\frac{\lambda [z^{1-q}H_q^{λ} u(z)]^\prime}{|\lambda|} = \frac{\lambda [z^{1-q}H_q^{λ-1} u(z)]^\prime}{|\lambda|} + 1$$

Since $z^{1-q}H_q^{λ-1} u(z)$ which is a convex function.

$$Re \left( \frac{\lambda [z^{1-q}H_q^{λ} u(z)]^\prime}{|\lambda|} \right) = Re \left( \frac{\lambda [z^{1-q}H_q^{λ-1} u(z)]^\prime}{|\lambda|} \right) > Re \frac{\lambda-1}{|\lambda|}.$$ 

And where,

$$\left( \begin{array}{c} a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+\tau)+q} = \frac{\alpha}{\lambda}, \zeta \geq 0, \tau \geq 0, \\
\delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \end{array} \right)$$

therefore, by definition of close-to-convex we have obtained the required result.

**Theorem 8.3.2** Let us assume that the functions

$$f_1(z), f_2(z) \in A(q, 1),$$

$$\ell_{\epsilon, \delta, \tau, q} [f_1(z)] < h_1(z) \text{ and } \ell_{\epsilon, \delta, \tau, q} [f_2(z)] < h_2(z).$$
Here $h_1(z), h_2(z)$ are convex univalent (or schlicht) in the disk $U$ and if $\frac{\lambda}{a \mu} \geq 0, \lambda > a \mu > 0$ i.

e. here after it is to be taken as

$$\ell_{e, \delta, \tau, q}[H_q^{\lambda-1}(h_1 * h_2)(z)]$$

$$< \frac{\lambda}{a \mu} z^\frac{\lambda}{a \mu} \int_0^z t^\frac{\lambda}{a \mu} [h_1(t) * h_2(t)] dt < [h_1(z) * h_2(z)]$$

**Proof** Since $\ell_{e, \delta, \tau, q}[f_1(z)] < h_1(z)$ And $l_{e, \delta, \tau, q}[f_2(z)] < h_2(z)$

I. e. here after it is to be taken as we have

$$\ell_{e, \delta, \tau, p}[f_1(z)] \ast \ell_{e, \delta, \tau, p}[f_2(z)] < h_1(z) \ast h_2(z).$$

And by [60], the Convolution of convex Univalent (or schlicht) functions is also the convex Univalent (or schlicht) function. Now, let

$$p(z) = \ell_{e, \delta, \tau, q}[H_q^{\lambda-1}(f_1 \ast f_1)(z)]$$

$$= [1 - a \mu \frac{H_q^{\lambda-1}[H_q^{\lambda-1}(f_1 * f_1)]}{z^q} + a \mu \frac{H_q^{\lambda-1}[H_q^{\lambda-1}(f_1 * f_1)]}{z^q}]$$

I. e. here after it is to be taken as $p(z)$ is Holomorphic (an analytic) function and $p(0) = 1$ in $U$.

Since we have

$$z[H_q^{\lambda-1}u(z)]' = \lambda H_q^{\lambda}f(z) - (\lambda - p)H_q^{\lambda-1}u(z)$$

$$\therefore p(z) + \frac{a \mu}{\lambda} p(z)' = \ell_{e, \delta, \tau, q}[H_q^{\lambda-1}(f_1 \ast f_1)(z)] + \frac{a \mu}{\lambda} \ell_{e, \delta, \tau, q}[H_q^{\lambda-1}(f_1 \ast f_1)(z)]'$$

$$= \left[1 - 2 \frac{a \mu}{\lambda} q + \left(\frac{a \mu}{\lambda}\right)^2 q^2\right] z^{-q} \left(H_q^{\lambda-1} f_1(z) \ast D^{\mu+q-1} f_2(z)\right)$$

$$+ \left[\frac{a \mu}{\lambda} q \left(1 - \frac{a \mu}{\lambda} q\right) + \frac{a \mu}{\lambda} \left(1 - q\right)\right] z^{-q} \left(H_q^{\lambda-1} f_1(z) \ast H_q^{\lambda-1} f_2(z)\right)'$$

$$+ \left(\frac{a \mu}{\lambda}\right)^2 z^{-q} \left(H_q^{\lambda-1} f_1(z) \ast H_q^{\lambda-1} f_2(z)\right)''.$$
\[
\phi(z) < \frac{\lambda}{\alpha \mu} \int_0^z \frac{t^{\alpha\mu-1}}{[h_1(t) * h_2(t)]} dt \leq [h_1(z) * h_2(z)].
\]

Where,
\[
\begin{pmatrix}
\alpha_{2k} \geq 0, n \geq 0, m \geq 0, \, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \zeta)}{\varepsilon(\delta + \tau) + q} = \frac{\alpha \mu}{\lambda}, \, \zeta \geq 0, \, \tau \geq 0,
\delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}
\end{pmatrix}
\]

**Theorem 8.3.3** Let \( f_1(z) \in \mathcal{L}_{\sigma, \eta, \delta, \tau, q}(\mathbb{C}; A_1, B_1) \) and \( f_2(z) \in \mathcal{L}_{\sigma, \eta, \delta, \tau, q}(\mathbb{C}; A_2, B_2) \)

Where \( \mathcal{L}_{\sigma, \eta, \delta, \tau, q}[f_1(z)] < \frac{1 + A_1 z}{1 + B_1 z} \) & \( \mathcal{L}_{\sigma, \eta, \delta, \tau, q}[f_2(z)] < \frac{1 + A_2 z}{1 + B_2 z} \)

\[-1 \leq B_1 < A_1 \leq 1; -1 \leq B_2 < A_2 \leq 1\]
\[\varepsilon(\delta + \tau) + q > \sigma(\eta + \zeta) > 0 \quad \text{And} \quad \frac{\lambda}{\alpha \mu} \geq 0.\]

\[
\therefore \mathcal{L}_{\sigma, \eta, \delta, \tau, q}[H_q^{\lambda-1}(f_1 * f_2)(z)] \leq 1 + (A_1 - B_1)(A_2 - B_2) \frac{\lambda}{\alpha \mu} \int_0^z \frac{t^{\alpha\mu-1}}{[1 - B_1 B_2 t z]^{-1}} dt = q(z).
\]

Where,
\[
q(z) = 1 + \frac{\lambda(A_1 - B_1)(A_2 - B_2) z}{\lambda + \alpha \mu}[1 - B_1 B_2 t z]^{-1} \binom{1}{2} \binom{1}{1 - B_1 B_2 z - 1}.
\]

**Proof** Since \( \frac{1 + A_1 z}{1 + B_1 z} \) and \( \frac{1 + A_2 z}{1 + B_2 z} \) are univalent (or schlicht i.e. a single valued function) convex

Holomorphic (an analytic) functions,
\[
\frac{1 + A_1 z}{1 + B_1 z} = \left[ 1 + (A_2 - B_2) \frac{z}{1 + B_2 z} \right] = 1 + (A_1 - B_1)(A_2 - B_2) \frac{z}{1 + B_1 B_2 z}.
\]

Thus, by Theorem 8.3.2
\[
\mathcal{L}_{\sigma, \eta, \delta, \tau, q}[H_q^{\lambda-1}(f_1 * f_2)(z)] \leq 1 + (A_1 - B_1)(A_2 - B_2) \frac{\lambda}{\alpha \mu} \int_0^z \frac{t^{\alpha\mu-1}}{[1 - B_1 B_2 t]^{-1}} dt
\]
\[
q(z) = \frac{1}{\sigma(\eta + \zeta)} \int_0^z \frac{t^{\alpha\mu-1}}{[1 - B_1 B_2 t]^{-1}} dt (1 + \frac{(A_1 - B_1)(A_2 - B_2) t}{1 - B_1 B_2 t}) dt
\]
\[
= 1 + (A_1 - B_1)(A_1 - B_2) \frac{\lambda}{\alpha \mu} \int_0^z \frac{t^{\alpha\mu-1}}{[1 - B_1 B_2 t]^{-1}} ds.
\]

And where,
\[
\begin{align*}
(a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \varsigma)}{\epsilon(\delta + \tau) + q} = \frac{a}{\lambda}, \varsigma \geq 0, \tau \geq 0, \\
\delta \geq 0, 0 < \varepsilon \leq \frac{1}{z}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{z}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{align*}
\]

Hence we obtained the required result. Putting \( A_1 = A_2 = B_1 = B_2 = 1 \) in Theorem 8.3.3, we have next corollary.

**Corollary 8.3.2** Let us assume that \( f_1(z), f_2(z) \in A(q, 1) \).

And let

\[
\ell_{\epsilon, \delta, \tau, \varphi}[f_1(z)] < \frac{1 + z}{1 - z} \quad \text{and} \quad \ell_{\epsilon, \delta, \tau, \varphi}[f_2(z)] < \frac{1 + z}{1 - z}
\]

\[
\therefore \ell_{\epsilon, \delta, \tau, \varphi}(f_1 * f_2)(z) < 1 + 4 \frac{\lambda}{\alpha \mu} \frac{z^q}{1 + z} \int_0^\frac{1}{1 + z} dt.
\]

Hence by putting \( \sigma(\eta + \varsigma) = 1, \epsilon(\delta + \tau) = 0 \) next result is obtained.

**Corollary 8.3.3** Let us assume that \( f_1(z), f_2(z) \in A(q, 1) \).

And let

\[
\ell_{0, 0, 0, q}[f_1(z)] < \frac{1 + z}{1 - z} \quad \text{and} \quad \ell_{0, 0, 0, q}[f_2(z)] < \frac{1 + z}{1 - z}
\]

\[
\therefore \ell_{0, 0, 0, q}(f_1 * f_2)(z) < 1 + 4q z^{-q} \int_0^\frac{1}{1 + z} dt.
\]

\[
F_c(z) = \frac{c + q}{z^c} \int_0^z t^{c - 1} u(t) dt = \sum_{m=0}^\infty \frac{c + q}{c + n} z^n * u(z) \quad (8.3.8)
\]

Where, \( u(z) \in A(q, 1) \) And \( c + q > 0 \). Now since

\[
H_{q}^{\lambda - 1} u(z) = \frac{z^q}{(1 - z)\lambda} * u(z) = z^q + \sum_{n=1}^\infty \frac{\Gamma(\lambda + n)}{\Gamma(\lambda) a_n} a_n q z^n + q,
\]

I. e. here after it is to be taken as

\[
z[H_{q}^{\lambda - 1} f_c(z)]' = (c + q)H_{q}^{\lambda - 1} f(z) - cH_{q}^{\lambda - 1} f_c(z). \quad (8.3.9)
\]

**Theorem 8.3.4** Let \( \mu, c \) be real numbers \( (\mu \geq 0) \) and \( c + q > 0 \).

If \( f_1(z), f_2(z) \in A(q, 1) \) satisfy

\[
\frac{H_{q}^{\lambda - 1}(f_1 + f_2)(z)}{z^q} < \frac{(A_1 - B_1)(A_2 - B_2)z}{1 - B_1 B_2} \]

I. e. here after it is to be taken as

\[
\frac{H_{q}^{\lambda - 1}[F_c(z) + G_c(z)]}{z^q} < q(z) < \frac{(A_1 - B_1)(A_2 - B_2)z}{1 - B_1 B_2} \]

Where \( F_c(z) \) is defined as
\[
F_c(z) = \frac{c+q}{z^c} \int_0^z t^{c-1} u(t) dt = \sum_{n=q}^{\infty} \frac{c+q}{c+n} z^n * u(z)
\]

\[
H_{p\lambda}^{-1} f(z) = \frac{z^q}{(1-z)\lambda} * u(z) = z^q \sum_{n=1}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} a_n + q z^{n+q}.
\]

I. e. here after it is to be taken as \( G_c(z) \) is defined as follows

\[
G_c(z) = \sum_{n=q}^{\infty} \frac{c+q}{c+n} z^n * f_2(z)
\]

And

\[
q(z) = 1 + (1 - B_1 B_2 z)^{-1} \frac{c+q}{c+n+1} (A_1 - B_1)
\]

\[
\times (A_2 - B_2)z 2F1 \left(1, 1; 2 + c + q; \frac{B_1 B_2 z}{B_1 B_2 z - 1}\right).
\]

Where,

\[
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \gamma)}{\epsilon(\delta + \tau + q)} = \frac{a_{2k}}{\lambda}, \gamma \geq 0, \tau \geq 0, \right)
\]

\[
\left( 0 \leq \epsilon, \eta \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \right).
\]

**Proof** Let us assume that

\[
p(z) = \frac{H_q^{\lambda-1}(F_c G_c)(z)}{z^q},
\]

\[
\therefore p(z) \text{ is Holomorphic in the disk U s. t. } p(0) = 1. \text{ Since we know}
\]

\[
Z \left[ H_q^{\lambda-1} f_c(z) \right]' = (c + q) H_q^{\lambda-1} u(z) - c H_q^{\lambda-1} f_c(z).
\]

\[
\therefore p(z) + \frac{zp'(z)}{c+q} = \frac{H_q^{\lambda-1}(F_c f_2)(z)}{z^q} \leq 1 + \frac{(A_1 - B_1)(A_2 - B_2)z}{1 - B_1 B_2 z}
\]

\[
\therefore \frac{H_q^{\lambda-1}(F_c G_c)(z)}{z^q} \leq q(z)
\]

\[
= (c + q) z^{-(c+q)} \int_0^z t^{c+q-1} \left( \frac{1 + A_1 t}{1 + B_1 t} \right) \left( \frac{1 + A_2 t}{1 + B_2 t} \right) dt
\]

\[
< 1 + \frac{(A_1 - B_1)(A_2 - B_2)z}{1 - B_1 B_2 z}
\]

Finally we obtain

\[
q(z) = 1 + (1 - B_1 B_2 z)^{-1} \frac{c+q}{c+n+1} (A_1 - B_1)
\]

\[
\times (A_2 - B_2)z 2F1 \left(1, 1; 2 + c + p; \frac{B_1 B_2 z}{B_1 B_2 z - 1}\right).
\]

If we put \( \delta = \tau = 0, p = A_1 = A_2 = 1, B_1 = B_2 = -1. \) In above Theorem 8.3.4, i. e. here after it is to be taken as we have next result.
Corollary 8.3.4 Let \( c + 1 > 0 \) where \( c \) a real number. If \( f_1(z), f_2(z) \in A(q, 1) \) and 
\[
\frac{(f_1 \ast f_2)(z)}{z} < 1 + \frac{4z}{1-z}
\]
\[
\therefore \frac{[F_c(z) \ast G_c(z)]}{z} < q(z) < 1 + \frac{4z}{1-z}
\]
Where it is,
\[
F_c(z) = \sum_{n=1}^{\infty} \frac{c-z}{c+n} z^n \ast f_1(z),
\]
\[
G_c(z) = \sum_{n=1}^{\infty} \frac{c-z}{c+n} z^n \ast f_2(z),
\]
\[
q(z) = 1 + 4(1-z)^{-1} \frac{c+1}{c+2} 2F1 \left(1, 1; c + 3; \frac{z}{z-1}\right)
\]

8.4 Convolution and Quasi-Convolution Properties

Let 
\[
u(z) = z - \sum_{k=2}^{\infty} a_k z^{2k} \quad (a_k \geq 0)
\]
Let \( P(A, B, \alpha) \) be Holomorphic (an analytic) in \( U \) that satisfies 
\[
u(z) < \frac{1+(1-\alpha)A+\alpha B}{1+Bz}
\]
Where it is, 
\[-1 \leq 2B < 2A \leq 1, 0 \leq \alpha < 1.
\]
Consider \( T^*_2 \) as subclass of \( T \) consisting
\[
u(z) = z - \sum_{k=2}^{\infty} a_k z^{2k}.
\]
we define
\[
T(n, m, \gamma, \eta, \zeta, A, B, \alpha)
\]
\[
= \left\{ u \in T^*: \frac{1-(\gamma)z[D^n u(z)]'+\gamma z[D^{n+m} u(z)]'}{(1-\gamma)D^n u(z)+\gamma D^{n+m} u(z)} \in P(A, B, \alpha) \right\}
\]
\[
T^*(n, m, \gamma, \eta, \zeta, A, B, \alpha) = \left\{ u \in T^*: \frac{1-(\gamma)z[D^n u(z)]'+\gamma z[D^{n+m} u(z)]'}{(1-\gamma)D^n u(z)+\gamma D^{n+m} u(z)} \in P(A, B, \alpha) \right\}
\]
\[
S^*(n, \eta, \zeta, A, B, \alpha) = T(n, m, 0, \sigma, \eta, \zeta, A, B, \alpha)
\]
\[
K(n, \eta, \zeta, A, B, \alpha) = T(n, m, 1, \sigma, \eta, \zeta, A, B, \alpha)
\]
\[
S^*_2(n, \eta, \zeta, A, B, \alpha) = T^*_2(n, m, 0, \sigma, \eta, \zeta, A, B, \alpha)
\]
\[
K^*_2(n, \eta, \zeta, A, B, \alpha) = T^*_2(n, m, 1, \sigma, \eta, \zeta, A, B, \alpha)
\]
\[
D^n u(z) = z + \sum_{k=2}^{\infty} [1 + (k - 1)\sigma(\eta + \zeta)] a_k z^{2k}
\]
\[
D^n u(z) = z + \sum_{k=2}^{\infty} [1 + (2k - 1)\sigma(\eta + \zeta)] a_k z^{2k}
\]

If \( u(z) = z - \sum_{k=2}^{\infty} a_k z^{2k} \) and 
\[
v(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \quad (a_{2k}, b_{2k} \geq 0)
\]
And where,
\[
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \zeta)}{\varepsilon(\delta + \tau) + q} = \frac{a}{\lambda}, \zeta \geq 0, \tau \geq 0, \frac{\delta}{\varepsilon} \geq 0, 0 \leq \varepsilon \leq 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \right)
\]

I. e. here after it is to be taken as the convolution is defined as follows

\[
u(z) * v(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k}
\]

(8.4.10)

In this section we are proposing few of generalized work given by [128].

Properties of schlicht i. e. a single valued functn with -ve coeffs of type

\[
u(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}
\]

are studied by the researchers like, [62], [62]. They have also worked on the convolution properties of schlicht i. e. a single valued function with -ve coeffs, [116] Schlicht i. e. a single valued function with missing coeffs.

**Theorem 8.4.1** If \( u(z) \) is as given in (8.4.1) & \( v(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \)

Where, \( a_{2k} \geq 0, b_{2k} \geq 0 \) and

\[
u(z), v(z) \in T_{2}^*(n, m, \gamma, \sigma, \eta, \zeta, A, B, \alpha),
\]

I. e. here after it is to be taken as

\[
u(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k} \in T_{2}^*(n, m, \gamma, \sigma, \eta, \zeta, A_1, B_1, \alpha).
\]

With

\[
A_1 \leq 1 - 2j, B_1 \geq \frac{j + A_1}{1 - j}
\]

Where,

\[
\frac{[6(1-\infty)(B-A)(B-C)]}{[3+2(4-\infty)B-2(1-\infty)A][3+(4-\infty)D-(1-\infty)C](1+3\mu)\lambda^m} \times \frac{1}{[(1-\gamma+y(1+3\mu)m)-2(\beta-A)(\beta-c)(1-\lambda)^{2}]
\]

\[

\sum_{k=2}^{\infty} \left[ (2k-1) + 2(2k-\alpha)B - 2(1-\infty)A \right] 
\times \left[ \left( 1 + (2k-1) \mu \right)^m (1 - \gamma + y)^{1 + (2k-1) \mu} \right] \frac{1}{2(\beta-A)(1-\lambda)^{m}} a_{2k} \leq 1
\]

(8.4.11)

\[
\sum_{k=2}^{\infty} \left[ (2k-1) + 2(2k-\alpha)B - 2(1-\infty)A \right] X^n 
\times (1 - \gamma + y X^m) \left[ 2(\beta-A)(1-\lambda)^{-1} \right] a_{2k} \leq 1
\]

(8.4.12)

\[
\sum_{k=2}^{\infty} \left[ (2k-1) + 2(2k-\alpha)B - 2(1-\infty)A \right] X^n
\]

\[
\text{Proof} \quad \text{Since } u \text{ and } v \text{ belong to, } T_{2}^*(n, m, \gamma, \sigma, \eta, \zeta, A, B, \alpha) \text{ therefore by lemma 8.2.1}
\]

\[
\sum_{k=2}^{\infty} \left[ (2k-1) + 2(2k-\alpha)B - 2(1-\infty)A \right] 
\times \left[ \left( 1 + (2k-1) \mu \right)^m (1 - \gamma + y)^{1 + (2k-1) \mu} \right] \frac{1}{2(\beta-A)(1-\lambda)^{m}} a_{2k} \leq 1
\]

(8.4.11)
where $X = 1 + (2k - 1)\sigma(\eta + \zeta)$, we contemplate to find $A_1, B_1$, as

$$-I \leq A_1 < B_1 \leq 1$$

for

$$q(z) \in T_s(n, m, \gamma, \sigma, \eta, \zeta, A_1, B_1, \infty),$$

∴\[ \sum_{k=2}^{\infty}[(2k - 1) + (2k - \infty)B_1 - (1 - \infty)A_1]X^n \]

\[ \times (1 - \gamma + \gamma X^m)[(B_1 - A_1)(1 - \infty)]^{-1}a_{2k}b_{2k} \leq 1. \] (8.4.14)

by using Cauchy-Schwarz inequality, we get

$$\sum_{k=2}^{\infty}V(a_{2k}b_{2k}) = (\sum_{k=2}^{\infty}Va_{2k})^{\frac{1}{2}}(\sum_{k=2}^{\infty}Vb_{2k})^{\frac{1}{2}} \leq 1,$$

(8.4.15)

$$V = [(2k - 1) + 2(2k - \infty)B - 2(1 - \infty)A]X^n \times (1 - \gamma + \gamma X^m)[2(B - A)(1 - \infty)]^{-1}$$

(8.4.16)

If

$$V_1(a_{2k}b_{2k}) \leq V(a_{2k}b_{2k})^{\frac{1}{2}},$$

i.e., hereafter it is to be taken as the above result is true, where

$$V_1 = [(2k - 1) + (2k - \infty)B_1 - (1 - \infty)A_1]X^n

(1 - \gamma + \gamma X^m)[(B_1 - A_1)(1 - \infty)]^{-1}$$

(8.4.17)

or

$$V_1(a_{2k}b_{2k})^{\frac{1}{2}} \leq V (k = 2, 3, \ldots)$$

According to (3.24)

$$a_{2k}^{\frac{1}{2}} \leq V^{-1}$$

(8.4.18)

Thus, to find $V_1$ as

$$V_1 = V^2$$

(8.4.19)

∴\[ [(2k - 1) + (2k - \infty)B_1 - (1 - \infty)A_1]X^n(1 - \gamma + \gamma X^m) \leq V^2[(B_1 - A_1)(1 - \infty)] \] (8.4.20)

$$A_1 = \frac{V^2[(1 - \infty)B_1 - [(2k - 1) + (2k - \infty)B]X^n(1 - \gamma + \gamma X^m)]}{(1 - \infty)[V^2X^n(1 - \gamma + \gamma X^m)]}.$$

(8.4.21)

Hence

$$V^2 \geq X^n(1 - \gamma + \gamma X^m) \text{ for } k \geq 1$$

From (8.4.21) we can get

$$\frac{(B_1 - A_1)}{(B_1 - 1)} \geq \frac{(2k - 1)X^n(1 - \gamma + \gamma X^m)}{(1 - \infty)[V^2X^n(1 - \gamma + \gamma X^m)]} \text{ for } k \geq 2$$

(8.4.22)

Thus r. h. s. of (8.4.22) decreases as k increases, i.e., hereafter it is to be taken as it has
maximum for \(k = 2\), i.e., hereafter it is to be taken as (8.4.22) is true if

\[
\frac{(B_1 - A_1)}{(1 - 1)} \leq \frac{12(1 - \infty)(B - A)^2}{[3 + 2(4 - \infty)B - 2(1 - \infty)A]^2(1 + 3a\mu(n - 1)(1 - \gamma + \gamma(1 - 3a\mu)^m) - 4(B - A)^2(1 - \infty)^2} = j
\]

(8.4.23)

Hence \(j < 1\). Putting \(A_1\) in (8.4.23),

\[
B_1 \geq \frac{j^1 + A_1}{1 - 1},
\]

(8.4.24)

Where it is obvious for all

\[
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \zeta)}{\epsilon(\delta + \tau)} = \frac{a\mu}{\lambda}, \zeta \geq 0, \tau \geq 0, \delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \right)
\]

\(-1 \leq A_1 < B_1 \leq 1\). The proof is completed.

**Corollary 8.4.1** Let us assume that the functions \(u(z)\) & \(v(z)\) are as given in (8.4.1)

Such as \(u(z), v(z) \in S^*_2(n, \sigma, \eta, \zeta, A, B, \alpha)\),

\[
q(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k} \in S^*_2(n, \sigma, \eta, \zeta, A_1, B_1, \alpha)
\]

With,

\[-1 \leq A_1 < B_1 \leq 1, -\frac{1}{2} \leq B < A \leq \frac{1}{2}\]

Where it is for

\[
A_1 \leq 1 - 2j_1, B_1 \geq \frac{j^1 + A_1}{1 - j_1},
\]

\[
j_1 = \frac{12(1 - \infty)(B - A)^2}{[3 + 2(4 - \infty)B - 2(1 - \infty)A]^2(1 + 3a\mu)^n - 4(B - A)^2(1 - \infty)^2}
\]

Putting \(n = \infty = 0\) in corol. 8.3.1, we obtain next corollary due to [128].

**Corollary 8.4.2** Let us assume that the functions \(u(z)\) & \(v(z)\) are as given in (8.4.1)

Defined as \(u(z), v(z) \in S^*_2(0,0,0,0, A, B, 0)\),

\[
q(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k} \in S^*_2(0,0,0,0, A_1, B_1, 0).
\]

Where it is for all

\[
a_{2k} \geq 0, b_{2k} \geq 0, -\frac{1}{2} \leq B < A \leq \frac{1}{2}
\]

With

\[
A_1 \leq 1 - 2j_2, B_1 \geq \frac{j^2 + A_1}{1 - j_2},
\]
Theorem 8.4.2 Let us assume that \( u(z) \in T_2^* (n, m, \gamma, \eta, \zeta, A, B) \) and \( v(z) \in T_2^* (n, m, \gamma, \eta, \zeta, C, D) \) i. e. here after it is to be taken as
\[ u(z) \ast v(z) \in T_2^* (n, m, \gamma, \eta, \zeta, E, F) \], where
\[ \Box \leq 1 - 2j, F \geq \frac{j+E}{1-j} \]
Where,
\[ j = \frac{[6(1-\infty)(B-A)(D-C)]}{[3+2(4-\infty)B-2(1-\infty)A][3+(4-\infty)D-(1-\infty)C][1+3a \mu]^n} \times \frac{1}{[1-\gamma+\gamma(1+3a \mu)^m]-2(B-A)(D-C)(1-\infty)^2} \]

Proof Using Thm 8.3.1, & Lemma 8.2.1,
\[ \frac{[2k(F+1)-(1-\infty F)+(1-\infty)E)]X^n(1-\gamma+\gamma X^m)}{(F-E)(1-\infty)} \]
\[ \leq \frac{[2k(2B+1)-(1+2a \beta B+2(1-\infty)A)]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\infty)} \times \frac{[2k(D+1)-(1+\alpha D+1-\infty)C)]X^n(1-\gamma+\gamma X^m)}{(D-C)(1-\infty)} = d, \] (8.4.25)
Where \( X = [1+(2k-1)a \mu], \ a \mu \geq 0. \)
I. e. here after it is to be taken as by simple calculations we have
\[ \frac{F-E}{F+1} \geq \frac{(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)[d-X^n(1-\gamma+\gamma X^m)]^2} \] (8.4.26)
i. e. here after it is to be taken as the r. h. s. of (8.4.26) decreasing in accordance with increase in k and it has max for k= 2, i. e. here after it is to be taken as we get
\[ \frac{F-E}{F+1} \geq \frac{[6(1-\infty)(B-A)(D-C)]}{[3+2(4-\infty)B-2(1-\infty)A][3+(4-\infty)D-(1-\infty)C][1+3a \mu]^n} \times \frac{1}{[1-\gamma+\gamma(1+3a \mu)^m]-2(B-A)(D-C)(1-\infty)^2} \]
\[ = j \] (8.4.27)

Corollary 8.4.3 Let us assume that \( u(z) \in S_2^* (n, \sigma, \eta, \zeta, A, B) \) and \( v(z) \in S_2^* (n, \sigma, \eta, \zeta, C, D) \) i. e. here after it is to be taken as
\[ u(z) \ast v(z) \in S_2^* (n, \sigma, \eta, \zeta, E, F) \], where
\[ E \leq 1 - 2j_1, \quad F \geq \frac{j_1 + E}{1 - j_1} \]

With

\[ \dot{j}_1 = \frac{6(1 - \infty)(B - A)(D - C)}{[3 + 2(4 - \infty)B - 2(1 - \infty)A]} \times \frac{1}{[(3 + 4 - \infty)D - (1 - \infty)C)(1 + 3\sigma(\eta + \zeta))^{n - 2}(B - A)(D - c)(1 - \infty)^2]} \]

Assuming \( \alpha = n = 0 \) and using the reference [128] we will obtain next result.

**Corollary 8.4.4** Let us assume that \( u(z) \in S_2^*(0,0,0,0,A,B,0) \) & \( v(z) \in S_2^*(0,0,0,0,C,D,0) \) i.e. hereafter it is to be taken as \( u(z) \ast v(z) \in S_2^*(0,0,0,0,E,F,0) \), where

\[ E \leq 1 - 2j_2, \quad F \geq \frac{j_2 + E}{1 - j_2} \]

Where,

\[ \dot{j}_2 = \frac{6(B-A)(D-C)}{(3 + 8B - A)(3 + 4D - C) - 2(B - A)(D - c)} \]

**Corollary 8.4.5** Let us assume that \( u(z) \in K_2^*(n, m, \sigma, \eta, \zeta, A, B, \alpha) \) & \( v(z) \in K_2^*(n, m, \sigma, \eta, \zeta, C, D, \alpha) \), i.e. hereafter it is to be taken as \( u(z) \ast v(z) \in K_2^*(n, m, \sigma, \eta, \zeta, E, F, \alpha) \), where

\[ E \leq 1 - 2j_3, \quad F \geq \frac{j_3 + E}{1 - j_3} \]

Where,

\[ \dot{j}_3 = \frac{6(1 - \infty)(B - A)(D - C)}{[3 + 2(4 - \infty)B - 2(1 - \infty)A][3 + 4 - \infty)(D - 1 - \infty)C]} \times \frac{1}{[(1 + 3\alpha)\mu + n - 2(B - A)(D - c)(1 - \infty)^2]} \]

Assuming \( \alpha = n = 0, \sigma = \frac{1}{2}, \eta = 1, \zeta = 1 \), and using the reference [128] we will obtain next result.

**Corollary 8.4.6** If \( u(z) \in K_2^*(\frac{0,1,1}{2,1,1}, A, B, 0) \) and \( v(z) \in K_2^*(\frac{0,1,1}{2,1,1}, C, D, 0) \). I.e. hereafter it is to be taken as \( u(z) \ast v(z) \in K_2^*(\frac{0,1,1}{2,1,1}, E, F, 0) \), Where,

\[ E \leq 1 - 2j_4, \quad F \geq \frac{j_4 + E}{1 - j_4} \]

With

\[ \dot{j}_4 = \frac{6(B-A)(D-C)}{(4[3 + 8B - 2A][3 + 4D - C] - 2(B - A)(D - c))}. \]
Theorem 8.4.3 Let us assume that the function
\[ u(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k} \]
Where, \( a_{2k} \geq 0 \in T_2(n, m, \gamma, \delta, \tau, A, B, \alpha) \) & \( g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \)
With \( |b_{2i}| \leq 1, \ i \geq 1 \), i.e. here after it is to be taken as \( u(z) * v(z) \in T(n, m, \gamma, \delta, \tau, A, B, \alpha) \).

Proof By assumption, we have
\[ \sum_{k=2}^{\infty} \left[ \left( 2 + 2(1-\alpha) \right) - 2(1-\alpha) \right] \left[ \frac{(1-\gamma+y[1+(2k-1)\lambda])^m}{2(B-A)(1-\alpha)} \right] a_{2k} b_{2k} \leq 1. \]
And \( \therefore |b_{2i}| \leq 1 \ for \ i \geq 1 \), i.e. here after it is to be taken as
\[ \sum_{k=2}^{\infty} \left[ \frac{(1+2k-1)\alpha \mu (1-\gamma+y[1+(2k-1)\lambda])^m}{2(B-A)(1-\alpha)} \right] a_{2k} b_{2k} \]
That is \( u(z) * v(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k} \in T(n, m, \gamma, \delta, \tau, A, B, \alpha) \).
Where,
\[ \left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+c)}{\epsilon(\delta+\tau)+q} = \frac{a\mu}{\lambda}, \zeta \geq 0, \tau \geq 0, \right. \\
\left. \delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \right) \]

Corollary 8.4.7 Let us assume that the function
\[ u(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}, \]
Where \( a_{2k} \geq 0 \in S_*(n, \sigma, \eta, \tau, A, B, \alpha) \) and
\[ v(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \]
With \( |b_{2i}| \leq 1, \ i \geq 1 \). I.e. here after it is to be taken as \( u(z) * v(z) \in S^*(n, \sigma, \eta, \tau, A, B, \alpha) \).
By putting \( n = \alpha = 0 \) and using reference [128] we get next corollary.

Corollary 8.4.8 Let us assume that the function
\[ u(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}, \]
Where,
\[ a_{2k} \geq 0 \in S_2^*(0,0,0,0,A,B,0) \] & \( v(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \)
With \( |b_{2i}| \leq 1, \ i \geq 1 \). ∴ \( u(z) * v(z) \in S^*(0,0,0,0,A,B,0) \)
Corollary 8.4.9 Let us assume that the function
\[ u(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}, \]
Where \( a_{2k} \geq 0 \in K_{2}^{*}(n, m, \sigma, \eta, \zeta, A, B, \alpha) \).
And
\[ u(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \]
With \( |b_{2i}| \leq 1, \ i \geq 1 \). i.e. here after it is to be taken as \( u(z) \ast v(z) \in K(n, m, \sigma, \eta, \zeta, A, B, \alpha) \), by putting \( n = \alpha = 0 \) and \( \sigma = \frac{1}{2}, \eta = \zeta = 1 \) and using reference [128] we get next corollary.

Corollary 8.4.10 Let \( u(z) \in K_{2}^{*} \left( 0,1, \frac{1}{2}, 2,1,1, A, B, 0 \right) \).
\[ u(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}, \] Where \( a_{2k} \geq 0 \), &
\[ v(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k} \] Where \( |a_{2i}| \leq 1, \ \forall \ i \geq 1 \).
I.e. here after it is to be taken as \( u(z) \ast v(z) \in K(0,1,1/2,1,1,A,B,0) \).
Where,
\[ \left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \eta \leq 1, -\frac{\sigma(n+\zeta)}{\epsilon(\delta+\tau)+q} = \frac{a_{\mu}}{\lambda}, \zeta \geq 0, \tau \geq 0, \right. \]
\[ \left. \delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \right). \]

Theorem 8.4.4 Let \( u, v \in T_{2}^{*}(n, m, \gamma, \sigma, \eta, \zeta, A, B, \infty) \), i.e. here after it is to be taken as
\[ q(z) = z - \sum_{k=2}^{\infty} \left( a_{2k}^2 + b_{2k}^2 \right) \in T_{2}^{*}(n, m, \gamma, \sigma, \eta, \zeta, A, B, \infty). \]
Here
\[ A_{1} \leq 1 - 2j \text{ and } B_{1} \geq \frac{A_{1}+j}{1-j} \]
with
\[ j = \frac{[24(1-\infty)(B-A)^2]}{[3+2(4-\infty)B-2(1-\infty)A]^{2}(1+3a\mu)^n[(1-\gamma+\gamma(1+3a\mu)^m]-8(B-A)^2(1-\infty)^2}]. \]

Proof By assumption,
\[ \sum_{k=2}^{\infty} \frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]x^n(1-\gamma+\gamma x^m)}{2(B-A)(1-\infty)} a_{2k} \leq 1 \]
\[ \sum_{k=2}^{\infty} \frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]x^n(1-\gamma+\gamma x^m)}{2(B-A)(1-\infty)} b_{2k} \leq 1 \]
Here \( X = 1 + (2k-1)a\mu \), thus
\[
\sum_{k=2}^{\infty} \left[ \frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]X^n(1-\gamma+yX^m)}{2(B-A)(1-\infty)} \right]^2 a_{2k} \leq 1.
\]
\[
\left( \sum_{k=2}^{\infty} \frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]X^n(1-\gamma+yX^m)}{2(B-A)(1-\infty)} \right)^2 b_{2k} \leq 1 \quad (8.4.28)
\]

I. e. here after it is to be taken as we may write
\[
\sum_{k=2}^{\infty} \frac{1}{2} \left[ \frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]X^n(1-\gamma+yX^m)}{2(B-A)(1-\infty)} \right]^2 (a_{2k}^2 + b_{2k}^2) \leq 1 \quad (8.4.29)
\]

Therefore, the inequality (8.4.29) holds if
\[
\frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]X^n(1-\gamma+yX^m)}{(B_1-A_1)(1-\infty)} \leq \frac{1}{2} \left[ \frac{[(2k-1)+2(2k-\infty)B-2(1-\infty)A]X^n(1-\gamma+yX^m)}{2(B-A)(1-\infty)} \right]^2 = \frac{\nu^2}{2}.
\]

And by simplification, the last inequality gives
\[
\frac{(B_1-A_1)}{(B_1+1)} \geq \frac{2(2k-1)X^n(1-\gamma+yX^m)}{(1-\alpha)\nu^2-2X^n(1-\gamma+yX^m)} \quad (8.4.30)
\]

the r. h. s. of (8.4.30) decreasing in accordance with increase in \(k\) & for \(k=2\) the following relation,
\[
\frac{(B_1-A_1)}{(B_1+1)} \geq \frac{[24(1-\infty)(B-A)^2]}{[3+2(4-\infty)B-2(1-\infty)A]^2(1+3\alpha\mu)^n[1-\gamma+y(1+3\alpha\mu)^m]-8(B-A)^2(1-\infty)^2]} = j \quad (8.4.31)
\]

Now by fixing \(A_1\) in (8.4.31), we have \(B_1 \geq \frac{A_1+j}{1-j} \& B_1 \leq 1\) give \(A_1 \leq 1 - 2j\).

with \(j\) as given in (8.4.31).

Where,
\[
\begin{align*}
a_{2k} &\geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \alpha \geq \frac{\sigma(\eta+\gamma)}{\varepsilon(\delta+t)+q} = \frac{a\mu}{\lambda}, \eta \geq 0, \tau \geq 0,
\delta &\geq 0, 0 < e \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, \frac{1}{2} \leq B \leq A \leq \frac{1}{2}.
\end{align*}
\]

**Corollary 8.4.11** Let \(u, v \in S_2(n, \sigma, \eta, \varsigma, A, B, \infty)\), i. e. here after it is to be taken as
\[
q(z) = z - \sum_{k=2}^{\infty} (a_{2k}^2 + b_{2k}^2)z^{2k} \in S_2(n, \sigma, \eta, \varsigma, A_1, B_1, \infty)
\]

Where \(A_1, B_1 and j_1\) as given in theorem 8.4.4. By putting \(n = \infty = 0\) in Corollary 8.4.11, we have the next corollary using reference [128].
Corollary 8.4.12 Let $u, v \in S_2^*(0,0,0,0,A,B,0)$, i. e. here after it is to be taken as

$$q(z) = z - \sum_{k=2}^{\infty} (a_{2k-2}^2 + b_{2k-2}^2) z^{2k} \in S_2^*(0,0,0,0,A_1,B_1,0).$$

Where $A_1$ and $B_1$ as given in theorem 8.4.4 with

$$j_2 = \frac{24(\theta - A)^2}{(3 + 8B - 2A)^2 - 8(\theta - A)^2}.$$

Let the class $T(n, q)$ of functions given by

$$u(z) = z^q - \sum_{k=n+q}^{\infty} a_k z^k$$

where $(n, q \in N, a_k \geq 0)$ (8.4.32)

$u(z)$ will be Holomorphic (an analytic) and Multivalent $u = \{z: |z| < 1\}$. Consider the generalized Ruscheweyh derivative $J_q^{\eta,\epsilon,\delta,\tau} u(z)$ defined as

$$J_q^{\eta,\epsilon,\delta,\tau} u(z) = z^\eta - \sum_{k=n+q}^{\infty} \Omega_q^{\eta,\epsilon,\delta,\tau}(k) a_k z^k$$

(8.4.33)

$$\Omega_q^{\eta,\epsilon,\delta,\tau} u(z) = \frac{\Gamma(k+q+2)\Gamma(k+q+1)\Gamma(k+q+2+\lambda-q\mu)\Gamma(k+q+2+\gamma-q\mu)\Gamma(k+q+2+\gamma-q\mu)\Gamma(k+q+2+\gamma-q\mu)}{\Gamma(k+q+1)\Gamma(k+q+2+\lambda-q\mu)\Gamma(k+q+2+\gamma-q\mu)\Gamma(k+q+2+\gamma-q\mu)\Gamma(k+q+2+\gamma-q\mu)}$$

(8.4.34)

$v, \epsilon, \delta, \tau \in R, \sigma = \epsilon, \eta = \delta, \zeta = \tau$ and $p = v = 1$.

We have Ruscheweyh derivative to univalent (or schl i. e. a single valued function) given as

$$AB(\sigma, \eta, \epsilon, \delta, \tau, \gamma, \alpha, p, n),$$

containing $u(z)$ given as (8.4.32) satisfying the condition

$$Re \left[ \frac{J_q^{\eta,\epsilon,\delta,\tau} u(z)}{(1-\gamma)z^\eta + \gamma z^\eta (J_q^{\eta,\epsilon,\delta,\tau} u(z))^\eta + \gamma (J_q^{\eta,\epsilon,\delta,\tau} u(z))^\eta} \right] > \alpha$$

(8.4.35)

And $J_q^{\eta,\epsilon,\delta,\tau} f(z)$ as defined in (8.4.33). Also Let

$$f_i(z) = z^q - \sum_{k=n+q}^{\infty} a_k z^k, \ (i = 1, 2).$$

F Belonging to $T(n, q)$ (an analytic) in $U$. I. e. here after it is to be taken as

$$(f_1 * f_2 * \cdots * f_n)(z) = z^q - \sum_{k=n+q}^{\infty} \prod_{i=1}^{n} a_{k,i} z^k$$

(8.4.36)

$$\prod_{i=1}^{n} a_{k,i} = a_{k,1} a_{k,2} \cdots a_{k,n}, \ (\square \in N)$$

Where,

$$\left( \begin{array}{c}
a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+\tau)+q} = \frac{am}{\lambda}, \zeta \geq \delta \geq 0, \\
\delta \geq 0, 0 < \epsilon \leq \frac{1}{2} \leq 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2} - \frac{1}{2} \leq B < A \leq \frac{1}{2}
\end{array} \right)$$

Theorem 8.4.5 Let us assume that $u(z) \in T(n, q) \Rightarrow$
If \( u(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \) if and only if
\[
\Sigma_{k=n+q}^{\infty} \left\{ 1 - \alpha [(1 - \gamma)k(k - 1) + \gamma k + 1] \Omega_q^{\sigma, \eta, \zeta, \epsilon, \delta, \tau}(k) a_k \right\} < 1 - \alpha \left( [(1 - \gamma)q(q - 1) + \gamma q + 1] \right)
\]
(8.4.37)
\[0 \leq \alpha < \frac{1}{(1 - \gamma)q(q - 1) + \gamma q + 1}\] and \( \Omega_q^{\sigma, \eta, \zeta, \epsilon, \delta, \tau}(k) \)

as defined in (8.4.34). The result holds true.

**Proof** If \( u(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \), i.e. here after it is to be taken as
\[
Re \left\{ \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \left[ (1 - \gamma)z^2 \left[ \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \right] + \alpha z \left[ \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \right] \right] \right] \right\} > \alpha \quad (z \in u).
\]

\[
\left| (1 - \alpha) \Omega_q^{\sigma, \eta, \zeta, \epsilon, \delta, \tau} u(z) - (1 - \gamma)z^2 \alpha \left[ \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \right] \right| > 0.
\]

\[
\therefore \quad \Omega_q^{\sigma, \eta, \zeta, \epsilon, \delta, \tau} u(z),
\]

\[|z|^q \left\{ 1 - \alpha \left( [(1 - \gamma)q(q - 1) + \gamma q + 1] \right) \right\} = 0.
\]

Letting \( z \to 1^- \) on real values yields
\[
\Sigma_{k=n+q}^{\infty} \left\{ 1 - \alpha [(1 - \gamma)k(k - 1) + \gamma k + 1] \Omega_q^{\sigma, \eta, \zeta, \epsilon, \delta, \tau}(k) a_k \right\}
\]
\[< 1 - \alpha \left( [(1 - \gamma)q(q - 1) + \gamma q + 1] \right)\]

Where
\[
\Omega_q^{\sigma, \eta, \zeta, \epsilon, \delta, \tau}(k) = \frac{\Gamma((k-2q+1)+\lambda)\Gamma(v+2+\lambda-a\mu)\Gamma(k+v-q+2)}{\Gamma(k+q+1)\Gamma(k+v-2q+2+\lambda-a\mu)\Gamma(v+2)\Gamma(1+a\mu)}
\]

Conversely, suppose (8.4.37) holds true, i.e. here after it is to be taken as
\[
W = \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \left[ (1 - \gamma)z^2 \left[ \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \right] + \alpha z \left[ \frac{\int_{q}^{\sigma, \eta, \zeta, \epsilon, \delta, \tau u(z)} \right] \right] \right] \right\} > \alpha \quad (z \in u).
\]

Where,
\[
\left(\begin{array}{c}
a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+\tau)+\eta} = \frac{a\mu}{\lambda}, \zeta \geq 0, \tau \geq 0, \\
\delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}
\end{array}\right)
\]
Corollary 8.4.13 Let \( u(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \), i.e. hereafter it is to be taken as

\[
a_k \leq \frac{1-\alpha[(1-\gamma)q(q-1)+yq+1]}{(1-\alpha[(1-\gamma)(n+q)(n+q+1)+y(n+q+1)+1])\omega_q^\sigma_{\eta,\zeta,\epsilon,\delta,\tau}(n+q)}, \quad k \geq n + q \quad (8.4.38)
\]

Also consider the class \( AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \) s.t.

\[
zf^r(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n).
\]

Theorem 8.4.6 The function \( u \in ABS(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \) if & only if

\[
\sum_{k=n+q}^{\infty} k(1-\alpha (\sigma(\eta + \zeta) + 1) - \alpha (k-1) (2 - \alpha + (k-2) [(1 - \alpha\mu)])
\times \omega_q^\sigma_{\eta,\zeta,\epsilon,\delta,\tau}(k)a_k z^k
\leq q[1 - \alpha(1 - \alpha\mu)(p - 1) + \alpha\mu q + 1].
\]

Where it is obviously,

\[
0 \leq \alpha < \frac{1}{\alpha((1-\alpha\mu)q(q-1)+\alpha\mu q+1)}, \quad 0 \leq \alpha \mu < \frac{1}{2}.
\]

Where it is obviously,

\[
\begin{aligned}
a_{2k} &\geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \zeta)}{\epsilon(\delta + \tau)+q} = \frac{\alpha\mu}{\lambda}, \quad \square \geq 0, \tau \geq 0, \\
\delta &\geq 0, 0 \leq \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{aligned}
\]

And \( \omega_q^\sigma_{\eta,\zeta,\epsilon,\delta,\tau}(k) \) as defined in (8.4.34).

Corollary 8.4.14 Let \( u(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \), i.e. hereafter it is to be taken as

\[
a_k \leq \frac{q[1-\alpha((1-\gamma)q(q-1)+yq+1)]}{(n+q)[1-\alpha((1-\alpha\mu)(n+q)(n+q+1)+\alpha\mu(n+q+1))]}\omega_q^\sigma_{\eta,\zeta,\epsilon,\delta,\tau}(n+q)
\]

\[
\leq \frac{q[z^2-q+2]}{(n+q)[2-((q+n)^2+2)]}\omega_q^\sigma_{\eta,\zeta,\epsilon,\delta,\tau}(n+q).
\]

Where,

\[
\begin{aligned}
a_{2k} &\geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta + \zeta)}{\epsilon(\delta + \tau)+q} = \frac{\alpha\mu}{\lambda}, \xi \geq 0, \tau \geq 0, \\
\delta &\geq 0, 0 \leq \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{aligned}
\]
Theorem 8.4.7 Let \( u \in AB(\sigma, \eta, \varsigma, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \) i.e. here after it is to be taken as

\[
|z|^q - \frac{[1-\alpha[1-\gamma](q-1)+\gamma q+1]}{1-\alpha[(1-\gamma)(q+1)+\gamma(q+1)+1]} z^{n+q}
\]

\[
\leq \left| J_{q}^{\sigma, \eta, \varsigma, \epsilon, \delta, \tau} u(z) \right| \leq |z|^q + \frac{[1-\alpha[1-\gamma](q-1)+\gamma q+1]}{1-\alpha[(1-\gamma)(q+1)+\gamma(q+1)+1]} z^{n+q}
\]

(8.4.39)

\[\therefore q|z|^{q-1} - \frac{(q+n)[1-\alpha[1-\gamma](q-1)+\gamma q+1]}{1-\alpha[(1-\gamma)(q+1)+\gamma(q+1)+1]} |z|^{n+p-1}
\]

\[\leq \left| J_{q}^{\sigma, \eta, \varsigma, \epsilon, \delta, \tau} u(z) \right| \leq q|z|^{q-1} + \frac{(q+n)[1-\alpha[1-\gamma](q-1)+\gamma q+1]}{1-\alpha[(1-\gamma)(q+1)+\gamma(q+1)+1]} |z|^{n+q-1}
\]

(8.4.40)

Where,

\[ v, \epsilon, \delta, \tau \in R, \ z \in u. \]

& \[ \Omega_{q}^{\sigma, \eta, \varsigma, \epsilon, \delta, \tau} (n + q) = \frac{\Gamma(n+1+\lambda-q)\Gamma(n+v+2+\lambda-q-\alpha \mu)\Gamma(n+v+2)}{\Gamma(n+1)\Gamma(n+v+2+\lambda-q-\alpha \mu)\Gamma(n+v+2)\Gamma(1+\lambda-q)}. \]

Proof For \( u \in AB(\sigma, \eta, \varsigma, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \), we have

\[ \sum_{k=n+q}^{\infty} a_{k} \Omega_{q}^{\sigma, \eta, \varsigma, \epsilon, \delta, \tau} (k) \]

\[ \leq \frac{[1-\alpha[1-\gamma](q-1)+\gamma q+1]}{1-\alpha[(1-\gamma)(q+1)+\gamma(q+1)+1]} |z|^q + \sum_{k=n+q}^{\infty} a_{k} \Omega_{q}^{\sigma, \eta, \varsigma, \epsilon, \delta, \tau} (k) \]

\[ \leq \frac{[1-\alpha[1-\gamma](q-1)+\gamma q+1]}{1-[(1-\gamma)(n+q+1)+\gamma(n+q)+1]} |z|^n+q \]

And

\[ \left| J_{q}^{\sigma, \eta, \varsigma, \epsilon, \delta, \tau} u(z) \right| \]
\[ \geq |z|^q - \frac{(1-\xi[(1-\gamma)q(q-1)+\gamma q+1])}{1+\xi((1-\gamma)(n+q)(n+q-1)+\gamma(n+q)+1)} |z|^{n+q}. \]

Where it is obviously for all,
\[
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(n+\xi)}{\lambda}, \xi \geq 0, \tau \geq 0, \delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \right).
\]

Similarly, we can prove the relation (8.4.40).

**Theorem 8.4.8** Let us assume that
\[ f_i(z) \in AB(\square, \eta, \zeta, \varepsilon, \delta, \gamma, \alpha, q, n) \]
\[ \therefore (f_1 * f_2 * \cdots * f_l)(z) \in AB(\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \alpha, q, n) \]
And
\[ 0 \leq \xi < \frac{1}{(1-\gamma)q(q-1)+\gamma q+1} - \frac{n}{T_1(n+q, I)}. \]

Where it is obviously,
\[ f_i(z) = z^q - \sum_{k=n+q}^{\infty} a_{k,i} z^k \quad (i = 1, 2, \cdots, \ell \in N). \]
\[ T_1(n + q, \ell) = \prod_{i=1}^{\ell} \frac{1-\alpha_i[(1-\gamma)q(n+q-1)+\gamma(n+q)+1]}{[1-\alpha_i((1-\gamma)q(q-1)+\gamma q+1)]} \]
\[ \Omega_{q}^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \tau}(n + q) - 1 \quad \text{For } 0 \leq \alpha_i < \frac{1}{(1-\gamma)q(q-1)+\gamma q+1}. \]

**Proof** Result is verified for \( \ell = 1 \) & \( \ell = 2 \)
\[
\sum_{k=n+q}^{\infty} \frac{(1-\alpha_1[(1-\gamma)q(k-1)+\gamma k+1])\Omega_{q}^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \tau}(k)}{1-\alpha_1[(1-\gamma)q(q-1)+\gamma q+1]} a_{k,1} \leq 1. \]
\[
\sum_{k=n+p}^{\infty} \frac{(1-\alpha_2[(1-\gamma)q(k-1)+\gamma k+1])\Omega_{q}^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \tau}(k)}{1-\alpha_2[(1-\gamma)q(q-1)+\gamma q+1]} a_{k,2} \leq 1
\]
By inequality given by Cauchy- Schwarz we get the following result
\[
\sum_{k=n+p}^{\infty} \left( \prod_{i=1}^{\ell_2} \frac{(1-\alpha_i[(1-\gamma)q(k-1)+\gamma k+1])}{1-\alpha_i[(1-\gamma)q(q-1)+\gamma q+1]} a_{k,i} \right)^2 \Omega_{p}^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \tau}(k) \leq 1.
\]
We will try to obtain \( \xi \) so that
\[
\sum_{k=n+q}^{\infty} \frac{(1-\xi[(1-\gamma)q(k-1)+\gamma k+1])\Omega_{q}^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \tau}(k)}{1-\xi[(1-\gamma)q(q-1)+\gamma q+1]} a_{k,1} a_{k,2} \leq 1
\]
\[ \text{S. t. } \frac{(1-\xi[(1-\gamma)q(k-1)+\gamma k+1])}{(1-\xi[(1-\gamma)q(q-1)+\gamma q+1])} \sqrt{a_{k,1} a_{k,2}} \]
Consequently, we will obtain $\xi$ so that
\[
\frac{[1-\xi[(1-\gamma)k(k-1)+\gamma k+1]]}{[1-\xi[(1-\gamma)q(q-1)+\gamma q+1]]} \leq \Omega_q^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} \prod_{i=1}^2 \frac{[1-\alpha_i[(1-\gamma)k(k-1)+\gamma k+1]]}{[1-\alpha_i[(1-\gamma)q(q-1)+\gamma q+1]]}.
\]
Thus \((f_1 * f_2)(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n),\) for
\[
0 < \xi \leq \frac{1}{(1-\gamma)q(q-1)+\gamma q+1} - \frac{n}{T_2(n+q)}, \text{ where}
\]
\[
T_2(k) = \Omega_q^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} \prod_{i=1}^2 \frac{1-\alpha_i[(1-\gamma)k(k-1)+\gamma k+1]}{1-\alpha_i[(1-\gamma)q(q-1)+\gamma q+1]} - 1.
\]
So for $k \geq n + q$ we get
\[
0 < \xi \leq \frac{1}{(1-\gamma)q(q-1)+\gamma q+1} - \frac{n}{T_1(n+q)},
\]
\[
= f_q^{\sigma,\eta,\zeta,\epsilon,\delta,\tau}(n + q) \prod_{i=1}^2 \frac{1-\alpha_i[(1-\gamma)q(q-1)+\gamma q+1]}{1-\alpha_i[(1-\gamma)q(q-1)+\gamma q+1]} - 1.
\]

\[
\forall p \in N. \text{ I.e. here after it is to be taken as we must show that } (f_1 * f_2 * \cdots * f_\ell)(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n),
\]
Where it is obvious for, \(0 < \xi \leq \frac{1}{(1-\gamma)q(q-1)+\gamma q+1} - \frac{n}{M_1(n+q, \ell+1)}\)
And
\[
M_1(n+q, \ell+1) = \left\{ 1 - \xi[(1-\gamma)(n+p)(n+q-1) + \gamma(n+q) + 1] \right\}
\times \left\{ 1 - \xi[(1-\gamma)q(q-1) + \gamma q+1] \right\} - 1.
\]
I.e. here after it is to be taken
\[
(f_1 * f_2 * \cdots * f_\ell)(z) = z^q - A_{n+q} z^{n+q}.
\]
Where it is obvious for,
\[
A_{n+q} = \prod_{i=1}^\ell \frac{[1-\alpha_i[(1-\gamma)q(q-1)+\gamma q+1]]}{1-\alpha_i[(1-\gamma)(n+q)(n+q-1) + \gamma(n+q) + 1]} \frac{1}{\alpha_i^{\sigma,\eta,\zeta,\epsilon,\delta,\tau}(n+q)} - 1
\]
\[
0 \leq \alpha_i < \frac{1}{(1-\gamma)q(q-1)+\gamma q+1}.
\]
Hence theorem is proved. Moreover
\[
f_i(z)
\]
\[ z^n - \frac{(1-\alpha_i[(1-x)q(q-1)+yq+1])}{(1-\alpha_i[(1-x)(n+q)(n+q-1)+y(n+q)+1])q(n+p)} z^{n+p}, \]

Similarly we can prove the result for \( AB \in S(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \xi, q, n) \) in next Theorem.

Where,
\[
\left( \begin{array}{c}
\alpha_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+\tau)+q} = \frac{a_{\mu}}{\lambda}, \zeta \geq 0, \tau \geq 0,
\delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{array} \right)
\]

**Theorem 8.4.9** If \( f_i(z) \in ABS(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha_i, q, n) \) for each \( (i = 1, 2, \ldots, \ell) \)
then, \((f_1 \ast f_2 \ast \ldots \ast f_\ell)(z) \in ABS(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n)\)
\[
\therefore 0 < \beta \leq \frac{1}{(1-\alpha_{q})q(q-1)+a_{q}q+1} - \frac{n}{r_{2}(n+q, \ell)}
\]
\[
T_2(n + q, \ell) = \prod_{i=1}^{\ell} \frac{(n+q)\sigma_{q}^{\eta, \zeta, \epsilon, \delta, \tau} (n+q)}{q(1-\alpha_q[(1-x)q(q-1)+yq+1])}
\]
\[
\times \{1 - \alpha_i[(1-x)(n+q)(n+q-1)+y(n+q)+1]\} - 1.
\]
I. e. here after it is to be taken as for the functions denoted by \( f_i(z) \forall (i = 1, 2, \ldots, \ell) \) where it is obviously,
\[
f_i(z) = z^n - \frac{(1-\alpha_i[(1-x)q(q-1)+yq+1])}{(1-\alpha_i[(1-x)(n+q)(n+q-1)+y(n+q)+1])q(n+p)} z^{n+p}.
\]
Where,
\[
\left( \begin{array}{c}
\alpha_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\epsilon(\delta+\tau)+q} = \frac{a_{\mu}}{\lambda}, \zeta \geq 0, \tau \geq 0,
\delta \geq 0, 0 < \epsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{array} \right)
\]
Put \( \alpha_i = \infty \forall (i = 1, 2, \ldots, \ell) \) in Them. 8.4.8, we get

**Corollary 8.4.15** If \( f_i(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n) \forall (i = 1, 2, \ldots, \ell \in N)\)
\[
\therefore (f_1 \ast f_2 \ast \ldots \ast f_\ell)(z) \in ABS(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \beta, q, n).
\]
And
\[
\beta = \frac{1}{(1-\gamma)q(q-1)+yq+1}
\]
\[
-\frac{n}{(1-\alpha(1-x)(n+q)(n+q-1)+y(n+q)+1)]q^{\sigma_{q}^{\eta, \zeta, \epsilon, \delta, \tau} (n+q)} - 1
\]
Where it is obviously,

\[ 0 \leq \alpha < \frac{1}{(1-\gamma)(q(q-1)+\gamma q+1)}. \]

I. e. hereafter it is to be taken as \( f_i(z) \), \( \forall (i = 1, 2, \ldots, \ell \in N) \) is

\[ f_i(z) = \frac{z^q - \frac{(1-\alpha)(1-\gamma)(q(q-1)+\gamma q+1))}{(1-\alpha)((1-\gamma)(q(q-1)+\gamma q+1))\Omega_q^{\gamma, \eta, \zeta, \varepsilon, \delta, \tau}(n+q)}}{z^{n+q}}. \]

Where,

\[ \begin{cases} a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta, \zeta)}{\varepsilon(\delta, \tau) + q} = \frac{a \mu \lambda}{\eta, \xi, \zeta, \delta, \tau} \geq 0, \tau \geq 0, \\ \delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \end{cases} \]

\[ \forall \alpha_i = \alpha \text{ for } (i = 1, 2, \ldots, \ell). \]

**Corollary 8.4.16** If \( f_i(z) \in AB(\sigma, \eta, \xi, \varepsilon, \delta, \gamma, \alpha, q, n) \) i. e. hereafter it is to be taken as for \( (i = 1, 2, \ldots, \ell \in N) \) \( f_1(z) \ast f_2 \ast \ldots \ast f_\ell(z) \in ABS(\sigma, \eta, \xi, \varepsilon, \delta, \gamma, \beta, \beta, q, n) \)

\[ \therefore \beta = \frac{1}{(1-\gamma)(q(q-1)+\gamma q+1)} \]

\[ - \frac{n}{(n+q)((1-\alpha)((1-\gamma)(q(q-1)+\gamma q+1))\Omega_q^{\gamma, \eta, \zeta, \varepsilon, \delta, \tau}(n+q))}. \]

Where,

\[ \begin{cases} a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta, \zeta)}{\varepsilon(\delta, \tau) + q} = \frac{a \mu \lambda}{\eta, \xi, \zeta, \delta, \tau} \geq 0, \tau \geq 0, \\ \delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2} \end{cases} \]

**Theorem 8.4.10** Let \( f_i(z) \) where \( (i = 1, 2, \ldots, \ell \in N) \) is defined as

\[ f_i(z) = \frac{1}{z^q} - \sum_{k=n+q}^{\infty} a_{k,i} \frac{1}{z^k} \in ABS(\sigma, \xi, \varepsilon, \delta, \tau, \gamma, \alpha, q, n). \]

Where \( (i = 1, 2, \ldots, \ell \in N) \) i. e. hereafter it is to be taken as arithmetic mean of \( f_i \) \( (i = 1, 2, \ldots, \ell \in N) \) is defined as given as follows

\[ h(z) = \frac{1}{\ell} \sum_{i=1}^{\infty} f_i(z), \]

It is also in \( AB(\sigma, \eta, \xi, \varepsilon, \delta, \gamma, \alpha, q, n) \) \( (i = 1, 2, \ldots, \ell \in N) \).
Proof By definition of $h(z)$

\[ \begin{align*}
\therefore \ h(z) &= \frac{1}{z} \sum_{i=1}^{\infty} \left( \frac{1}{z^q} - \sum_{k=n+q}^{\infty} a_{k,i} \cdot \frac{1}{z^{k-q}} \right) \\
&= \frac{1}{z^q} - \sum_{k=n+q}^{\infty} \left( \frac{1}{z} \sum_{i=1}^{\infty} a_{k,i} \right) \cdot \frac{1}{z^{k-q}}.
\end{align*} \]

Using Theorem 8.4.5,

\[ \begin{align*}
\sum_{k=n+q}^{\infty} \left\{ 1-\alpha \left[ (1-\gamma)k(k-1) + \gamma k + 1 \right] \Omega_{q}^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} (k) \right\} a_{k,i} \\
= \frac{1}{z} \sum_{i=1}^{\infty} \left( \sum_{k=n+q}^{\infty} \left\{ 1-\alpha \left[ (1-\gamma)k(k-1) + \gamma k + 1 \right] \Omega_{q}^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} (k) \right\} a_{k,i} \\
\leq \frac{1}{z} \sum_{i=1}^{\infty} (1-\alpha) \left[ (1-\gamma)q(q-1) + \gamma q + 1 \right].
\end{align*} \]

I.e. here after it is to be taken as we obtain $h(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n)$.

Where,

\[ \begin{align*}
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\gamma+\epsilon)}{\epsilon(\delta+\tau)+\epsilon} = \frac{a_{\mu}}{\lambda}, \zeta \geq 0, \tau \geq 0, \right) \\
\left( \gamma \geq 0, 0 < \epsilon \leq \frac{1}{z}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{z}, -\frac{1}{z} \leq B < A \leq \frac{1}{z} \right).
\end{align*} \]

Theorem 8.4.11 Let us assume that $u(z) \& v(z) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n)$

\[ \therefore \ h(z) = tu(z) + (1-t)g(z), \ 0 \leq t \geq 1. \]

where

\[ (f_1 * f_2 * \ldots \ * f_l) \in AB(\sigma, \eta, \zeta, \epsilon, \delta, \tau, \gamma, \alpha, q, n). \]

Proof By definition of $h(z)$

\[ h(z) = \frac{1}{z} - \sum_{k=n+q}^{\infty} \left[ ta_{k} + (1-t)b_{k} \right] \cdot \frac{1}{z^{k-q}}, \]

Where it is obvious,

\[ u(z) = \frac{1}{z} - \sum_{k=n+q}^{\infty} a_{k} \cdot \frac{1}{z^{k-q}}, \]

And

\[ v(z) = \frac{1}{z} - \sum_{k=n+q}^{\infty} b_{k} \cdot \frac{1}{z^{k-q}}, \]

\[ (a_{k}, b_{k} \geq 0). \]

Using theorem 8.4.5

\[ \begin{align*}
\sum_{k=n+q}^{\infty} \left\{ 1-\alpha \left[ (1-\gamma)k(k-1) + \gamma k + 1 \right] \Omega_{q}^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} (k) \right\} ta_{k} + (1-t)b_{k} \\
= t \sum_{k=n+q}^{\infty} \left\{ 1-\alpha \left[ (1-\gamma)k(k-1) + \gamma k + 1 \right] \Omega_{q}^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} (k) \right\} a_{k} \\
+ (1-t) \sum_{k=n+q}^{\infty} \left\{ 1-\alpha \left[ (1-\gamma)k(k-1) + \gamma k + 1 \right] \Omega_{q}^{\sigma,\eta,\zeta,\epsilon,\delta,\tau} (k) \right\} b_{k} \leq 1.
\end{align*} \]
Where,
\[
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(n+\zeta)}{\mu} = \frac{\sigma\eta}{\tau}, \zeta \geq 0, \tau \geq 0, \right)
\]
\[
\delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\]
I. e. here after it is to be taken as \(h(z) \in AB(\sigma, \eta, \zeta, \varepsilon, \delta, \tau, \gamma, \alpha, q, n)\).

Letting
\[
P_{\zeta}^q(z) = \frac{1}{z^q} - \sum_{k=n+q}^{\infty} \frac{\Gamma(q+\zeta+k)\Gamma(q+\zeta+k)}{\Gamma(q+\zeta+k)} a_k \frac{1}{z^k}
\]
Where, \(q \geq 0, \zeta > -1\), by referring the reference \([19]\), obtained the thm 8.4.12.

**Theorem 8.4.12** Let \(f \in AB(\sigma, \eta, \zeta, \varepsilon, \delta, \tau, \gamma, \alpha, q, n)\) be defined by \((8.4.32)\) and \(q \geq 0, \zeta > -1\) i. e. here after it is to be taken as \(P_{\zeta}^q(z)\) defined above also contained in \(AB(\sigma, \eta, \zeta, \varepsilon, \delta, \tau, \gamma, \alpha, q, n)\).

**Proof** Thm 8.4.5 gives,
\[
\sum_{k=n+q}^{\infty} \frac{1-\alpha[(1-\gamma)k+1]}{1-\alpha[(1-\gamma)q+1]} \times \Omega_q^{\sigma, \eta, \zeta, \varepsilon, \delta, \tau}(k) \frac{\Gamma(q+\zeta+k)\Gamma(q+\zeta+k)}{\Gamma(q+\zeta+k)} a_k
\]
\[
\leq \sum_{k=n+p}^{\infty} \frac{1-\alpha[(1-\gamma)k+1]}{1-\alpha[(1-\gamma)q+1+\gamma k+1]} \times \Omega_q^{\sigma, \eta, \zeta, \varepsilon, \delta, \tau}(k) a_k \leq 1.
\]
Then for \(k \geq n+p, P_{\zeta}^q(z) \in AB(\sigma, \eta, \zeta, \varepsilon, \delta, \tau, \gamma, \alpha, q, n)\).

Where,
\[
\left( a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(n+\zeta)}{\mu} = \frac{\sigma\eta}{\tau}, \zeta \geq 0, \tau \geq 0, \right)
\]
\[
\delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\]

**Theorem 8.4.13** Let \(P_{\zeta}^q(z)\) be defined, having Taylor series expansion like,
\[
P_{\zeta}^q(z) = \frac{1}{z^q} - \sum_{k=n+q}^{\infty} \frac{\Gamma(q+\zeta+k)\Gamma(q+\zeta+k)}{\Gamma(q+\zeta+k)} a_k \frac{1}{z^k}.
\]
I. e. here after it is to be taken as \( F_\zeta^q(z) \) is a star like Holomorphic (an analytic) function of order \( \beta \) in

\[
|z| \leq R_1(\sigma, \eta, \zeta, \varepsilon, \delta, \gamma, \alpha, \beta, q, n) =
\inf_{k \geq n+q} \left[ \frac{(q-\beta)\Gamma(q+\xi+k)\Gamma(\zeta+q)}{(k-\beta)\Gamma(\xi+k)\Gamma(q+\zeta+q)} \frac{1 - \alpha[(1-\gamma)k(k-1)+\gamma k+1]}{1 - \alpha[(1-\gamma)q(q-1)+\gamma q+1]} \frac{1}{k-p} \right]
\times \left[ \Omega_q^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma}(k) \right]^{1-k-q}
\]

Where,

\[
\begin{align*}
&a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \frac{\sigma(\eta+\zeta)}{\varepsilon(\delta+\gamma)} = \frac{a\mu}{\lambda}, \zeta \geq 0, \tau \geq 0, \\
&\delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B < A \leq \frac{1}{2}.
\end{align*}
\]

**Proof** We have to show that

\[
|z| \leq (q-\beta)\Gamma(q+\xi+k)\Gamma(\zeta+q) \frac{1 - \alpha[(1-\gamma)k(k-1)+\gamma k+1]}{1 - \alpha[(1-\gamma)q(q-1)+\gamma q+1]} \Omega_q^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma}(k).
\]

\[
\therefore |z|^{k-q} \leq \frac{(q-\beta)\Gamma(q+\xi+k)\Gamma(\zeta+q)}{(k-\beta)\Gamma(\xi+k)\Gamma(q+\zeta+q)} \frac{1 - \alpha[(1-\gamma)k(k-1)+\gamma k+1]}{1 - \alpha[(1-\gamma)q(q-1)+\gamma q+1]} \Omega_q^{\sigma, \eta, \zeta, \varepsilon, \delta, \gamma}(k).
\]

Hence theorem proved.