Chapter 6

STUDY OF SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

6.0 Introduction

In this chapter we investigated and discussed a subclass of Univalent (or schlicht i.e. a single valued function) functions

\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^{n+1}, \quad a_{n+1} \geq 0. \]

We obtained some results of Convolutions with negative coefficients. We have also obtained the Coefficient Bounds and discuss for sharpness of the results obtained

6.1 Sub classes of Univalent (or schlicht i.e. a single valued function) Functions with Negative Coefficients

Let \( S \) represents the class of normalized multivalent functions \( f \) given by

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \quad a_n \geq 0. \]

Holomorphic (a regular) in \( D = \{ z ; |z| < 1 \} \). Let \( T \) represents the sub class of \( S \) given by

\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n; \quad a_n \geq 0. \]

The properties of these functions were studied by Silverman and Berman [62], Schild and Silverman [66], S. M. Khairnar and S. R. Kulkarni [28], S. M. Khairnar and Meena More [26], [53]. M. Darus [13], M. Darus [16] studied the same class. Let \( K = \{ w ; \) Holomorphic (a regular) belongs to \( E , w(0) = 0; \ |w(z)| < 1 \) belonging to \( E \} \). Also let \( G (A, B) \) represents the class of the Holomorphic (an analytic) function in \( D \) which is of the form

\[ \frac{1 + Aw(z)}{1 + Bw(z)} \quad -1 \leq A < B \leq 1. \]

Here \( w(z) \) contained in \( K \). Now we are going to study another subclass of \( T \) and we denote it by \( T_1 \), which consists as
Define

\[ S^*(A, B) = \left\{ f \in S \& \frac{zf'}{f} \in G(A, B) \right\}. \]

And

\[ H(A, B) = \left\{ f \in S \& \frac{(zf')'}{f'} \in G(A, B) \right\}. \]

Further we shall define

\[ T_1^*(A, B) = \left\{ f \in H_1 \& z \frac{f'}{f} \in G(A, B) \right\}. \]

And

\[ C_1(A, B) = \left\{ f : f \in H_1 \& \frac{(zf')'}{f'} \in G(A, B) \right\}. \]

Their Convolution is defined by

\[ h(z) = f(z)g(z) \]

\[ h(z) = -\sum_{n=2}^{\infty} a_{n+1}z^{n+1}, \quad a_{n+1} \geq 0, b_{n+1} \geq 0. \]

Silverman and Berman [62] have defined the class \( T_1^*(A, B) \) and obtained some interesting results. In this paper we have obtained some results of Convolution for the class \( T_1^*(A, B) \) and \( C_1(A, B) \). We need following lemmas.

**Lemma**

6.1.1 A function \( f(z) \) as given follows

\[ f(z) = z - \sum_{n=2}^{\infty} a_{n+1}z^{n+1}, \quad a_{n+1} \geq 0. \]
It is in $T_1^*(A, B)$ if & only if 
\[
\sum_{n=2}^{\infty} \frac{n(B + 1) + (B - A)}{B - A} a_{n+1} \leq 1.
\]

**Proof** We have been observed

\[
f(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1}
\]

\[
f'(z) = 1 - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^n
\]

\[
z f'(z) = 1 - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^{n+1}
\]

\[
\frac{zf'(z)}{f(z)} = \frac{z - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^{n+1}}{z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1}} = \frac{1 - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^n}{1 - \sum_{n=2}^{\infty} a_{n+1} z^n} = 1 + Aw(z)
\]

\[
1 - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^n + Bw(z)[1 - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^n] = 1 - \sum_{n=2}^{\infty} a_{n+1} z^n + Aw(z)[1 - \sum_{n=2}^{\infty} a_{n+1} z^n] = \sum_{n=2}^{\infty} a_{n+1} z^n - \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^n
\]

\[
= Aw(z) - Bw(z) + Bw(z) \sum_{n=2}^{\infty} a_{n+1} (n + 1) z^n - Aw(z) \sum_{n=2}^{\infty} a_{n+1} z^n
\]

\[
\sum_{n=2}^{\infty} na_{n+1} z^n = w(z) \left( B - A - \sum_{n=2}^{\infty} (Bn + B - A) a_{n+1} z^n \right)
\]

\[
\therefore w(z) = \frac{\sum_{n=2}^{\infty} na_{n+1} z^n}{(B - A) - \sum_{n=2}^{\infty} (Bn + B - A) a_{n+1} z^n}.
\]

Noting that,

\[
|w(z)| < 1
\]

\[
\left| \frac{\sum_{n=2}^{\infty} na_{n+1} z^n}{(B - A) - \sum_{n=2}^{\infty} (Bn + B - A) a_{n+1} z^n} \right| < 1.
\]
Letting $|z| = r \to 1,$
\[
\sum_{n=2}^{\infty} na_{n+1} \leq \frac{B - A - \sum_{n=2}^{\infty} (Bn + B - A)a_{n+p}}{B - A} \leq 1.
\]

That is
\[
\sum_{n=2}^{\infty} na_{n+1} \leq B - A - \sum_{n=2}^{\infty} (Bn - B - A)a_{n+1}
\]
That is
\[
\sum_{n=2}^{\infty} (n + Bn + B - A)a_{n+1} \leq B - A.
\]
That is
\[
\sum_{n=2}^{\infty} \frac{n(B+1) + (B-A)}{B-A} a_{n+1} \leq 1.
\]

**Lemma 6.1.2** A function $f(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1}$

Where $a_{n+1} \geq 0$ is in $C^1 (A, B)$ iff
\[
\sum_{n=2}^{\infty} \frac{(n+1)[n(1+B) + (1-A)]}{B-A} a_{n+1}
\]

**Proof** Here function is taken like,
\[
f(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1}
\]
\[
f'(z) = 1 - \sum_{n=2}^{\infty} a_{n+1} (n+1) z^n
\]
\[
z f''(z) = z - \sum_{n=2}^{\infty} a_{n+1} (n+1) z^{n+1}
\]

\[
\frac{d}{dz} [zf'(z)] = [zf'(z)]' = 1 - \sum_{n=2}^{\infty} a_{n+1} (n+1)^2 z^n
\]
\[
[z f''(z)]' = \frac{1 - \sum_{n=2}^{\infty} (n+1)^2 a_{n+1} z^n}{1 - \sum_{n=2}^{\infty} (n+1) a_{n+1} z^n} = \frac{1 + Aw(z)}{1 + Bw(z)}
\]
\[
\sum_{n=2}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=2}^{\infty} (n+1)^2 a_{n+1} z^n
\]
\[
= Aw(z) \left[ 1 - \sum_{n=2}^{\infty} (n+1) a_{n+1} z^n \right] - Bw(z) \left[ 1 - \sum_{n=2}^{\infty} (n+1)^2 a_{n+1} z^n \right]
\]
\[
\sum_{n=2}^{\infty} [(n+1) - (n+1)^2] a_{n+1} z^n = w(z) \left[ (A-B) + \sum_{n=2}^{\infty} [B(n+1)^2 - A(n+1)] a_{n+1} z^n \right]
\]

\[w(z) = \frac{\sum_{n=2}^{\infty} n(n+1)a_{n+1} z^n}{(B-A) - \sum_{n=2}^{\infty} (n+1)[B(n+1) - A] a_{n+1} z^n}.\]

Noting that;
\[|w(z)| < 1\]
\[\left| \frac{\sum_{n=2}^{\infty} n(n+1)a_{n+1} z^n}{(B-A) - \sum_{n=2}^{\infty} (n+1)[B(n+1) - A] a_{n+1} z^n} \right| < 1.\]

Letting \[|z| = r \rightarrow 1,\]
\[\frac{\sum_{n=2}^{\infty} n(n+1)a_{n+1}}{(B-A) - \sum_{n=2}^{\infty} (n+1)[B(n+1) - A] a_{n+1}} \leq 1\]
\[\sum_{n=2}^{\infty} n(n+1)a_{n+1} \leq (B-A) - \sum_{n=2}^{\infty} (n+1)[B(n+1) - A] a_{n+1}\]
\[\sum_{n=2}^{\infty} (n+1)[n + B(n+1) - A] a_{n+1} \leq B - A.\]

That is
\[\sum_{n=2}^{\infty} \frac{(n+1)[n(B+1) + (B-A)]}{B-A} a_{n+1} \leq 1.\]

We define,
\[h(z) = f(z) \ast g(z) = \sum_{n=2}^{\infty} a_{n+1} z^{n+1}.\]

Assuming \(f(z)\) and \(g(z)\) to be numbers of the classes \(T^+_1(A,B)\) and \(C_1(A,B)\)

6.2 Analysis

**Theorem 6.2.1** If a function \(f(z)\) as given follows  
\[f(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1}\]  And  
\[g(z) = z - \sum_{n=2}^{\infty} b_{n+1} z^{n+1}.\]
Where, \( a_{n+1} \geq 0 \) and \( b_{n+1} \geq 0 \), be the elements of the classes \( T_1^*(A, B) \& C_1(A, B) \) i.e. here after it is to be taken as

\[
h(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_{n+1} b_{n+1} z^{n+1}.
\]

It is an element of \( T_1^*(A, B) \) with \(-1 \leq A_i < B_i \leq 1\)

\[
A_i \leq 1 - 2k \quad \text{And} \quad B_i \geq \frac{A_i + k}{1 - k}.
\]

And that bounds to \( A_i, B_i \) be sharp

**Proof** Applying lemma numbered 6.2.1 we obtained

\[
\sum_{n=2}^{\infty} \frac{n(B+1) + (B-A)}{B-A} a_{n+1} \leq 1
\] (6.2.1)

\[
\sum_{n=2}^{\infty} \frac{n(B+1) + (B-A)}{(1+B) - (1+A)} b_{n+1} \leq 1
\] (6.2.2)

We intend to find values of \( A_i, B_i \) s.t. \(-1 \leq A_i < B_i \leq 1\)

For \( h(z) = f(z) * g(z) \in T_1^*(A_i, B_i) \)

Equivalently we intend to find \( A_i, B_i \) s.t.

\[
\sum_{n=2}^{\infty} \frac{n(1+B) + (B-A)}{B-A} a_{n+1} b_{n+1} \leq 1.
\] (6.2.3)

On combining Equations (6.2.1) and (6.2.2) & using Cauchy Schwarz inequality we obtain

\[
\sum_{n=2}^{\infty} u \sqrt{a_{n+1} b_{n+1}} \leq \left( \sum_{m=2}^{\infty} u a_{m+1} \right)^{\frac{1}{2}} \left( \sum_{m=2}^{\infty} u b_{m+1} \right)^{\frac{1}{2}} \leq 1.
\] (6.2.4)

Where we have assumed,

\[
u = \frac{m(1+B) + (B-A)}{B-A}.
\]

Inequality (6.2.3) is true if \( u_i a_{m+1} b_{m+1} \leq u \sqrt{a_{m+1} b_{m+1}}. \)

Where it is obviously for,

\[
u_i = \frac{m(1+B_i) + (B_i-A_i)}{B_i-A_i}.
\] (6.2.5)
For \( m = 2, 3, 4 \ldots \)

That is if \( u \sqrt{(a_{m+1}b_{m+1})} \leq u \) from (6.2.5). But from Equation (6.2.1) we have

\[
\sqrt{(a_{m+1}b_{m+1})} \leq \frac{1}{u}.
\]

Therefore it is enough to find \( u \) s. t.

\[
\frac{1}{u} \leq \frac{u}{u_i} \quad \therefore u_i \leq u^2.
\]

\[
\frac{m(1+B_i)+(B_i-A_i)}{B_i-A_i} \leq \frac{m(1+B)+(B-A)}{B-A} \cdot \frac{m(1+B)+(B-A)}{B-A} = u^2 \quad (6.2.6)
\]

\[
m(1+B_i)+(B_i-A_i) \leq u^2(B_i-A_i)
\]

\[
A_1 \leq \frac{b_i(u^2-1)-m(1+B_i)}{u^2-1}.
\]

It is easy to verify and check that \(-1 \leq A < B \leq 1\).

Where we take,

\[
f(z) = g(z) = z - \frac{B-A}{1+2B-A} z^3 \in T_1^+(A,B).
\]

It follows that

\[
h(z) = f(z) \ast g(z) = z - \left[ \frac{B-A}{1+2B-A} \right]^2 z^3.
\]

I. e. here after it is to be taken as from Equation (6.2.6)

\[
\frac{m(1+B_i)}{B_i-A_i} + 1 \leq \alpha
\]

\[
\frac{(1+B_i)m}{B_i-A_i} \leq \alpha - 1
\]

\[
\frac{(1+2B_i-A_i)}{B_i-A_i} = \left[ \frac{(1+B)-(B-A)}{B-A} \right]^2.
\]

It showing that

\[
h(z) \in T_1^+(1-2k,l)
\]

With constant given by

\[
k = \frac{1}{\alpha - 1} = \frac{(B-A)(B'-A')}{(1+2B-A)(1+2B'-A') - (B-A)(B'-A')}.
\]
Theorem 6.2.2 If \( f(z) \in T^*_1(A, B) \) and \( g(z) \in T^*_1(A', B') \) e. here after it is to be taken as

\[
f(z) \ast g(z) \in T^*_1(A, B)
\]

Where,

\[
A_i \leq 1 - 2k \quad \text{And} \quad B_i \geq \frac{A_i + k}{1 - k}.
\]

With

\[
k = \frac{1}{\alpha - 1} = \frac{(B - A)(B' - A')}{(1 + 2B - A)(1 + 2B' - A') - (B - A)(B' - A')}.
\]

Proof By theorem 6.2.1 we have

\[
\frac{m(1 + B_1) + (B_1 - A_1)}{B_1 - A_1} \leq \frac{m(1 + B) + (B - A)}{B - A}, \quad \frac{m(1 + B') + (B' - A')}{B' - A'} = \alpha
\]

\[
\frac{m(1 + B_1)}{B_1 - A_1} + 1 \leq \alpha
\]

\[
\frac{1 + B_1}{B_1 - A_1} \leq \frac{\alpha - 1}{m}
\]

\[
\frac{B_1 - A_1}{1 + B_1} \geq \frac{m}{\alpha - 1}.
\]  \hspace{1cm} (6.2.7)

The function \( \frac{m}{\alpha - 1} \) is decreasing with increasing \( m \). From above Equation (6.2.7) it is clear that

\[
\frac{B_1 - A_1}{1 + B_1} \geq k
\]  \hspace{1cm} (6.2.8)

\[
1 > k = \frac{1}{\alpha - 1} = \frac{(B - A)(B' - A')}{(1 + 2B - A)(1 + 2B' - A') - (B - A)(B' - A')}.
\]

Fixing \( A_1 \) in Equation (6.2.8) we get \( B_1 - A_1 \geq k + kB_1 \)

\[
B_1 \geq \frac{A_1 + k}{1 - k}. \quad \text{We require} \quad B_1 \leq 1.
\]

We immediate obtain

\[
A_i \leq 1 - 2k.
\]
If we take 

\[ f(z) = z^3 - \frac{B - A}{1 + 2B - A} \] \( z \in T_1^+(A, B) \)

\[ g(z) = z^p = \frac{B' - A}{1 + 2B' - A} \] \( z^3 \in T_1^+(A', B') \).

We see that 

\[ h(z) = f(z) \ast g(z) = z^3 - \left( \frac{(B - A)(B' - A')}{(1 + 2B - A)(1 + 2B' - A')} \right) \] .

I. e. here after it is to be taken as 

\[ \frac{(1 + B_1) + (B_1 - A_1)}{B_1 - A_1} = \frac{(1 + 2B - A)(1 + 2B' - A')}{(B - A)(B' - A')} \].

When \( A_i = 1 - 2k \) and \( B_i = 1 \). Showing that our result is best possible

**Theorem 6.2.3** If \( f(z) \in C_1(A, B) \) and \( g(z) \in C_1(A', B') \).

I. e. here after it is to be taken as \( f(z) \ast g(z) \in C_1(A_i, B_i) \) where 

\[ k = \frac{1}{\alpha - 1} = \frac{(B - A)(B' - A')}{(1 + 2B - A)(1 + 2B' - A') - (B - A)(B' - A')} . \] (6.2.9)

\[ A_i = 1 - 2k \quad \text{And} \quad B_i \geq \frac{A_i + k}{1 - k} . \]

The above result is best possible

**Remarks** Choosing 

\[ f(z) = z^p - \frac{B - A}{2(1 + 2B - A)} \] \( z \in C_1(A, B) \)

\[ g(z) = z^p - \left( \frac{B' - A}{2(1 + 2B' - A')} \right) \] \( z^3 \in C_1(A', B') \).

Where, 

\( A_i = 1 - 2k \) And \( B_i = 1 \).

\[ : k = \frac{1}{\alpha - 1} = \frac{(B - A)(B' - A')}{(1 + 2B - A)(1 + 2B' - A') - (B - A)(B' - A')} . \]

We can show that our Estimates in above theorem are sharp.

**Theorem 6.2.4** If \( f(z) \in T_1^+(A, B) \) and \( g(z) \in T_1^+(A', B') \) i. e. here after it is to be taken as 

\[ h(z) = f(z) \ast g(z) \in C_1(A, B) . \]
Where it is obviously for all, \( A_i = 1 - 2k \) and \( B_i \geq \frac{A_i + k}{1 - k} \).

\[
\therefore k = \frac{1}{\alpha - 1} = \frac{2(B - A)(B' - A')}{(1 + 2B - A)(1 + 2B' - A') - 2(B - A)(B' - A')}.
\]

The result is best possible.

**Remarks**

\[
f(z) = z - \left[ \frac{B - A}{1 + 2B - A} \right] z^3 \in T^*_1(A', B')
\]

\[
g(z) = z^p - \left[ \frac{B' - A'}{1 + 2B' - A'} \right] z^3 \in T^*_1(A', B').
\]

This shows that our bounds are best possible.

**Theorem 6.2.5** If

\[
f(z), g(z) \in T^*_1(A, B).
\]

I. e. here after it is to be taken as

\[
h(z) = z - \sum_{n=2}^{\infty} \left[ (a_{n+1})^2 + (b_{n+1})^2 \right] z^n \in T_1(A, B).
\]

Where it is obviously for all \( A_i = 1 - 2k \) and \( B_i \geq \frac{A_i + k}{1 - k} \).

With

\[
k = \frac{2(B - A)^2}{(1 + 2B - A)^2 - 2(B - A)^2}.
\]

Our result is best possible.

**Proof** Since \( f(z), g(z) \in T^*_1(A, B) \)

\[
\sum_{m=2}^{\infty} \frac{m(1 + B) + (B - A)}{B - A} a_{m+1} \leq 1 \quad (6.2.10)
\]

\[
\sum_{m=2}^{\infty} \frac{m(1 + B) + (B - A)}{B - A} b_{m+1} \leq 1 \quad (6.2.11)
\]

\[
\sum_{m=2}^{\infty} \left[ \frac{m(1 + B) + (B - A)}{B - A} a_{m+1} \right]^2 \leq 1 \quad (6.2.12)
\]
Comparing above Equations (6.2.13) and (6.2.14) we say that

\[
\frac{m(1 + B_i) + (B_i - A_i)}{B_i - A_i} \leq \frac{1}{2} \left[ \frac{m(1 + B) + (B - A)}{B - A} \right] = \frac{u^2}{2}. \tag{6.2.15}
\]

On simplification Equation gives us

\[
\frac{m(1 + B_i) + (B_i - A_i)}{B_i - A_i} \leq \frac{u^2}{2} \quad \text{and} \quad \frac{(1 + B_i)m}{B_i - A_i} + 1 \leq \frac{u^2}{2}.
\]

On simplification we get \(2m(1 + B_i) \leq (B_i - A_i)(u^2 - 2)\)

\[
\frac{B_i - A_i}{1 + B_i} \geq \frac{2m}{u^2 - 2} = \beta(m). \tag{6.2.16}
\]

Here \(\beta(m)\) is a decreasing function of \(m\). On putting \(m = 1\) in Equation (6.2.16) we get

\[
\frac{B_i - A_i}{1 + B_i} \geq \frac{2}{1 + 2B - A} = \frac{2(B - A)^2}{(1 + 2B - A)^2} = k. \tag{6.2.17}
\]

Keeping \(A_i\) fixed in Equation-(17) we get \(\frac{B_i - A_i}{1 + B_i} \geq k, \quad B_i - A_i \geq k(1 + B_i),\)

\[B_i \geq \frac{A_i + k}{1 - k} \quad \text{and} \quad B_i \leq 1 \Rightarrow A_i < 1 - 2k.\]

The function

\[f(z) = g(z) = z - \left[ \frac{B - A}{1 + 2B - A} \right] z^3.\]

This shows the best possibility of result.