Chapter 3

Mate Functions in Near Subtraction Semigroups

This chapter contains two sections. The first section deals with the mate functions in a (right) near subtraction semigroup $X$. In Ring Theory a ring $(R, +, \cdot)$ is said to be Von-Neumann regular [48], if for every $a$ in $R$ there exists $b \in R$ such that $a = aba$. S. Suryanarayanan and N. Ganesan in their paper “Pseudo-Stable near rings”, Indian J. Pure and Appl. Math 19(12) (December, 1988) 1206 – 1216 introduced the concept of mate functions in a near ring with view to handling the regularity structure with considerable ease. Motivated by this, using some axiom of choice, we introduce the concept of mate functions in a near subtraction semigroup.

A function $f : X \to X$ is called a mate function for $X$, if $a = af(a)a$ for all $a$ in $X$. If, in addition, $f(a)af(a) = f(a)$ for all $a$ in $X$ then $f$ is called a mutual mate function for $X$. We show that if $X$ admits a mate function then it certainly has a mutual mate function. We establish that $X$ possesses a unique mutual mate function if and only if $E \subseteq C(E)$. We derive some properties of $X$, when $X$ has a unique mutual mate function.

In the second section, we define a near subtraction semigroup $X$ to be $P_1$ if $aX = aXa$ for every $a$ in $X$. We also discuss certain properties of a zero symmetric $P_1$ near subtraction semigroup with a mate function. We also obtain a characterisation of $P_1$ near subtraction semigroups. Some of the results in

3.1 Mate Functions

We shall now give the definition of a mate function in a near subtraction semigroup and illustrate this concept with suitable examples. We discuss some properties of ‘mates’. We also obtain characterisations of mate functions.

Definition 3.1.1. Let $\mathbb{B}$ be a near subtraction semigroup. If there exists a map $f: \mathbb{B} \rightarrow \mathbb{B}$ such that $a = af(a)a$ for all $a$ in $\mathbb{B}$, we call $f'$ a mate function for $\mathbb{B}$. $f(a)$ is called a mate of $a$.

Definition 3.1.2. We say that $\mathbb{B}$ is an

(i) $S$-near subtraction semigroup if $a \in Xa$ for all $a \in X$

(ii) $S'$-near subtraction semigroup if $a \in aX$ for all $a \in X$

(iii) $SS'$-near subtraction semigroup if it is both an $S$-near subtraction semigroup and an $S'$-near subtraction semigroup.

Examples 3.1.3. (a) We consider the near subtraction semigroup $(X, -, \cdot)$ with $X = \{0, a, b, 1\}$ where we define ‘$-$’ and ‘$\cdot$’ as follows:

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The map $f : X \rightarrow X$ defined by $f(0) = 0$, $f(a) = a$; $f(b) = b$; $f(1) = 1$ is a mate function for $X$. 
(b) Let $X = \{0, a, b, c\}$ in which \(\cdot\) and \(\cdot\) are defined by

\[
\begin{array}{ccc}
- & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & b & 0 & b & b \\
c & c & c & 0 & c \\
\end{array}
\quad
\begin{array}{ccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & 0 & 0 & b \\
c & 0 & a & b & c \\
\end{array}
\]

This near subtraction semigroup has no mate function since $b$ has no mate $[bf(b) \neq b]$.

(c) We consider the near subtraction semigroup $(X, \cdot)$ where $X = \{0, a, b, c\}$ in which \(\cdot\) and \(\cdot\) are defined by

\[
\begin{array}{ccc}
- & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & c & b \\
b & b & 0 & 0 & b \\
c & c & 0 & c & 0 \\
\end{array}
\quad
\begin{array}{ccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & 0 & 0 & b \\
c & 0 & a & b & c \\
\end{array}
\]

Then $(X, \cdot)$ is an $S$ near subtraction semigroup but not $S'$.

(d) Let $X = \{0, a, b, 1\}$ in which \(\cdot\) and \(\cdot\) are defined as follows:

\[
\begin{array}{ccc}
- & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 1 & b \\
b & b & 0 & 0 & b \\
1 & 1 & 0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
\cdot & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & a & 0 & 1 & b \\
1 & 0 & a & 1 & b \\
\end{array}
\]

Then $(X, \cdot)$ is an $SS'$ near subtraction semigroup.

**Theorem 3.1.4.** Let $X$ be an $S$-near subtraction semigroup. Let $f$ be a map from $X$ into $X$. Then the following statements are equivalent.

(i) $f$ is a mate function for $X$. 

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(ii) $f(a)a$ is an idempotent and $Xa = Xf(a)a$ for all $a \in X$.

(iii) For every pair of principal $X$-systems $Xa$, $Xb$ and for every $X$-homomorphism, $g : Xa \to Xb$ we have $g(xa) = xag(f(a)a)$ for all $x \in X$.

Proof. (i) $\Rightarrow$ (ii): $(f(a)a)^2 = (f(a)a)(f(a)a) = f(a)(af(a)a) = f(a)a$ for all $a \in X$. Therefore, $f(a)a \in E$ for every $a \in X$. Also, $Xa = Xaf(a)a \subseteq Xf(a)a \subseteq Xa$. Hence, $Xa = Xf(a)a$ for all $a \in X$.

(ii) $\Rightarrow$ (iii): Since $Xa = X(f(a)a)$, we observe that for every $x \in X$, there is some $n$ in $X$, such that $xa = nf(a)a$. For $a, b$ in $X$ we consider an $X$-homomorphism $g : Xa \to Xb$. Obviously, then we have $g(xa) = g(nf(a)a) = g(nf(a)af(a)a) = nf(a)ag(f(a)a) = xag(f(a)a)$ and (iii) follows.

(iii) $\Rightarrow$ (i): In (iii) we take $b = a$, $g$ to be the identity $X$-homomorphism and $x$ be such that $xa = a$. This is possible since $X$ is an $S$-near subtraction semigroup. We get $a = xa = g(xa) = g(xaf(a)a) = xag(f(a)a) = xaf(a)a = af(a)a$ and (i) follows.

In the following result we give a simple characterisation of mate functions

**Theorem 3.1.5.** Let $X$ be an $S'$-near subtraction semigroup. Then a map $f : X \to X$ is a mate function for $X$ if and only if $af(a) \in E$ and $aX = af(a)X$ for every $a$ in $X$.

Proof. For the ‘only if’ part assume ‘$f$’ is a mate function for $X$. Then $(af(a))^2 = (af(a))af(a) = (af(a))f(a) = af(a)$. Therefore $af(a) \in E$. Also $aX = af(a)aX \subseteq af(a)X \subseteq aX$. Hence $aX = af(a)X$, for all $a \in X$. 

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For the ‘if’ part, let \( a \in X \). Since \( X \) is an \( S' \) near subtraction semigroup, \( a \in aX \). Also \( aX = af(a)X \). Then there exists some \( n \) in \( X \) such that \( a = af(a)n \). But \( af(a) \in E \). Therefore \( af(a)a = af(a)(af(a)n) = (af(a))^2n = af(a)n = a \). Hence \( f \) is a mate function for \( X \). \( \square \)

We shall now define the concept of a mutual mate function in a near subtraction semigroup

**Definition 3.1.6.** We say that, a mate function \( f : X \to X \) is a mutual mate function if \( x \) is also a mate of \( f(x) \) for every \( x \) in \( X \). We refer to each of \( x \) and \( f(x) \) as mutual mates of each other. If a mutual mate function happens to be an involution, we call it an involutary mate function for \( X \).

**Lemma 3.1.7.** If \( X \) admits a mate function ‘\( f \)’ then it certainly has a mutual mate function.

**Proof.** Let us define \( g : X \to X \) such that \( g(x) = f(x)xf(x) \) for every \( x \) in \( X \).

\[
xg(x)x = x(f(x)xf(x))x = xf(x)(xf(x)x) = xf(x)x = x
\]

and this guarantees that \( g \) is a mate function for \( X \).

Further \( g(x)gx = (f(x)xf(x))xf(x) = (f(x)x)(f(x)x)(f(x)x)f(x) = (f(x)x)^3f(x) \) [since \( f(x)x \in E \) = \( f(x)xf(x) = g(x) \). This guarantees that \( g \) is a mutual mate function for \( X \). \( \square \)

We omit the proofs of the following two corollaries as they are immediate consequences of Theorems 3.1.4 and 3.1.5

**Corollary 3.1.8.** Let \( X \) be an \( SS' \) near subtraction semigroup. Then the following statements are equivalent.
(i) \( f \) is a mate function for \( X \)

(ii) \( f(a)a \in E \) and \( Xa = Xf(a)a \) for every \( a \in X \)

(iii) \( af(a) \in E \) and \( aX = af(a)X \) for every \( a \in X \)

(iv) For every pair of principal \( X \)-systems \( X, Xb \) and for every \( X \)-homomorphism \( g : Xa \to Xb, g(xa) = xag(f(a)a) \) for all \( x \in X \).

**Corollary 3.1.9.** Let \( X \) be an \( SS' \) near subtraction semigroup. Then the following statements are equivalent.

(i) \( f \) is a mutual mate function for \( X \)

(ii) as in Corollary 3.1.8

(iii) as in Corollary 3.1.8

(iv) \( f(a)X = f(a)aX \)

(v) \( Xf(a) = Xaf(a) \) for every \( a \) in \( X \).

**Theorem 3.1.10.** Let \( X \) be a nil near subtraction semigroup with a mate function \( f \) and let \( g : X \to X \) be such that \( g(x) = f(x)[xf(x) - x^{k-1}] \) for every \( x \) in \( X \) where \( k \) (depending on \( x \)) is some definite integer \( > 1 \) such that \( x^k = 0 \). Then \( g \) is a mate function for \( X \). If \( f \) is a mutual function for \( X \), then so is \( g \).

**Proof.** For every \( x \) in \( X \), we have \( xg(x)x = xf(x)[xf(x) - x^{k-1}]x = xf(x)[xf(x)x - x^k] = xf(x)[x - 0] = xf(x)x = x \). Therefore \( g \) is mate function for \( X \).

Suppose \( f \) is a mutual mate function for \( X \). Then for every \( x \) in \( X \), \( g(x)xg(x) = f(x)[xf(x) - x^{k-1}]xg(x) = f(x)[xf(x)x - x^k]g(x) = f(x)[xf(x)x - 0]g(x) = f(x)xg(x) = f(x)xf(x)[xf(x) - x^{k-1}] = f(x)[xf(x) - x^{k-1}] = g(x) \). Hence \( g \) is a mutual mate function for \( X \). \( \square \)
As an immediate consequence we have the following corollary

**Corollary 3.1.11.** If $X$ is a near subtraction semigroup with a mutual mate function $f$ and if $x^2 = 0$ for some $x$ in $X$, then the element $f(x)(xf(x) - x)$ is a mutual mate of $X$.

**Proof.** It follows from Theorem 3.1.10 by substituting $k = 2$. \hfill $\square$

**Lemma 3.1.12.** Suppose $xy = 0$ for some $x, y$ in a near subtraction semigroup $X$. Then $(yx)^2X = \{y0\}$ and in particular $(yx)^r = y0$, for every $r \geq 2$. If $X$ is zero symmetric and reduced then $X$ has $(\ast, \text{IFP})$.

**Proof.** Since $xy = 0$, we get $(yx)^2 = (yx)(yx) = y(xy)x = y0x = y0$. Hence for all $n$ in $X$, we have $(yx)^2n = y0n = y0$. This yields $(yx)^2X = \{y0\}$ and by taking $n = (yx)^{r-2}$ where $r > 2$, we get $(yx)^2(yx)^{r-2} = y0$. Hence $(yx)^r = y0$ for every $r \geq 2$.

Also when $X$ is reduced and zero-symmetric, $xy = 0 \Rightarrow (yx)^2 = y0 = 0 \Rightarrow yx = 0$ and $(xny)^2 = (xny)(xny) = xn(yx)ny = xn(0)ny = 0 \Rightarrow xny = 0$

Hence $X$ has $(\ast, \text{IFP})$. \hfill $\square$

**Theorem 3.1.13.** Suppose $ab = b^2$ and $a^2 = ba$ for some $a, b$ in $X$. Further let $u_1 = a - b$, $u_2 = au_1$, $u_3 = bu_1$, $u_4 = b - a$, $u_5 = au_4$, $u_6 = bu_4$. If there exist $x_i$’s in $X$ such that $u_i = x_iu_i'$ where $r \geq 2$, $i = 1, 2, 3, 4, 5, 6$ then $a = b$.

**Proof.** We have $u_1a = (a - b)a = a^2 - ba = 0$ and $u_1b = (a - b)b = ab - b^2 = 0$. Therefore $u_1a = 0 = u_1b$. Also we have $u_2^2 = (au_1)(au_1) = a(u_1a)u_1 = a0u_1 = a0$ and $u_3^2 = (bu_1)^2 = (bu_1)(bu_1) = b(u_1b)u_1 = b0u_1 = b0$. Further since $u_2 = x_2u_2$, we get $u_2 = x_2a0$. Therefore $u_2^2 = x_2a0u_2 = x_2a0 = u_2$. Thus $u_2 = u_2^2 = a0$. 27
Similarly, since \( u_3 = x_3 u_2^2 \), we have \( u_3 = x_3 b0 \Rightarrow u_3^2 = x_3 b0 u_3 = x_3 b0 = u_3 \).

Therefore \( u_3 = u_3^2 = b0 \). We now observe that, \( u_1^2 = u_1 u_1 = (a - b)u_1 = a u_1 - b u_1 = u_2 - u_3 = a0 - b0 = (a - b)0 = u_10 \). Further since \( u_1 = x_1 u_1^2 \), we have as before \( u_1 = x_1 u_10 \Rightarrow u_1^2 = x_1 u_10 u_1 = x_1 u_10 = u_1 \). Therefore \( u_1 = u_1^2 = u_10 \). Clearly then \( u_1a = u_10a = u_10 = u_1 \). That is, \( u_1 = u_1a \). But \( u_1a = 0 \) Therefore \( u_1 = 0 \).

That is, \( a - b = 0 \).

We shall now show that \( b - a = 0 \). We observe that, \( u_4 = b - a \)

\( \Rightarrow u_4a = (b - a)a = ba - a^2 = a^2 - a^2 = 0 \). Similarly, \( u_4b = (b - a)b = b^2 - ab = b^2 - b^2 = 0 \). Also, \( u_5^2 = (au_4)(au_4) = a(u_4a)(u_4) = a0u_4 = a0 \). And

\( u_5^2 = (b u_4)^2 = b(u_4 b)u_4 = b0u_4 = b0 \). Further since \( u_5 = x_5 u_5^2 \), we have \( u_5 = x_5 a0 \) and therefore \( u_5^2 = x_5 a0 u_5 = x_5 a0 = u_5 \). Thus \( u_5 = u_5^2 = a0 \).

Similarly, since \( u_6 = x_6 u_6^2 \), we have \( u_6 = x_6 b0 \Rightarrow u_6^2 = x_6 b0 u_6 = x_6 b0 = u_6 \). It follows that \( u_4^2 = (u_4)(u_4) = (b - a)u_4 = bu_4 - au_4 = u_6 - u_5 = b0 - a0 = (b - a)0 = u_40 \). Since \( u_4 = x_4 u_4^2 \), we have, as before, \( u_4 = x_4 u_40 \Rightarrow u_4^2 = x_4 u_40 u_4 = x_4 u_40 \).

Therefore \( u_4 = u_4^2 = u_40 \). Clearly then \( u_4a = u_40 a = u_40 = u_4 \). But \( u_4a = 0 \).

Therefore \( u_4 = b - a = 0 \). Appealing to Lemma 2.2.4(viii), we get \( a = b \).

In the following theorem we obtain a necessary and sufficient condition for a near subtraction semigroup to possess a unique mutual mate function.

**Theorem 3.1.14.** Let \( X \) admit mate functions. Then \( X \) possesses a unique mutual mate function if and only if \( E \subseteq C(E) \).

**Proof.** For the ‘only if’ part, we suppose that, \( f \) is the unique mutual mate function for \( X \). Clearly then \( f \) is involutary as both \( X \) and \( f(f(x)) \) serve as
mutual mates of \( f(x) \) for all \( x \) in \( X \). Also \( f \) fixes every element of \( E \). Then it is clear that for every \( xy \) in \( E \), both \( yf(xy) \) and \( f(xy)x \) serve as mutual mates of \( xy \).

The uniqueness of \( f \) (as the mutual mate function) for \( X \) demands that these mutual mates of \( xy \) must be identical with \( f(xy) \). It follows that, 
\[
(f(xy))^2 = (f(xy)x)(yf(xy)) = f(xy)xyf(xy) = f(xy).
\]
This forces \( f(xy) \in E \). Since \( f \) is involutorial and since it fixes every idempotent, we get, \( xy = f(f(xy)) = f(xy) \in E \). This guarantees that \((E, \cdot)\) is a sub semigroup of \((X, \cdot)\).

We make use of this result to observe that \( f(yx) = yx \) also can serve as mutual mate of \( xy \) for all \( x, y \) in \( E \). Again from the uniqueness of \( f \) we get \( xy = f(xy) = f(yx) = yx \) and the ‘only if’ part follows.

For the ‘if’ part we first observe that Lemma 3.1.7 guarantees the existence of a mutual mate function \( f \) for \( X \). To prove \( g = f \), we freely make use of the following (i) the assumption that, \( E \subseteq C(E) \) and (ii) for every \( x \in X \) and for every mate function \( f \) of \( X \) both \( f(x)x \) and \( xf(x) \) are in \( E \).

We have for all \( a \) in \( X \), 
\[
\begin{align*}
g(x) &= g(x)yg(x) = g(x)(xf(x)x)g(x) = g(x)x(f(x)x)g(x) \\
&= (f(x)x)(g(x)x)g(x) = f(x)x(g(x)xg(x)) = f(x)yg(x) = f(x)(xg(x)) \\
&= (f(x)xg(x))yg(x) = f(x)(xf(x))g(x) = f(x)(xg(x))(xf(x)) = f(x)(xg(x)x)f(x) \\
&= f(x)xf(x) = f(x).
\end{align*}
\]
This guarantees that, \( f \) is unique as the mutual mate function for \( X \).

\[\square\]

**Proposition 3.1.15.** Let \( f \) be a mate function for \( X \). Then any \( X \)-system \( A \) of \( X \) is idempotent.
Proof. As $A$ is an $X$-system of $X$, $XA \subset A$. Clearly then $A^2 = AA \subset XA \subset A$. Also for any $a$ in $A$, $a = af(a)a = a(f(a)a) \in AXA \subset AA = A^2$. Therefore $A \subset A^2$. Hence $A = A^2$ is idempotent. \hfill \Box

**Theorem 3.1.16.** Let $X$ admit a unique mutual mate function `$f$'. Then $X$ has the following properties:

(i) $X$ is zero symmetric

(ii) $f$ has the reversal law. That is, if $k$ is any positive integer, then for $a_1, a_2, \cdots, a_k \in X$, we have $f(a_1 a_2 \cdots a_k) = f(a_k) f(a_{k-1}) \cdots f(a_1)$

(iii) $f(a^k) = (f(a))^k$ for any positive integer $k$ and for any $a$ in $X$

(iv) If $X$ has no non-zero nilpotent elements, $e \in E$ and $x \in X$ are such that $exe = xe$ then $e \in C(X)$

(v) If $X$ has no non-zero nilpotent elements, $E \subseteq C(X)$.

**Proof.** (i) For every $n$ in $X$, define $f_n : X \to X$, such that $f_n$ agrees with $f$ in $X^*$ and that $f_n(0) = n0$. Obviously $f_n$ serves as a mutual mate function for $X$ and the uniqueness of $f$ (as the mutual mate function of $X$) demands that $f_n = f$. Clearly, $f$ fixes every idempotent and as such $0 = f(0) = f_n(0) = n0$ for all $x$ in $X$. Hence $X$ is zero symmetric.

(ii) Let us prove this by simple induction on the number of elements $k$. When $k = 1$ the result holds trivially. We shall assume that the results holds for any set of $k$ elements of $X$. Let $a_1, a_2, \cdots, a_k \in X$ and $a = a_1 a_2 \cdots a_k$ for convenience. Now by assumption, $f(a) = f(a_1 \cdot a_2 \cdots \cdot a_k) = f(a_k) f(a_{k-1}) \cdots f(a_2) f(a_1)$. Let `$b$' be any element of $X$. To get the desired result by simple induction we need
only to prove that \( f(ab) = f(b)f(a) \) (For this, we make use of Theorem 3.1.4 and Theorem 3.1.14).

Now \( ab = (af(a)a)(bf(b)b) = a(f(a)a)(bf(b))b = a(bf(b))(f(a)a)b = abf(b)f(a)ab \). Also \( f(b)f(a) = (f(b)b)(f(a)a)f(a) = f(b)(bf(b))(f(a)a)f(a) = f(b)(f(a)a)b(f(b))f(a) = f(b)f(a)abf(b)f(a) \). This guarantees that \( f(b)f(a) \) is a mutual mate of \( ab \). Since \( f(ab) \) is the unique mutual mate of \( ab \), we must have \( f(ab) = f(b)f(a) \) and the result follows.

(iii) This follows by taking \( a = a_1 = a_2 = a_3 \cdots = a_k \) in (ii).

(iv) \( exe = xe \Rightarrow (ex - xe)e = 0 \Rightarrow e(ex - xe) = 0 \) (by Lemma 3.1.12)
\( \Rightarrow ex(ex - xe) = 0 \) by IFP. Also \( xe(ex - xe) = x0 = 0 \) (from (i)). Thus
\( ex(ex - xe) - xe(ex - xe) = 0 \Rightarrow (ex - xe)^2 = 0 \). Similarly \( (xe - ex)^2 = 0 \).
Since \( X \) has no non-zero nilpotent elements we get \( ex - xe = 0, \; xe - ex = 0 \).
Therefore \( ex = xe \). Hence \( e \in C(X) \).

(v) For every \( e \) in \( E \) and for every \( a \) in \( X \) we have, \( (af(a)e - af(a))e = 0, \)
\( (af(a) - af(a)e)e = 0 \). Since \( f(a)a, \; af(a) \) and \( e \in E \), we invoke Lemma 3.1.14 to get \( (eaf(a) - af(a))e = 0, \; (af(a) - eaf(a)e = 0 \). By IFP, we have
\( (eaf(a) - af(a))ae = 0, \; (af(a) - eaf(a))ae = 0 \Rightarrow eaf(a)ae - af(a)ae = 0, \)
\( af(a)ae - eaf(a)ae = 0 \Rightarrow eae - ac = 0, \; ae - eae = 0 \). Therefore \( eae = ae \) and rest of the proof is taken care of by (iv).

\[ \square \]

**Theorem 3.1.17.** Suppose that \( X \) has IFP and a mate function ‘\( f \)’. Then the following statements are true.

(i) For any \( x, \; y \in X \) and \( e^2 = e, \; xy = xy \)
(ii) \(a \in Xa^2 \cap a^2X\) for all \(a \in X\)

(iii) \(X\) has no non zero nilpotent elements

(iv) \(X\) has the strong IFP.

Proof. (i) Let \(x, y \in X\) and \(e^2 = e\). Since \((x - xe)e = 0\) by IFP, we have \((x - xe)ye = 0 \Rightarrow xye - xeye = 0\). Similarly \(xye - xye = 0\). Therefore \(xye = xye\).

(ii) Let \(a \in X\). Since \(f\) is a mate function for \(X\) we have \(a = af(a)a\). Since \(af(a)\) and \(f(a)a\) are idempotents (by Lemmas 3.1.4 and 3.1.5). Now 
\[
af(a) = (f(a)a)^2 = (f(a)a)(f(a)a) = f(a)a(f(a)a) = f(a)f(a)a(af(a)a) \quad \text{[by (i)]}
\]
\[
= (f(a))^2a^2 \in Xa^2.
\]
That is, \(f(a)a \in Xa^2\). And 
\[
af(a) = (af(a))^2 = (af(a))(af(a))
\]
\[
= af(a)(af(a)) = a(af(a))f(a)(af(a)) \quad \text{[by (i)]}
\]
\[
= a^2f(a)(f(a)af(a)) = a^2(f(a))^2
\]
\[
\in a^2X.
\]
That is, \(af(a) \in a^2X\). Therefore \(a = af(a)a \in Xa^2 \cap a^2X\) for all \(a \in X\).

(iii) Suppose ‘\(a\)’ is a nilpotent element of \(X\). Then \(a^k = 0\) for some positive integer \(k\). Since \(a \in Xa^2 \cap a^2X\) there exist \(x, y \in X\) such that \(a = xa^2 = a^2y\).
\[
a = a^2y = a(a^2y)y = a^3y^2 = a^2(a^2y)y^2 = a^4y^3 \cdots = a^ky^{k-1} = 0y^{k-1} = 0.
\]
That is, \(X\) has no non-zero nilpotent elements.

(iv) Let \(I\) be any ideal of \(X\). Let \(a, b \in X\) with \(ab \in I\). Since \(X\) has a mate function ‘\(f\)’, \(bf(b)\) is idempotent. Now 
\[
axb = axbf(b)b = ax(bf(b))b
\]
\[
= abf(b)x(bf(b)b) = abf(b)xb \in abX \subseteq I \quad \text{for all} \quad x \in X.
\]
Consequently \(axb \in I\). And the desired result now follows. (by Definition 2.2.34).

Proposition 3.1.18. Let \(X\) be a zero symmetric near subtraction semigroup with a mate function ‘\(f\)’. Then \(X\) has \((\ast, \text{IFP})\) if and only if \(L = \{0\}\) where \(L\) is the set of all nilpotent elements of \(X\).
Proof. Suppose $X$ has $(\ast, \text{IFP})$. If $a^2 = 0$ for any $a$ in $X$, then by IFP $af(a)a = 0$. That is $a = 0$. Hence $L = \{0\}$.

For the converse we assume, $L = \{0\}$. Suppose $ab = 0$ for some $a, b$ in $X$. Then $(ba)^2 = (ba)(ba) = b(ab)a = b0a = 0$. Since $X$ has no non-zero nilpotent elements we get $ba = 0$. Also for any $n$ in $X$, $(anb)^2 = (anb)(anb) = an(ba)nb = an0nb = 0$ as $ba = 0$. Consequently $anb = 0$. That is, $X$ has $(\ast, \text{IFP})$. \hfill $\Box$

We furnish below another characterisation of mate functions

**Proposition 3.1.19.** Let $X$ be a zero symmetric near subtraction semigroup. Then $X$ has a mate function if and only if every $X$-system of $X$ is idempotent and $X$ is an $S$-near subtraction semigroup.

**Proof.** For the ‘only if’ part, we suppose $X$ has a mate function ‘$f$’. For $a \in X$, $a = af(a)a \in Xa$ and hence $X$ is an $S$-near subtraction semigroup. Also proposition 3.1.15 demands that every $X$-system of $X$ is idempotent.

For the ‘if’ part, as $X$ is an $S$ near subtraction semigroup $a \in Xa$ for every $a \in X$. Since $Xa$ is an $X$-system of $X$ it is idempotent (by assumption). Thus $Xa = (Xa)^2 = XaXa = (Xa)a \subset (Xa)a = Xa^2$. Therefore $a \in Xa \subset Xa^2$. Hence there exists $b \in X$ such that $a = ba^2$. Therefore $a^2 = 0 \Rightarrow a = 0$ for all $a \in X$. Now Proposition 2.2.28 demands that $L = \{0\}$

Therefore by Proposition 3.1.18 $X$ has $(\ast, \text{IFP})$. Again we get $a^2 = aba^2$ and this implies $(a - aba)a = 0$. Further, $a(a - aba) = 0$ and $aba(a - aba) = 0$. Consequently $(a - aba)^2 = (a - aba)(a - aba) = a(a - aba) - aba(a - aba) = 0$. Similarly $(aba - a)^2 = 0$. As $L = \{0\}$ we get $a = aba$. By setting $b = f(a)$ we see that $a = af(a)a$ and hence $f$ is a mate function for $X$. \hfill $\Box$
Theorem 3.1.20. Let $X$ be a zero symmetric near subtraction semigroup with
$(\star, \text{IFP})$ and let $f$ be a mate function for $X$. If $S = (0 : Xa)$ then $Xa = (0 : S)$.

Proof. We observe that, in view of Proposition 3.1.18, $L = \{0\}$. Let $y \in Xa$
Then $y = xa$ for some $x \in X$. Let $s \in S = (0 : Xa)$. Therefore we get $sy = 0$
$\Rightarrow s(xa) = 0 \Rightarrow (xa)s = 0$ [by $(\star, \text{IFP})] \Rightarrow xa \in (0 : S) \Rightarrow y \in (0 : S)$.
Consequently, $Xa \subseteq (0 : S)$

To prove the reverse inclusion, we consider $y \in (0 : S)$. Then $yS = \{0\}$.
Again $(y - yf(a)a)f(a)a = 0, (yf(a)a - y)f(a)a = 0 \Rightarrow (y - yf(a)a)Xf(a)a$
$= \{0\}, (yf(a)a - y)Xf(a)a = \{0\}$ [Since $Xf(a)a = Xa] \Rightarrow (y - yf(a)a)Xa$
$= \{0\}, (yf(a)a - y)Xa = \{0\} \Rightarrow y - yf(a)a, yf(a)a - y \in S$. Consequently
we get $y(y - yf(a)a) = 0, y(yf(a)a - y) = 0$ and $yf(a)a(y - yf(a)a) = 0,$
$yf(a)a(yf(a)a - y) = 0$ and hence $(y - yf(a)a)^2 = 0, (yf(a)a - y)^2 = 0$
$\Rightarrow y - yf(a)a = 0, yf(a)a - y = 0$ [since $L = \{0\}] \Rightarrow y = yf(a)a \in Xa$.
Thus $(0 : S) \subseteq Xa$ and the desired result now follows. $\Box$

Remark 3.1.21. It is worth noting that in view of Proposition 2.2.23 when
$X(= X_0)$ with $(\star, \text{IFP})$ admits mate functions every principal $X$-system of $Xa$ of
$X$ is an annihilator ideal. In fact $Xa = (0 : (0 : Xa))$.

We are now in a position to prove one of our principal theorems of this chapter

Theorem 3.1.22. Let $X(= X_0)$ admit a mate function ‘$f$’. Then the following
statements are equivalent.

(i) Every principal $X$-system of $X$ is an invariant $X$-system

(ii) $aX = aXa$ for all $a$ in $X$ (That is, $X$ is a $P_1$-near subtraction semigroup)
(iii) For all $X$-systems $A$ and $B$ of $X$, $A \cap B = AB$

(iv) $Xa \cap Xb = Xab$ for all $a, b$ in $X$

(v) Every $X$-system of $X$ is completely semiprime $X$-system

(vi) $X$ has property $P_4$

(vii) $X$ has strong IFP.

**Proof.** (i) $\Rightarrow$ (ii): We observe that $Xa$ is a principal $X$-system of $X$, for every $a \in X$, hence

$$(Xa)X \subseteq Xa \quad (3.1)$$

Since $f$ is a mate function for $X$, $a = af(a)a$. By relation (3.1), for every $n \in X$, there exists $n' \in X$ such that

$$(f(a)a)n = n'a \quad (3.2)$$

Now, $an = (af(a)a)n = a(f(a)a)n = an'a$ [by equation (3.2)]. Therefore $aX \subseteq aXA$. Consequently $aX = aXa$.

(ii) $\Rightarrow$ (iii): Let $a \in A \cap B$. Obviously $a = af(a)a \in A(XB) \subseteq AB$. Thus $A \cap B \subseteq AB$. Now let $n \in AB$. Then $n = a_1b_1$ where $a_1 \in A$ and $b_1 \in B$. We have $n = a_1b_1 \in a_1X = a_1Xa_1$ and therefore $n = a_1xa_1$ (for some $x$ in $X$) $= a_1(xa_1) \in a_1(XA) \subseteq a_1A \subseteq A$. Therefore $n \in A$. This guarantees $AB \subseteq A$. Again since $n = a_1b_1 \in XB \subseteq B$, we get $AB \subseteq B$. Collecting all these pieces we get $AB \subseteq A \cap B$ and (iii) follows.

(iii) $\Rightarrow$ (iv): For $a, b$ in $X$, $Xa \cap Xb = XaXb$, by taking $A = Xa$ and $B = Xb$ in (iii). We have obviously $Xa = Xa \cap X = XaX$ and this yields $Xab = XaXb$ and (iv) follows.
(iv) ⇒ (v): For every $a \in X$, $a = af(a)a \in Xa = Xa \cap Xa = Xa^2$ and therefore $a^2 = 0 \Rightarrow a = 0$. By Proposition 2.2.28 $L = \{0\}$ and Proposition 3.1.18 guarantees that $X$ has $(\ast, \text{IFP})$. Let $A$ be any $X$-system of $X$ and let $a^2 \in A$. Obviously $a \in Xa = Xa \cap Xa = Xa^2 \subset XA \subset A$. Therefore $a^2 \in A \Rightarrow a \in A$ and (v) follows.

(v) ⇒ (vi): Let $ab \in I$, where $I$ is any ideal of $X$. Now $(ba)^2 = (ba)(ba) = b(ab)a \in XIX \subset XI \subset I$. Since $X = X_0$, $I$ is an $X$-system by Proposition 2.2.18. Thus $(ba)^2 \in I$ and (v) demands that $ba \in I$ and (vi) follows.

(vi) ⇒ (vii): Let $I$ be any ideal of $X$ and let $ab \in I$. Now (vi) ⇒ $ba \in I$ and therefore $ban \in IX \subset I$ for all $n$ in $X$. Again (vi) guarantees that $(an)b \in I$ and (vii) follows.

(vii) ⇒ (i): As $\{0\}$ is an ideal of $X$, (vii) ⇒ $a^2(= aa) = 0 \Rightarrow af(a)a = 0 \Rightarrow a = 0$. Therefore by Proposition 2.2.28 $L = \{0\}$ and Proposition 3.1.18 guarantees that $X$ has $(\ast, \text{IFP})$. In view of Remark 3.1.21 it follows that $Xa$ is an ideal of $X$. Consequently every principal $X$-system of $X$ is an invariant $X$-system of $X$. □

3.2 $P_1$-Near Subtraction semigroups

In this section we introduce the concept of $P_1$-near subtraction semigroups, obtain a characterisation theorem and prove elementary important properties.

Definition 3.2.1. We say that $X$ is a $P_1$ near subtraction semigroup if $aX = aXa$ for every $a$ in $X$.  

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We furnish below examples of $P_1$-near subtraction semigroups

**Examples 3.2.2.** (a) We consider the near subtraction semigroup $(X, -, \cdot)$ with $X = \{0, a, b, 1\}$ where we define '$-' and '$\cdot'$ as follows:

\[
\begin{array}{c|cccc}
- & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
b & b & b & 0 & 0 \\
1 & 1 & b & a & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
b & b & 0 & b & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

Then $(X, -, \cdot)$ is a $P_1$ near subtraction semigroup.

(b) Let $X = \{0, a, b, c\}$ in which '$-' and '$\cdot'$ are defined by

\[
\begin{array}{c|cccc}
- & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & b & b & 0 & b \\
c & c & c & c & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & b & c \\
b & b & 0 & b & b \\
c & 0 & a & b & c \\
\end{array}
\]

This is not a $P_1$-near subtraction semigroup.

**Proposition 3.2.3.** A homomorphic image of a $P_1$ near subtraction semigroup is also a $P_1$ near subtraction semigroup.

**Proof.** Let $X$ be a $P_1$-near subtraction semigroup and let $h : X \to X'$ be an epimorphism. Let $a' \in X'$, then there exists $a \in X$ such that $a' = h(a)$. For any $n' \in X'$, there exists $n \in X$ such that $n' = h(n)$. Therefore $a'n' = h(a)h(n) = h(an)$. Since $X$ is a $P_1$-near subtraction semigroup, $aX = aXa$. Then $an = aba$ for some $b \in X$. Therefore $a'n' = h(aba) = h(a)h(b)h(a) = a'b'a$ where $b' = h(b) \in X'$ and hence $a'X' \subset a'X'a'$. 

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In a similar fashion we get $a'X'a' \subseteq a'X'$. Hence $a'X'a' = a'X'$ and the desired result now follows.

Theorem 3.2.4. If $X$ is a zero symmetric $P_1$ near subtraction semigroup with a mate function, then

(i) $X$ has no non-zero nilpotent elements
(ii) For $x, y \in X$, $xy = 0 \Rightarrow yx = 0$
(iii) $X$ has IFP
(iv) If $e \in E$ and $x \in X$ are such that $exe = xe$ then $e \in C(X)$
(v) $E \subseteq C(X)$
(vi) $LX = \{0\}$.

Proof. (i) Let $'a'$ be nilpotent element of $X$. Then there exists a positive integer $k$ such that $a^k = 0$. Since $X$ has a mate function $'f'$ such that $a = af(a)a$. That is, $a = (af(a))a \in aX = aXa = (aX)a = (aX)a = aXa^2 \subseteq Xa^2$. That is, $a \in Xa^2$. Repeating like this we get $a \in Xa^k = \{0\}$. Consequently $X$ has no non-zero nilpotent elements. That is, $L = \{0\}$.

Since $L = \{0\}$ Proposition 3.1.18 guarantees that (ii) and (iii) hold good.

Again Theorem 3.1.16(iv), (v) guarantee that (iv) and (v) hold good.

(vi) For all $a$ in $X$, $aX = (aXa) = aXa^r$ for every integer $r \geq 1$. If $a \in L$, $a^r = 0$ for some positive integer. Clearly then $aX = \{0\}$, and (vi) follows.

A near subtraction semigroup $X$ with a mate function $f$ has the property that for every $a$ in $X$, there is some $a'$ in $X$ such that $a - aa'a, aa'a - a \in L$. To verify
this we need only to observe that $0 \in L$ and take $f(a)$ for $a'$. Motivated by this we give the following

**Definition 3.2.5.** (S. Suryanarayanan [46]) We say that $X$ has $(L, \ast)$ property, if for every $a$ in $X$ there is some $a'$ in $X$ such that $a - aa'a \in L$, $aa'a - a \in L$ and $a'$ is called an $L$-associate of $a$.

We shall now discuss the behaviour of the $L$-associates of elements of a near subtraction semigroup $X$ with the $(L, \ast)$ property

**Example 3.2.6.** Let $X = \{0, a, b, 1\}$ in which ‘$-$’ and ‘$.$’ are defined by

\[
\begin{array}{c|cccc}
- & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 1 & b \\
b & b & 0 & 0 & b \\
1 & 1 & 0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & a & 0 & 1 & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

The map $f : X \to X$ is a mate function. Then $X$ has $(L, \ast)$ property.

**Lemma 3.2.7.** Let $X$ have the following properties:

$(\alpha)$ $(L, \ast)$ property

$(\beta)$ $xy = 0 \Rightarrow yx = 0$ for $x, y$ in $X$

$(\gamma)$ $XL = \{0\}$.

If $a'$ is an $L$-associate of $a$ in $X$ then we have

$(i)$ $aa', (a'a)^2 \in E$

$(ii)$ $a'aa'$ is also an $L$-associate of $a$. 
Proof. (i) Since $X$ has $(L, \ast)$ property, $a - aa'a, aa'a - a \in L$. We have by assumption $(\gamma)$, $X(a - aa'a) = \{0\}$ and consequently $a'(a - aa'a) = 0$, $a'(aa'a - a) = 0$. Again by assumption $(\beta)$, we have $(a - aa'a)a' = 0$, $(aa'a - a)a' = 0$. That is, $aa' - aa'aa' = 0, aa'aa' - aa' = 0$. Thus $aa' = aa'aa'$ which forces $aa' \in E$. Also $(a'a)^4 = a'(aa')(aa') = a'(aa')^3a = a'aa'a = (a'a)^2$ and hence $(a'a)^2 \in E$.

(ii) Further $a(a'a)^2 = aa'aa'a = (aa')^2a = aa'$. Thus $a - a(a'aa')a = a - a(a'a)^2 = a - aa'a \in L$. That is, $a - a(a'aa')a \in L$. In a similar fashion we get $a(a'aa')a - a \in L$. The desired result now follows.

We furnish below another characterisation of $P_1$-near subtraction semigroups

**Proposition 3.2.8.** If $X$ is a zero symmetric $P_1$-near subtraction semigroup then $X$ has the following properties:

(i) $LX = \{0\}$

(ii) $(\ast, \text{IFP})$

(iii) $(L, \ast)$ property.

Proof. (i) For all $a$ in $X$, $aX = aXa = aXa^r$ for every integer $r \geq 1$. Thus whenever $a \in L$, $aX = \{0\}$ and (i) follows.

(ii) $ab = 0 \Rightarrow aXb = (aXa)b = aXab = aX0 = \{0\}$. Also $ab = 0 \Rightarrow ba \in L$ and hence $baX = \{0\}$. Since $ba \in bX = bXb$, $ba = byb$ for some $y$ in $X$ and we observe that $by \in L$. This yields $ba = byb \in byX = \{0\}$ and (ii) follows.
(iii) Taking $r = 2$ in (i), we have $a^2 = aa'a^2$ for some $a'$ in $X$. Therefore $(a - aa'a)a$ and Theorem 3.2.4 guarantees that $X$ has $(\ast, \text{IFP})$. Therefore $a(a - aa'a) = 0$ and $aa'(a - aa'a) = 0$. These force $(a - aa'a)^2 = 0$ and hence $a - aa'a \in L$. In a similar fashion we can prove $aa'a - a \in L$. Thus $X$ has $(L, \ast) \text{ property.}$

**Remark 3.2.9.** “$LX = \{0\}$” and $ab = 0 \Rightarrow ba = 0$ together guarantees that “$XL = \{0\}$” for a zero symmetric $P_1$ near subtraction semigroup.

**Proposition 3.2.10.** Let $X$ be a zero symmetric near subtraction semigroup. Then $X$ has $(L, \ast)$ property with $XL = \{0\}$ if and only if $X$ admits mate function and $L = \{0\}$.

**Proof.** For the ‘if’ part, we observe that when $f$ is a mate function for $X$, $f(a)$ serves as an $L$-associate of $a$ for all $a$ in $X$. This guarantees that $X$ has the $(L, \ast)$ property. Also since $L = \{0\}$ and $X = X_0$, we get $XL = \{0\}$ trivially.

For the converse part, we first note that if $f$ is a map from $X$ into $X$ with $f(a)$ serving as a choosen $L$-associate of $a$ for every $a$ in $X$. Then $a - af(a)a \in L$. Since $X$ is an $S$-near subtraction semigroup, we have $a - af(a)a \in X(a - af(a)a) = \{0\}$. This guarantees that $f$ is a mate function for $X$. Also if $a \in L$, $Xa = \{0\}$ and since $a \in Xa$, we get $L = \{0\}$. □