Chapter 5

$S_1$-Near Subtraction Semigroups

This chapter contains three sections. In the first section we introduce the concept of $S_1$ near subtraction semigroups. In [10] Dheena and Rajeswari define $X$ to be left weakly regular if $a \in (Xa) \ast (Xa)$ for all $a \in X$. Also $X$ is said to be left weak weakly regular if for any $x \in N$, $x = ax$ for all $a \in N$. Motivated by this we introduce the concept of $S_1$-near subtraction semigroups. We call $X$ an $S_1$-near subtraction semigroup if for every $a \in X$, there exists $x \in X^*$ such that $axa = xa$.

We discuss some of the properties of $S_1$-near subtraction semigroups. We prove that every ideal and every $X$-system of an $S_1$-near subtraction semigroup is an $S_1$-near subtraction semigroup. The characterisation of $S_1$-near subtraction semigroups is obtained. The conditions under which every Boolean near subtraction semigroup is an $S_1$-near subtraction semigroup are discussed. It is interesting to observe that every zero symmetric $S_1$-near subtraction semigroup without zero divisors is subdirectly irreducible.

In the second section, we introduce the notation $X_{S_1}(a) = \{x \in X^*/axa = xa\}$ where $a \in X$. We obtain a condition for an $S_1$-near subtraction semigroup to admit mate functions. We prove an interesting result $"X_{S_1}(a)$ has no non-zero zero
divisors if and only if it is a multiplicative system”. We obtain a characterisation of zero symmetric near subtraction semigroups and discuss the conditions under which $X^* = X_{S_1}(a), a \in X$.

In the third section, we introduce the concept of strong $S_1$-near subtraction semigroups by defining that $X$ is a strong $S_1$ if $X^* = X_{S_1}(a), a \in X$. We obtain the characterisation of strong $S_1$ near subtraction semigroups. We show that every strong $S_1$ near subtraction semigroup is an $S_1$ near subtraction semigroup and we observe that an $S_1$-near subtraction semigroup need not be a strong $S_1$-near subtraction semigroup. We prove some important properties of strong $S_1$-near subtraction semigroups and discuss the behaviour of identities and annihilators in a strong $S_1$-near subtraction semigroup.

Certain results in chapter 5 are included in author’s paper “$S_1$-near subtraction semigroups” [41] published in journal “Ultra scientist of physical science”.

### 5.1 $S_1$-near Subtraction Semigroups

Let us now define $S_1$-near subtraction semigroups

**Definition 5.1.1.** We say that $X$ is an $S_1$-near subtraction semigroup if for every $a \in X$ there exists $x \in X^*$ such that $axa = xa$.

**Examples 5.1.2.** (a) We consider the near subtraction semigroup $(X, -, \cdot)$ with $X = \{0, a, b, 1\}$ where we define ‘$-$’ and ‘$\cdot$’ as follows:
This is an $S_1$-near subtraction semigroup. [It may be seen that $aba = ba$, $bab = ab$, $1a1 = a1$, $0a0 = a0$].

(b) We consider the near subtraction semigroup $X = \{0, a, b, 1\}$ in which ‘$-$’ and ‘$\cdot$’ are defined by

\[
\begin{array}{c|cccc}
- & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
b & b & b & 0 & 0 \\
1 & 1 & b & a & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & b & 0 & b & b \\
1 & 1 & 0 & 1 & 0 \\
\end{array}
\]

This is not an $S_1$ near subtraction semigroup. [Since, for $b \in X$ there exists no $x \in X^*$ such that $bxb = xb$].

We furnish below some properties of $S_1$-near subtraction semigroup

**Proposition 5.1.3.** Let $X$ be an $S_1$-near subtraction semigroup.

(i) If $ax = 0$ then $xa = 0$

(ii) If $ax \in E$ then $xa \in E$

(iii) If the right cancellation law is valid in $X$ then $xa \in E$ implies $ax \in E$ for all $a \in X$ and for some $x \in X^*$.

**Proof.** Let $a \in X$. Since $X$ is an $S_1$-near subtraction semigroup, there exists $x \in X^*$ such that

\[ axa = xa \quad (5.1) \]
(i) If $ax = 0$, then from equation (5.1), we get $xa = 0a = 0$. Thus $xa = 0$.

(ii) If $ax \in E$, then

$$ (ax)^2 = ax $$ \hfill (5.2)

Now $(xa)^2 = (xa)xa = (axa)xa$ [by equation (5.1)] = $(ax)^2a = (ax)a$ [by equation (5.2)] = $xa$. That is, $(xa)^2 = xa$ and hence $xa \in E$.

(iii) If $xa \in E$ then

$$ (xa)^2 = xa $$ \hfill (5.3)

Now $(ax)^2a = (axa)xa = (xa)xa$ [by equation (5.1)] = $(xa)^2 = xa$ [by equation (5.3)] = $axa$. That is, $(ax)^2a = (ax)a$. Since the right cancellation law is valid in $X$, $(ax)^2 = ax$. Thus $ax \in E$.

\[\square\]

**Proposition 5.1.4.** Let $X$ be an $S_1$-near subtraction semigroup without non-zero zero divisors. If $X$ is commutative then $X$ is Boolean.

**Proof.** Let $a \in X$. Since $X$ is an $S_1$-near subtraction semigroup, there exists $x \in X^*$ such that $axa = xa$. Since $X$ is commutative, $a(ax) = ax \Rightarrow a^2x = ax \Rightarrow (a^2 - a)x = 0$. Since $X$ has no non-zero zero divisors, $a^2 - a = 0$. In a similar fashion we get $a - a^2 = 0$. Consequently $X$ is Boolean. \[\square\]

**Proposition 5.1.5.** Let $X$ be an $S_1$ near subtraction semigroup. If $X$ has no non-zero zero divisors then every $X$-system and every ideal of $X$ is an $S_1$ near subtraction semigroup in its own right.

**Proof.** Let $A$ be an $X$-system of $X$ and let $a \in A$. Since $X$ is an $S_1$-near subtraction semigroup there exists $x \in X^*$ such that $axa = xa$. Let $n = xa$. Since $A$ is an $X$-system of $X$, $XA \subset A$. Therefore $n \in A$. Since $X$ has no non-zero zero
divisors, \( n \neq 0 \). Now \( ana = a(xa)a = (axa)a = (xa)a = na \). Thus \( A \) is an \( S_1 \)-near subtraction semigroup.

Let \( I \) be an ideal of \( X \) and let \( a \in I \). Since \( X \) is an \( S_1 \)-near subtraction semigroup, there exists \( x \in X^* \) such that \( axa = xa \). Let \( i = ax \). Since \( I \) is an ideal of \( X \), \( IX \subseteq I \). Therefore \( i \in I \).

Our hypothesis demands that \( i \neq 0 \). Now \( aia = a(ax)a = a(axa) = a(xa) = (ax)a = ia \). Thus \( I \) is an \( S_1 \)-near subtraction semigroup.

We shall now obtain a characterisation of \( S_1 \)-near subtraction semigroups.

**Theorem 5.1.6.** Let \( X \) be a nil near subtraction semigroup. Then \( X \) is an \( S_1 \)-near subtraction semigroup if and only if \( X \) is zero symmetric.

**Proof.** For the ‘only if’ part, we take \( a \in X \). Since \( X \) is an \( S_1 \)-near subtraction semigroup there exists \( x \in X^* \) such that,

\[
axa = xa \tag{5.4}
\]

We shall prove that

\[
ax^ka = x^ka \tag{5.5}
\]

for all positive integers \( k \). We use induction on \( k \). Equation (5.4) demands that equation (5.5) is true for \( k = 1 \).

We assume that the result is true for \( k = s - 1 \). If \( k = s \) then

\[
ax^s a = ax^{s-1}(xa) = ax^{s-1}axa \quad \text{[by equation (5.4)]} = (ax^{s-1})xa = (x^{s-1})xa = x^{s-1}(axa) = x^{s-1}(xa) = x^s a.
\]

Thus \( ax^k a = x^k a \) for all positive integer \( k \).
Since $X$ is nil, $x^t = 0$ for some positive integer $t$. Since $ax^t a = x^t a$, $a0a = 0a$
$\Rightarrow a0 = 0$. Thus $X$ is zero symmetric.

For the ‘if’ part, let $a \in X$. Since $X$ is nil, there exists a positive integer $k > 1$ such that $a^k = 0$. This implies $xa = 0$ where $x = a^{k-1}$. Therefore $axa = a(xa) = a0 = 0$ [since $X = X_0$] $= xa$. Thus $X$ is an $S_1$ near subtraction semigroup.

We shall now obtain conditions under which a Boolean near subtraction semigroup becomes an $S_1$-near subtraction semigroup.

**Theorem 5.1.7.** Let $X$ be a Boolean near subtraction semigroup. Each of the following statements implies that $X$ is an $S_1$-near subtraction semigroup.

(i) $X$ is zero symmetric.

(ii) $X$ has $(\ast, \text{IFP})$ and identity 1.

(iii) $Xa = aXa$ for all $a \in X$ ($X$ is a $P'_1$ near subtraction semigroup).

(iv) $X$ is subcommutative.

**Proof.** (i) Let $X$ be a zero symmetric near subtraction semigroup. Let $a \in X$. If $a \neq 0$, we take $x = a$. Then $axa = a^2a = aa$ [since $X$ is Boolean] $= xa$.
That is, $axa = xa$. If $a = 0$, then for any $x \in X^*$, $axa = 0 = xa$ [since $X = X_0$].
Consequently $X$ is an $S_1$-near subtraction semigroup.

(ii) We assume that $X$ has $(\ast, \text{IFP})$ and identity 1. Let $a \in X$. Since $X$ is Boolean $a^2 = a \Rightarrow a^2 - a = 0$ and $a - a^2 = 0 \Rightarrow a(1 - a) = 0$, $a(a - 1) = 0$.
Since $X$ has $(\ast, \text{IFP})$, we get $(a - 1)a = 0$, $(1 - a)a = 0$ and hence $(a - 1)xa = 0$,
$(1-a)xa = 0$ for all $x \in X$. In particular, $(a-1)xa = 0$, $(1-a)xa = 0$ for any $x \in X^* \Rightarrow axa - xa = 0$, $xa - axa = 0$. Therefore $axa = xa$. Thus $X$ is an $S_1$-near subtraction semigroup.

(iii) Let $a \in X$. Since $Xa = aXa$ for any $x \in X$, there exists $y \in X$ such that $xa = aya$. Therefore $axa = a(xa) = a(aya) = a^2ya = aya$ [since $X$ is Boolean] $= xa$ and hence $X$ is an $S_1$ near subtraction semigroup.

(iv) Let $a \in X$. Since $X$ is subcommutative $Xa = aX$. Therefore for any $x \in X$, there exists $y \in X$ such that $xa = ay$. Therefore $axa = a(xa) = a(ay) = a^2y = ay$ [since $X$ is Boolean] $= xa$. That is $axa = xa$ for all $x \in X$. In particular $axa = xa$ for any $x \in X^*$. Thus $X$ is an $S_1$-near subtraction semigroup.

In the following theorem we obtain another characterisation of $S_1$ near subtraction semigroups.

**Theorem 5.1.8.** Let $X$ be a zero-symmetric commutative near subtraction semigroup without non-zero zero divisors. Then $X$ is an $S_1$-near subtraction semigroup if and only if $X$ is Boolean.

**Proof.** For the ‘if’ part, let $a \in X$. For any $x \in X^*$, $axa - xa = a(xa) - xa = a(ax) - ax$ [since $X$ is commutative] $= ax - ax$ [since $X$ is Boolean] $= 0$. That is, $axa - xa = 0$. Similarly $xa - axa = 0$. Therefore $axa = xa$. It follows that $X$ is an $S_1$-near subtraction semigroup.

Proof of ‘only if’ part follows from proposition 5.1.4.
**Proposition 5.1.9.** Let $X$ be an $S_1$-near subtraction semigroup without non-zero zero divisors. Then $X$ has a mate function if and only if $X$ is Boolean.

**Proof.** Let $a \in X$, since $X$ is an $S_1$-near subtraction semigroup there exists $x \in X^*$ such that $axa = xa$.

For the ‘only if’ part, we assume that, $X$ has a mate function $f$. Then $af(a)xa = af(a)(xa) = af(a)axa = axa$. This implies, $af(a)xa - axa = 0$. That is, $(af(a) - a)xa = 0$. Since $X$ has non-zero zero divisors, $af(a) - a = 0$. Similarly we can show that $a - af(a) = 0$. That is, $af(a) = a$. Therefore $(af(a))a = a^2$. This implies that $a = a^2$. Consequently $X$ is Boolean.

Proof of ‘if’ part is obvious. \(\Box\)

**Remark 5.1.10.** Proposition 5.1.9 gives a necessary and sufficient condition for an $S_1$-near subtraction semigroup to admit mate functions.

**Theorem 5.1.11.** Let $X = X_0$ have no zero divisors. If $X$ is an $S_1$-near subtraction semigroup then $X$ is subdirectly irreducible.

**Proof.** Let $X$ be an $S_1$-near subtraction semigroup. Let $\{I_k\}$ be a sequence of non-zero ideals of $X$. Since $X$ has no zero divisors, Proposition 5.1.5 demands that $\{I_k\}$ is a sequence of $S_1$-near subtraction semigroups. We prove that $\bigcap I_k$ is an $S_1$-near subtraction semigroup. Let $a \in I_1 \cap I_2$. Since $I_1$ and $I_2$ are $S_1$ near subtraction semigroups, there exist non-zero elements $i_1$ and $i_2$ respectively in $I_1$ and $I_2$ such that,

$$ai_1a = i_1a \quad (5.6)$$

$$ai_2a = i_2a \quad (5.7)$$
Let us take $i_3 = i_1i_2$. Since $X$ has no zero divisors, $i_1i_2 \neq 0 \Rightarrow i_3 \neq 0$. Since $i_3 = i_1i_2 \in XI_2$ and since $X = X_0$, it follows that $i_3 \in I_2$. Since $i_3 = i_1i_2 \in I_1X$ and since $I_1$ is a non-zero divisor, it follows that $i_3 \in I_1$. Therefore $i_3 \in I_1 \cap I_2$. We prove that $ai_3a = i_3a$. Now $ai_3a = a(i_1i_2)a = ai_1(i_2a) = ai_1(ai_2a)$ [by equation (5.7)]

$= (ai_1ai_2a = (i_1a)i_2a$ [by equation (5.6)] $= i_1(ai_2a) = i_1(i_2a) = (i_1i_2)a = i_3a$.

That is, $ai_3a = i_3a$. Thus $I_1 \cap I_2$ is an $S_1$-near subtraction semigroup. Proceeding this way, we get $\bigcap I_k$ is an $S_1$-near subtraction semigroup. Since every $S_1$-near subtraction has at least one non-zero element, $\bigcap I_k \neq 0$. Thus $X$ is subdirectly irreducible.

\[ \square \]

### 5.2 $S_1$-near Subtraction Semigroups and the Subsets $X_{S_1}(a)$

In this section we introduce the notation $X_{S_1}(a), a \in X$ and discuss the properties of $S_1$ near subtraction semigroup with the subset $X_{S_1}(a)$.

**Notation 5.2.1.** For any $a \in X$, we denote $\{x \in X^*/axa = xa\}$ by $X_{S_1}(a)$.

**Remark 5.2.2.** It easily follows that $X$ is an $S_1$ near subtraction semigroup if and only if $X_{S_1}(a) \neq \phi$ for all $a \in X$.

**Examples 5.2.3.** (a) We consider the near subtraction semigroup $(X, -, \cdot)$ where $X = \{0, a, b, 1\}$, '−' and '⋅' are defined as follows:

\[
\begin{array}{c|cccc}
- & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
\hline
a & a & 0 & a & 0 \\
\hline
b & b & b & 0 & 0 \\
\hline
1 & 1 & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
\hline
a & a & 0 & a & 0 \\
\hline
b & b & 0 & b & 0 \\
\hline
1 & 0 & a & b & 1 \\
\end{array}
\]

Clearly this is an $S_1$ near subtraction semigroup. We observe that $X_{S_1}(x) \neq \phi$ for all $x \in X$. $[X_{S_1}(0) = X_{S_1}(a) = X_{S_1}(b) = X_{S_1}(1) = \{a, b, 1\}]$. 
(b) We consider the near subtraction semigroup $X = \{0, a, b, 1\}$ where ‘−’ and ‘·’ are defined as follows:

$$
\begin{array}{c|cccc}
- & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 1 & b \\
b & b & 0 & 0 & b \\
1 & 1 & 0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & a & 0 & 1 & b \\
1 & 0 & a & b & 1 \\
\end{array}
$$

We note that $X_{S_1}(b) = \phi$.

**Proposition 5.2.4.** Let $X$ be an $S_1$-near subtraction semigroup. If $ab = ba$ for all $a, b \in X$ and if $X_{S_1}(a) \cap X_{S_1}(b) \neq \phi$ then $X_{S_1}(a) \cap X_{S_1}(b) \subset X_{S_1}(ab)$.

**Proof.** Let $a, b \in X$. Suppose $ab = ba$. Let $x \in X_{S_1}(a) \cap X_{S_1}(b) \Rightarrow x \in X_{S_1}(a)$ and $x \in X_{S_1}(b) \Rightarrow axa = xa$ and $bxb = xb$. Now $(ab)x(ab) = (ba)x(ab) = b(axa)b = b(xa)b = bx(ab) = bx(ba) = (bx)b = (xb)a = x(ba) = x(ab)$. That is, $(ab)x(ab) = x(ab) \Rightarrow x \in X_{S_1}(ab)$. Thus $X_{S_1}(a) \cap X_{S_1}(b) \subset X_{S_1}(ab)$. $\square$

We furnish below a condition for the existence of a mate function in an $S_1$-near subtraction semigroup in terms of the set $X_{S_1}(a)$

**Proposition 5.2.5.** Let $X$ be an $S_1$-near subtraction semigroup. If $a \in X_{S_1}(a)a$ for all $a \in X$ then $X$ admits a mate function.

**Proof.** Let $a \in X$. By hypothesis $a = xa$ for some $x \in X_{S_1}(a)$. Since $x \in X_{S_1}(a)$, $axa = xa$. Therefore $a = axa$. If we set $x = f(a)$, then we get $a = a f(a) a$. The desired result now follows. $\square$
Proposition 5.2.6. Let $X$ be an $S_1$ near subtraction semigroup. Then $a \in X_{S_1}(b)$ and $ba \in E$ if and only if $b \in X_{S_1}(a)$ and $ab \in E$ for $a, b \in X^*$.

Proof. Let $a, b \in X^*$. We assume that $a \in X_{S_1}(b)$ and $ba \in E$. Now $a \in X_{S_1}(b) \Rightarrow bab = ab$. Since $ba \in E$, we get $aba = (ab)a = (bab)a = (ba)^2 = ba$. It follows that $b \in X_{S_1}(a)$. Also $(ab)^2 = (aba)b = (ba)b = bab = ab$ and hence $ab \in E$.

Proof of the converse is similar. \(\square\)

Proposition 5.2.7. If $X$ is zero symmetric then there exists $a \in X^*$ such that $X_{S_1}(a)$ contains all left zero divisors of $X$.

Proof. Let $x$ be a left zero divisor of $X$. Then there exists $a \in X^*$ such that $xa = 0$. Therefore $axa = a(xa) = 0 = 0$ [since $X = X_0 = xa$. Consequently $x \in X_{S_1}(a)$]. \(\square\)

Lemma 5.2.8. Let $X$ be an $S_1$-near subtraction semigroup. Then $X_{S_1}(a)$ has no non-zero zero divisors if and only if $X_{S_1}(a)$ is a multiplicative system.

Proof. Since $X$ is an $S_1$-near subtraction semigroup, $X_{S_1}(a) \neq \emptyset$ for all $a \in X$.

For the ‘only if’ part, let $x, y \in X_{S_1}(a)$. Then $x, y \in X^*$ and $axa = xa, aya = ya$.

It follows that $a(xy)a = ax(ya) = ax(aya) = (axa)ya = (xa)ya = x(aya) = x(ya) = (xy)a$. That is, $a(xy)a = (xy)a$.

Further since $X_{S_1}(a)$ has no non-zero zero divisors, $xy \neq 0$. Consequently $xy \in X_{S_1}(a)$. Thus $X_{S_1}(a)$ is a multiplicative system.
For the ‘if’ part, let \( x, y \in X_{S_1}(a) \). Since \( X_{S_1}(a) \) is a multiplicative system, \( xy \in X_{S_1}(a) \). As \( X_{S_1}(a) \subset X^* \), it follows that \( xy \neq 0 \) and hence \( X_{S_1}(a) \) has no non-zero zero divisors.

\[ \square \]

**Lemma 5.2.9.** If \( X \) is an \( S_1 \)-near subtraction semigroup then \( X_{S_1}(a) \subset X_{S_1}(a^k) \) for all positive integers \( k > 1 \) and for all \( a \in X \).

**Proof.** Let \( x \in X_{S_1}(a) \). Then \( axa = xa \). Therefore \( a^2xa^2 = a(axa)a = a(axa) = (xa)a = xa^2 \). That is, \( a^2xa^2 = xa^2 \). Continuing in the same vein we get, \( a^kxa^k = xa^k \) for all positive integers \( k \). It follows that \( X_{S_1}(a) \subset X_{S_1}(a^k) \).

In the following theorem we derive some important properties of an \( S_1 \)-near subtraction semigroup

**Theorem 5.2.10.** Let \( X \) be an \( S_1 \)-near subtraction semigroup, without non-zero zero divisors. Then we have the following:

(i) \( aX_{S_1}(a) = X_{S_1}(a) \) for all \( a \in X^* \)

(ii) \( X_{S_1}(a)a \subset X_{S_1}(a) \) for all \( a \in X^* \)

(iii) If \( X \) is finite then \( X_{S_1}(a)a = X_{S_1}(a) \) for all \( a \in X^* \)

(iv) If \( X = X_0 \) then \( X_{S_1}(a)X^* \subset X_{S_1}(a) \) for all \( a \in X \)

(v) \( [X_{S_1}(a)]^k \subset X_{S_1}(a^k) \) for all positive integers \( k > 1 \) and for all \( a \in X \).

**Proof.** (i) Let \( z \in aX_{S_1}(a) \). Then there exists \( x \in X_{S_1}(a) \) such that \( z = ax \). Since \( x \in X_{S_1}(a) \), \( axa = xa \). Now \( axa = a(ax)a = a(axa) = a(xa) = (ax)a = za \). Since
$X$ has no non-zero zero divisors, $ax \neq 0 \Rightarrow z \in X^*$. It follows that $z \in X_{S_1}(a)$ and therefore

$$aX_{S_1}(a) \subseteq X_{S_1}(a). \quad (5.8)$$

Let $y \in X_{S_1}(a)$. Then $aya = ya$. That is, $(ay - y)a = 0$, $(y - ay)a = 0$. Since $X$ has no non-zero zero divisors, $ay - y = 0$, $y - ay = 0$. This implies $y = ay \in aX_{S_1}(a)$ and therefore

$$X_{S_1}(a) \subseteq aX_{S_1}(a) \quad (5.9)$$

From relations (5.8) and (5.9) we get $aX_{S_1}(a) = X_{S_1}(a)$.

(ii) Let $z \in X_{S_1}(a) \Rightarrow aza = za$. Therefore $a(za)a = (aza)a = (za)a$. That is $a(za)a = (za)a$. Since $X$ has no non-zero zero divisors $za \neq 0$. Thus $za \in X_{S_1}(a)$ and hence $X_{S_1}(a)a \subseteq X_{S_1}(a)$.

(iii) Suppose $X$ is a finite near subtraction semigroup. Let $x_1, x_2, \ldots, x_s$ be all the elements of $X_{S_1}(a)$. From (ii) we get $x_1a, x_2a, \ldots, x_sa \in X_{S_1}(a)$, for all $a \in X^*$. We prove that they all are distinct. Suppose $x_ka = x_la$ for $k \neq l \Rightarrow (x_k - x_l)a = 0$, $(x_l - x_k)a = 0$. Since $X$ has no non-zero zero divisors $x_k - x_l = 0$, $x_l - x_k = 0$. That is, $x_k = x_l$ which is a contradiction to $k \neq l$. Therefore $x_1a, x_2a, \ldots, x_sa$ are $s$-distinct elements lying in $X_{S_1}(a)$.

Thus every element $y \in X_{S_1}(a)$ can be written as $y = x_la$ for some $x_l$ in $X_{S_1}(a) \Rightarrow y \in X_{S_1}(a)a$. Therefore $X_{S_1}(a) \subseteq X_{S_1}(a)a$ and using (ii) we get $X_{S_1}(a)a = X_{S_1}(a)$. 83
(iv) For any \( a, n \in X^* \), \( x \in X_{S_1}(a) \Rightarrow axa = xa \Rightarrow (ax - x)a = 0 \), \( (x - ax)a = 0 \). Since \( X \) has no non-zero zero divisors, \( ax - x = 0 \), \( x - ax = 0 \). Therefore \( (ax - x)na = 0 \), \( (x - ax)na = 0 \). This implies that \( axna - xna = 0 \), \( xna - axna = 0 \). That is, \( a(na) = (an)a \). Since \( X \) has no non-zero zero divisors, \( xna \neq 0 \). Therefore \( xna \notin X_{S_1}(a) \) for all \( a \in X \).

(v) Since \( X \) has no non-zero zero divisors, \( X_{S_1}(a) \) has no non-zero zero divisors for all \( a \in X \). Therefore from Lemma 5.2.8, we get \( X_{S_1}(a) \) is a multiplicative system. That is, \( xy \in X_{S_1}(a) \) for all \( x, y \in X_{S_1}(a) \). Therefore \( [X_{S_1}(a)]^2 \subset X_{S_1}(a) \).

Proceeding this way,

\[
[X_{S_1}(a)]^k \subset X_{S_1}(a) \quad (5.10)
\]

for all positive integers \( k > 1 \). Lemma 5.2.9 demands that

\[
X_{S_1}(a) \subset X_{S_1}(a^k) \quad (5.11)
\]

Combining relations (5.10) and (5.11) we get \([X_{S_1}(a)]^k \subset X_{S_1}(a^k)\). \( \Box \)

We now discuss the conditions under which \( X^* = X_{S_1}(a), a \in X \). The following is a simple characterisation of zero symmetric near subtraction semigroups.

**Proposition 5.2.11.** \( X \) is zero symmetric if and only if \( X^* = X_{S_1}(0) \).

**Proof.** For the ‘only if’ part, let \( x \in X^* \). Since \( X = X_0 \), \( x0 = 0 \Rightarrow 0x0 = 0 = x0 \Rightarrow x \in X_{S_1}(0) \). Therefore \( X^* \subset X_{S_1}(0) \). Clearly then \( X^* = X_{S_1}(0) \).

For the ‘if’ part, we observe that \( X^* = X_{S_1}(0) \Rightarrow 0x0 = x0 \) for all \( x \in X^* \) \( \Rightarrow x0 = 0 \) and hence \( X \) is zero symmetric. \( \Box \)
Proposition 5.2.12. Let $X$ be an $S_1$ near subtraction semigroup. If $X$ has a left identity ‘$e$’ then $X^* = X_{S_1}(e)$.

Proof. Let $x \in X^*$. Since $e$ is a left identity of $X$, $ex = x \Rightarrow exe = xe \Rightarrow x \in X_{S_1}(e)$. Therefore $X^* \subseteq X_{S_1}(e)$. Since $X$ is an $S_1$-near subtraction semigroup $X_{S_1}(e) \subseteq X^*$ and hence $X^* = X_{S_1}(e)$. \qed

Proposition 5.2.13. Let $X$ be an $S_1$ near subtraction semigroup. If $X$ is an $S$-near subtraction semigroup then there exists $a \in X$ such that $X^* = \bigcup \{X_{S_1}(a)/a \in X^*\}$.

Proof. Let $x \in X^*$. Since $X$ is an $S$-near subtraction semigroup, there exists $a \in X$ such that $x = ax$. This implies $xa = axa$. That is, $x \in X_{S_1}(a)$. Consequently $X^* \subseteq \bigcup \{X_{S_1}(a)/a \in X^*\}$ and the desired result now follows. \qed

5.3 Strong $S_1$-Near Subtraction Semigroups

In this section we define a strong $S_1$-near subtraction semigroup, study some of its important properties and also prove a characterisation theorem.

Definition 5.3.1. We say that $X$ is a strong $S_1$-near subtraction semigroup if $X^* = X_{S_1}(a)$ for all $a \in X$.

Examples 5.3.2. (a) We consider the near subtraction semigroup $X = \{0, a, b, 1\}$ in which ‘$-$’ and ‘.’ are defined as follows:
This is a strong $S_1$-near subtraction semigroup [In fact, a commutative and Boolean near subtraction semigroup is a strong $S_1$-near subtraction semigroup].

(b) Let $X = \{0, a, b, 1\}$ in which ‘−’ and ‘⋅’ are defined by

\[
\begin{array}{c|cccc}
- & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 1 & b \\
b & b & 0 & 0 & b \\
1 & 1 & 0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & a & 0 & b & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

We observe that it is not a strong $S_1$-near subtraction semigroup (since $X_{S_1}(b) \neq X^*$).

Definition 5.3.1 readily yields the following

**Proposition 5.3.3.** $X$ is a strong $S_1$-near subtraction semigroup if and only if $axa = xa$ for all $a \in X$ and for all $x \in X^*$.

**Proof.** Straight forward. \qed

The following Corollary is an immediate consequence of Proposition 5.3.3 and Definition 5.3.1.

**Corollary 5.3.4.** Every strong $S_1$-near subtraction semigroup is an $S_1$-near subtraction semigroup.

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**Remark 5.3.5.** Converse of Corollary 5.3.4 is not valid. For example we consider the near subtraction semigroup \((X, -, \cdot)\) with \(X = \{0, a, b, c\}\), where we define \('-'\) and \('\cdot'\) as follows:

\[
\begin{array}{c|cccc}
- & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & b & b & 0 & b \\
c & c & c & c & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & b & c \\
b & b & 0 & 0 & 0 \\
c & 0 & a & b & c \\
\end{array}
\]

We observe that this is an \(S_1\)-near subtraction semigroup. However it is not a strong \(S_1\)-near subtraction semigroup. [Since \(bab \neq ab, bcb \neq cb\)].

**Proposition 5.3.6.** If \(X\) is a strong \(S_1\)-near subtraction semigroup then \(X\) is zero symmetric.

**Proof.** Since \(X\) is a strong \(S_1\)-near subtraction semigroup, Proposition 5.3.3 demands that \(axa = xa\) for all \(a \in X\) and for all \(x \in X^*\). Putting \(a = 0\) we get \(0x0 = x0\) for all \(x \in X^*\) and this implies \(x0 = 0\) for all \(x \in X^*\). Thus \(X\) is zero symmetric. \(\square\)

**Remark 5.3.7.** The converse of Proposition 5.3.6 is not valid. For example we consider the near subtraction semigroup \((X, -, \cdot)\) where \(X = \{0, a, b, c\}\) in which \('-'\) and \('\cdot'\) are defined as follows:

\[
\begin{array}{c|cccc}
- & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & b & b & 0 & b \\
c & c & c & c & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & b & c \\
b & b & 0 & 0 & 0 \\
c & 0 & b & c & c \\
\end{array}
\]

It is zero symmetric near subtraction semigroup. But it is not a strong \(S_1\)-near subtraction semigroup [since \(cac \neq ac\)].
We furnish below an important characterisation of strong $S_1$-near subtraction semigroup which is used frequently in this chapter.

**Theorem 5.3.8.** $X$ is a strong $S_1$-near subtraction semigroup if and only if $axa = xa$ for all $a, x \in X$.

**Proof.** If $X$ is a strong $S_1$-near subtraction semigroup then Proposition 5.3.6 demands that $X$ is zero symmetric. Therefore $a0a = (a0)a = 0a$ for all $a \in X$.

The rest of the proof is taken care of by Proposition 5.3.3.

**Corollary 5.3.9.** Every $X$-system and every ideal of a strong $S_1$-near subtraction semigroup is also a strong $S_1$-near subtraction semigroup in its own right.

**Proof.** Follows from Theorem 5.3.8.

**Theorem 5.3.10.** Let $X$ be a Boolean near subtraction semigroup. Each of the following statements implies that $X$ is a strong $S_1$-near subtraction semigroup.

(i) $X$ is commutative.

(ii) $X$ is an IFP near subtraction semigroup with identity.

(iii) $X$ is a $P'$-near subtraction semigroup. (That is, $aXa = Xa$).

(iv) $X$ is subcommutative.

**Proof.** (i) Let $X$ be a commutative near subtraction semigroup and let $a, b \in X$.

Now $aba = a(ba) = a(ab)$ [since $X$ is commutative] = $a^2b = ab$ [since $X$ is Boolean] = $ba$. That is, $aba = ba$. Thus $X$ is strong $S_1$-near subtraction semigroup.

Each of (ii), (iii) and (iv) follows from the proof of Theorem 5.1.7(ii), (iii) and (iv) respectively.

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We prove some properties of strong $S_1$-near subtraction semigroup in the following theorem.

**Theorem 5.3.11.** Let $X$ be a strong $S_1$-near subtraction semigroup. Then

(i) $ab$ and $ba$ ∈ $E$ for all $a, b ∈ X$

(ii) $X$ has $(\ast, \text{IFP})$

(iii) $X$ has strong IFP

(iv) $X$ has property $(P_3)$.

**Proof.** Let $X$ be a strong $S_1$-near subtraction semigroup. Then it follows from Theorem 5.3.8 that $axa = xa$ for all $a, x ∈ X$.

(i) Let $a, b ∈ X$. Now $ab = bab = (ba)b = (aba)b = (ab)^2$. That is, $ab = (ab)^2$ ⇒ $ab ∈ E$. In similar fashion we get $ba ∈ E$.

(ii) Suppose $xy = 0$ for $x, y ∈ X$. Now $yx = xyx = (xy)x = 0x = 0$. That is, $yx = 0$. Also for every $n ∈ X$, $xny = x(ny) = x(yny) = (xy)ny = 0ny = 0$. That is, $xny = 0$. Thus $X$ has $(\ast, \text{IFP})$.

(iii) Let $I$ be an ideal of $X$. Suppose $ab ∈ I$ for $a, b ∈ X$, Proposition 5.3.6 guarantees that $X$ is zero symmetric and therefore $XI ⊂ I$. Since $I$ is an ideal of $X$, $IX ⊂ I$. Now for any $n ∈ X$, $anb = (an)b = (nan)b = na(nb) = na(bnb) = n(ab)nb ∈ XIX = (XI)X ⊂ IX ⊂ I$. That is, $anb ∈ I$. Appealing to Definition 2.2.34 we get, $X$ has strong IFP.

(iv) Let $I$ be an ideal of $X$. Suppose $xy ∈ I$ for $x, y ∈ X$. As in (iii) above, $IX ⊂ I$ and $XI ⊂ I$. Now $(yx)^2 = (yx)(yx) = y(xy)x ∈ XIX = (XI)X ⊂ IX ⊂ I$. 

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Appealing to (i) we get, \( yx = (yx)^2 \in I \). That is, \( yx \in I \). Consequently \( X \) has \( P_4 \).

**Theorem 5.3.12.** Any homomorphic image of a strong \( S_1 \)-near subtraction semigroup is a strong \( S_1 \)-near subtraction semigroup.

**Proof.** Let \( X \) be a strong \( S_1 \)-near subtraction semigroup and let \( f : X \to X' \) be a homomorphism. Since \( X \) is a strong \( S_1 \), Theorem 5.3.8 demands that \( aba = ba \) for all \( a, b \in X \). Let \( x', y' \in X' \). Then there exist \( x, y \in X \) such that \( f(x) = x' \) and \( f(y) = y' \). Clearly then \( x'y'x' = y'x' \) and the desired result now follows.

**Theorem 5.3.13.** Let \( X \) be a strong \( S_1 \)-near subtraction semigroup. If \( X \) has no non-zero zero divisors then we have the following:

(i) \( X \) is simple

(ii) \( X \) is distributive

(iii) \( X \) is Boolean

(iv) \( X \) has a mate function.

**Proof.** (i) Suppose \( X \) has a non trivial ideal. Let \( i \) be a non-zero element of \( I \).

Now let \( y \in X \). Since \( X \) is a strong \( S_1 \)-near subtraction semigroup, Theorem 5.3.8 demands that, \( iyi = yi \Rightarrow (iy - y)i = 0, (y - iy)i = 0 \). Since \( X \) has no non-zero zero divisors we get \( iy - y = 0, y - iy = 0 \). Therefore \( y = iy \in IX \subset I \) [since \( I \) is an ideal of \( X \) \( \Rightarrow y \in I \). It follows that \( I = X \). Thus \( X \) is simple.

(ii) Let \( x, y \in X \). Let \( b \in X^* \). Since \( X \) is a strong \( S_1 \), \( b(x - y)b = (x - y)b = xb - yb = bxb - byb = (bx - by)b \). That is, \( b(x - y)b = (bx - by)b \).
Clearly then \((b(x - y) - (bx - by))b = 0\), \(((bx - by) - b(x - y))b = 0\). Since \(X\) has no non-zero zero divisors, \(b(x - y) - (bx - by) = 0\) Similarly \((bx - by) - b(x - y) = 0\). Therefore we get \(b(x - y) = bx - by\). Consequently \(X\) is distributive.

(iii) Let \(a \in X^*\). Since \(aaa = aa \Rightarrow a^3 = a^2 \Rightarrow (a^2 - a)a = 0\) and \((a - a^2)a = 0\).
Since \(X\) has no non-zero zero divisors, \(a^2 - a = 0, a - a^2 = 0\). Consequently \(X\) is Boolean.

(iv) From (iii) we get, \(X\) is Boolean and we know that every Boolean near subtraction semigroup \(X\) has a mate function by Proposition 5.1.9. The desired result now follows. \(\Box\)

**Theorem 5.3.14.** Let \(X\) be a strong \(S_1\)-near subtraction semigroup. Then \(X\) has a mate function if and only if \(X\) is Boolean.

**Proof.** For the ‘only if’ part, let us assume \(f\) is a mate function for \(X\). If \(a \in X\) then, \(af(a)a = a\). Since \(X\) is a strong \(S_1\)-near subtraction semigroup, \(af(a)a = f(a)a\). That is, \(a = f(a)a\). Now \(a^2 = aa = a(f(a)a) = a\). Thus \(X\) is Boolean.

This proof of ‘if’ part is obvious. \(\Box\)

**Proposition 5.3.15.** Let \(X\) be a strong \(S_1\)-near subtraction semigroup. If \(X\) has no non-zero nilpotent elements then \(X\) is commutative.

**Proof.** Let \(y, z \in X\). Since \(X\) is a strong \(S_1\)-near subtraction semigroup, Theorem 5.3.8 demands that \(yz\) and \(zy = yz\). Now \((yz - zy)yz = (yz)^2 - (zy)(yz)\) \(= (yz)^2 - zy(zyz) = yz - (zy)^2z\) [since \(yz \in E\), Theorem 5.3.11(i)]
\(= yz - (zy)z = yz - yz = 0\). That is,

\[(yz - zy)yz = 0 \quad (5.12)\]

Also \((zy - yz)zy = (zy)^2 - yz(zy) = (zy)^2 - yz(yzy) = yz - (yz)^2y\) [since \(zy \in E\] = \(zy - (yz)y\) [since \(yz \in E\] = \(zy - yzy = 0\). That is,

\[(zy - yz)zy = 0 \quad (5.13)\]

Theorem 5.3.11(ii) demands that, \(X\) has \((\ast,\ \text{IFP})\). Consequently equations (5.12) and (5.13) imply \(yz(yz - y) = 0\) and \(zy(yz - y) = 0\). Therefore \((yz - zy)(yz - zy) = yz(yz - zy) - yz(yz - zy) = 0 \Rightarrow (yz - zy)^2 = 0\). Since \(X\) has no non-zero nilpotent elements \(yz - zy = 0\). Similarly we get, \(zy - yz = 0\). That is, \(yz = zy\). Thus \(X\) is commutative. \(\square\)

**Proposition 5.3.16.** Let \(X\) be a strong \(S_1\)-near subtraction semigroup. If \(X\) admits mate functions then it is commutative.

**Proof.** Since \(X\) admits mate function and strong \(S_1\) it becomes Boolean by Theorem 5.3.14. Therefore \(X\) has no non-zero nilpotent elements. Now Proposition 5.3.15 guarantees that \(X\) is commutative. \(\square\)

**Proposition 5.3.17.** Let \(X\) be an \(S\)-near subtraction semigroup. If \(X\) is strong \(S_1\) as well as strictly prime then \(X\) has no non-zero zero divisors.

**Proof.** We first observe from Proposition 5.3.6 that \(X\) is zero symmetric. Let \(x, y \in X\) such that \(xy = 0\). Clearly \(XX\) and \(XY\) are \(X\)-systems of \(X\). We prove that \(XXXY = \{0\}\). Let \(z \in XXXY\). Then there exist \(a, b \in X\) such that \(z = axby = ax(by) = ax(yby)\) [since \(X\) is a strong \(S_1\)-near subtraction semigroup]
= a(xy)by = a(0)by = a0 = 0 [since X = X_0]. That is, z = 0. Therefore
XxXy = \{0\}. Since X is strictly prime, either Xx = \{0\} or Xy = \{0\}. Since X
is an S-near subtraction semigroup x \in Xx and y \in Xy. Therefore either x = 0
(or) y = 0. Thus X has no non-zero zero divisors. \qed

We furnish below certain properties of identities and annihilators in a strong
S_1-near subtraction semigroup

**Proposition 5.3.18.** Let X be a strong S_1-near subtraction semigroup. Then
the following are true.

(i) Every right identity of X is left identity of X

(ii) xy is a left identity if and only if x and y are left identities for all x, y \in X

(iii) If (0 : xy) = \{0\} then xy is the identity for all x, y \in X

(iv) (0 : xy) = (0 : yx) for all x, y \in X.

**Proof.** Since X is a strong S_1-near subtraction semigroup, aba = ba
for all a, b \in X.

(i) If e is the right identity of X then xe = x for all x \in X. Now xe = exe
= e(xe) = ex \Rightarrow x = ex. That is, ex = x for all x \in X. Thus e is a left
identity of X.

(ii) Let x, y \in X. We assume that xy is a left identity. Let n \in X. Therefore
xyn = n \Rightarrow y(xyn) = yn \Rightarrow (yxy)n = yn \Rightarrow (xy)n = yn \Rightarrow n = yn. Further
xn = x(yn) = n. Therefore xn = n. Thus x and y are left identities.

Conversely, we assume that x and y are left identities. Therefore xn = n and
yn = n for all n \in X. Now (xy)n = x(yn) = xn = n. That is, (xy)n = n for all
n \in X. Thus xy is a left identity.
(iii) Let $z \in X$. Now $(z - zxy)xy = zxy - z(xy)^2 = zxy - zxy \ [\text{since } xy \in E, \ 
\text{Theorem 5.3.11(i)}] = 0$. Therefore $z - zxy \in (0 : xy)$. Similarly $zxy - z \in (0 : xy)$. 
Since $(0 : xy) = \{0\}$, $z - zxy = 0$ and $zxy - z = 0$. That is, $zxy = z$. Thus $xy$ is a right identity of $X$. Now (i) guarantees that, it is a left identity as well and (iii) follows.

(iv) Let $n \in (0 : xy) \Rightarrow nxy = 0$. Now $nyx = n(yx) = n(xy) = (nxy)x = 0x = 0$. That is, $nyx = 0 \Rightarrow n \in (0 : yx)$. Therefore

\[(0 : xy) \subseteq (0 : yx) \tag{5.14}\]

In a similar fashion, we can obtain the reverse inclusion,

\[(0 : yx) \subseteq (0 : xy) \tag{5.15}\]

From relations (5.14) and (5.15) we get $(0 : xy) = (0 : yx)$. $\square$

**Proposition 5.3.19.** Every strong $S_1$-near subtraction semigroup is a $P'_1$-near subtraction semigroup.

**Proof.** Since $X$ is a strong $S_1$-near subtraction semigroup, Theorem 5.3.8 demands that, $aba = ba$ for all $a, b \in X$. Obviously then, $Xa = aXA$ for all $a \in X$. Thus $X$ is a $P'_1$-near subtraction semigroup. $\square$

**Proposition 5.3.20.** If $X$ is a strong $S_1$-near subtraction semigroup then $aXbX = abX$ for all $a, b \in X$.

**Proof.** Since $X$ is a strong $S_1$ near subtraction semigroup, $aba = ba$ for all $a, b \in X$. Let $y \in aXbX$. Then there exist $n, n' \in X$ such that $y = anbn' = a(nb)n' = a(bn)b)n' = (ab)nbn' \in abX$. That is, $y \in abX$. Therefore

\[aXbX \subseteq abX \tag{5.16}\]
Next, let $z \in abX$. Then there exists $m \in X$ such that $z = abm = a(bm) = a(mbm) = ambm \in aXbX$. That is, $z \in aXbX$. Therefore

$$abX \subset aXbX$$ (5.17)

Combining relations (5.16) and (5.17) we get $aXbX = abX$. \qed

We conclude this chapter with the following result

**Proposition 5.3.21.** Let $X$ be a strong $S_1$-near subtraction semigroup. Then the following are equivalent.

(i) $X$ is an $S'$-near subtraction semigroup

(ii) $X$ is an $S$-near subtraction semigroup

(iii) $X$ is Boolean

(iv) $X$ admits mate function.

**Proof.** Since $X$ is a strong $S_1$-near subtraction semigroup, $aba = ba$ for all $a, b \in X$.

(i) $\Rightarrow$ (ii): Let $a \in X$. Since $X$ is an $S'$-near subtraction semigroup, $a \in aX$. Then there exists $b \in X$ such that $a = ab$. Therefore $ba = b(ab) = bab = ab = a$. That is, $a = ba \in Xa$. Thus $X$ is an $S$-near subtraction semigroup.

(ii) $\Rightarrow$ (iii): Let $a \in X$. Since $X$ is an $S$-near subtraction semigroup, $a \in Xa$. Then there exists $b \in X$ such that $a = ba$. This implies $a^2 = a(ba) = aba = ba = a$. That is, $a^2 = a$. Thus $X$ is Boolean.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): Obvious. \qed