Chapter 3

Best Proximity Point for Mappings Satisfying Generalized contractive condition of rational type on a Metric space

3.1 Introduction

In this chapter, we introduce the notion of generalized contractive condition of a rational type and prove the existence of best proximity point in the setting of metric space which generalizes the result of Eldred and Veeramani [24] and Jaggi [36]. Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$ and a map
$T : A \cup B \to A \cup B$ is called a cyclic mapping if $T(A) \subseteq B$ and $T(B) \subseteq A$.

If the fixed point equation $Tx = x$ does not possess a solution then it is natural to find an $x \in A$ satisfying

$$d(x, Tx) = d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}.$$

A point $x \in A$ is called a best proximity point for $T$ if $d(x, Tx) = d(A, B)$.

Erdred et.al [24] introduced cyclic contraction maps.

**Definition 3.1.1 ([24]).** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. A map $T : A \cup B \to A \cup B$ is called a cyclic contraction if it satisfies

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;

(ii) for some $k \in (0, 1)$ we have $d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$, for all $x \in A, y \in B$;

Using the concept of cyclic contraction Erdred et.al [24] proved the existence of best proximity point.

**Theorem 3.1.2.** [24] Let $A$ and $B$ be nonempty closed subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be cyclic contraction. If either $A$ or $B$ is boundedly compact then there exists $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$.

Jaggi [36] proved the following fixed point theorem.

**Theorem 3.1.3 ([36]).** Let $T$ be a continuous self map defined on a complete metric space $(X, d)$. Suppose that $T$ satisfies the following contractive
condition.
\[ d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y) \]
for all \( x, y \in X, x \neq y \) and for some \( \alpha, \beta \in [0, 1) \) with \( \alpha + \beta < 1 \) then \( T \) has a unique fixed point in \( X \).

In this thesis we introduce the notion of generalized contractive condition of a rational type and prove the existence of best proximity point in the setting of metric space which generalizes Theorem 3.1.2 by Eldred and Veeramani and Theorem 3.1.3 Jaggi.

### 3.2 Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the context of our results.

Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\) Define

\[ d(A, B) = \inf\{d(x, y) : x \in A, y \in B\} \]

**Definition 3.2.1 ([24]).** A subset \( K \) of a metric space \((X, d)\) is said to be boundedly compact if each bounded sequence in \( K \) has a subsequence converging to a point in \( K \).

**Definition 3.2.2.** Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\). A mapping \( T : A \cup B \to A \cup B \) is said to satisfy generalized contractive condition of a rational type if

(i) \( T(A) \subseteq B \) and \( T(B) \subseteq A \);
(ii)  
\[ d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y) \]
\[ + (1 - (\alpha + \beta + \gamma + \delta))d(A, B) \]

for all \( x \in A, y \in B \) with \( \alpha + \beta + \gamma + \delta < 1 \) where \( 0 \leq \alpha, \beta, \gamma, \delta < 1 \).

Note that if \( \alpha = \beta = \gamma = 0 \) then \( T \) is a cyclic contraction.

3.3 Best Proximity Theorems for Mappings Satisfying Generalized contractive condition of rational Type

First we give simple but very useful approximation result.

**Proposition 3.3.1.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Suppose that \( T : A \cup B \to A \cup B \) is cyclic and satisfies

\[ d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y) \]
\[ + (1 - (\alpha + \beta + \gamma + \delta))d(A, B) \]

\( \forall x \in A, y \in B \) with \( \alpha + \beta + \gamma + \delta < 1 \) where \( 0 \leq \alpha, \beta, \gamma, \delta < 1 \). Then for any \( x_0 \in A \cup B \) we have \( d(x_n, Tx_n) \to d(A, B) \) where \( x_{n+1} = Tx_n \), \( n = 0, 1, 2, \ldots \).
Proof.

\[ d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \]
\[ \leq \frac{\alpha d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) \]
\[ + \delta d(x_n, x_n) + (1 - (\alpha + \beta + \gamma + \delta))d(A, B) \]
\[ = \alpha d(x_{n-1}, x_n)d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \]
\[ + \delta d(x_n, x_n) + (1 - (\alpha + \beta + \gamma + \delta))d(A, B) \]
\[ = (\alpha + \gamma)d(x_n, x_{n+1}) + (\beta + \delta)d(x_{n-1}, x_n) \]
\[ + (1 - (\alpha + \beta + \gamma + \delta))d(A, B) \]

Therefore,

\[ d(x_n, x_{n+1}) \leq \frac{\beta + \delta}{1 - (\alpha + \gamma)}d(x_{n-1}, x_n) + (1 - \frac{\beta + \delta}{1 - (\alpha + \gamma)})d(A, B). \]

Put \( k = \frac{\beta + \delta}{1 - (\alpha + \gamma)} \) then \( k < 1 \). Therefore

\[ d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) + (1 - k)d(A, B) \]
\[ = k[kd(x_{n-2}, x_{n-1}) + (1 - k)d(A, B)] + (1 - k)d(A, B) \]
\[ = k^2d(x_{n-2}, x_{n-1}) + (1 - k^2)d(A, B). \]

Inductively we have,

\[ d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + (1 - k^n)d(A, B) \]

As \( n \to \infty \) we obtain \( d(x_n, x_{n+1}) \to d(A, B). \]

The following result of Eldred et.al [24] is a special case of the above Proposition 3.3.1
Corollary 3.3.2. Let $A$ and $B$ be nonempty subsets of a metric space $X$. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map. Then starting with any $x_0 \in A \cup B$ we have $d(x_n, Tx_n) \to d(A, B)$ where $x_{n+1} = Tx_n$, $n = 0, 1, 2 \cdots$.

Proposition 3.3.3. Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$. Let $T : A \cup B \to A \cup B$ be cyclic and satisfies
\[
d(Tx, Ty) \leq \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,y) + (1 - (\alpha + \beta + \gamma + \delta))d(A, B)
\]
\[
\forall x \in A, y \in B \text{ with } \alpha + \beta + \gamma + \delta < 1 \text{ where } 0 \leq \alpha, \beta, \gamma, \delta < 1. \text{ Let } x_0 \in A \text{ and define } x_{n+1} = Tx_n. \text{ Suppose } \{x_{2n}\} \text{ has a convergent subsequence in } A. \text{ Then there exists } x \in A \text{ such that } d(x, Tx) = d(A, B).
\]

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some $x \in A$

Then $d(A, B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})$.

Thus $d(x, x_{2n_k-1}) \to d(A, B)$.

Now,
\[
d(A, B) \leq d(x_{2n_k}, Tx) \leq \frac{\alpha d(x_{2n_k-1}, Tx_{2n_k-1})d(x, Tx)}{d(x_{2n_k-1}, x)} + \beta d(x_{2n_k-1}, Tx_{2n_k-1}) + \gamma d(x, Tx) + \delta d(x_{2n_k-1}, x) + (1 - (\alpha + \beta + \gamma + \delta))d(A, B)
\]

Taking limit as $n \to \infty$ we get
\[d(A, B) \leq d(x, Tx) \leq \frac{\alpha d(A, B)d(x, Tx)}{d(A, B)} + \beta d(A, B) + \gamma d(x, Tx) + \delta d(A, B) + (1 - (\alpha + \beta + \gamma + \delta))d(A, B)\]

That is

\[d(A, B) \leq d(x, Tx) \leq (\alpha + \gamma)d(x, Tx) + (1 - (\alpha + \gamma))d(A, B) \quad (3.3.1)\]

From (3.3.1) \(d(x, Tx) \leq (\alpha + \gamma)d(x, Tx) + (1 - (\alpha + \gamma))d(A, B)\) we have

\[(1 - (\alpha + \gamma))d(x, Tx) \leq (1 - (\alpha + \gamma))d(A, B)\]

\[d(x, Tx) \leq d(A, B) \quad (3.3.2)\]

From (3.3.1) and (3.3.2) we get \(d(A, B) \leq d(x, Tx) \leq d(A, B)\).
Thus \(d(x, Tx) = d(A, B)\).

The following result of Jaggi [36] is a special case of the above Proposition 3.3.3.

**Corollary 3.3.4.** Let \(T\) be a continuous self map defined on a complete metric space \((X, d)\). Suppose that \(T\) satisfies the following contractive condition.

\[d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y)\]

for all \(x, y \in X, x \neq y\) and for some \(\alpha, \beta \in [0, 1)\) with \(\alpha + \beta < 1\) then \(T\) has a unique fixed point in \(X\).

**Proof.** Let \(A = B = X \Rightarrow A \cup B = X\). Then \(T\) is cyclic map. Define \(x_{n+1} = Tx_n\). Then \(\{x_n\}\) is a convergent sequence in \(A\) and hence \(\{x_{2n}\}\) is a
convergent sequence in $A$. Then by Proposition 3.3.3 there exists $x \in A$ such that $d(x, Tx) = d(A, B) = 0$. Therefore $Tx = x$.

The following result of Eldred et.al [24] is a special case of the above Proposition 3.3.3

**Corollary 3.3.5.** Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$. Let $T : A \cup B \to A \cup B$ be cyclic contraction map. Let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in $A$. Then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.

**Proposition 3.3.6.** Let $A$ and $B$ be nonempty subsets of a metric space $X$.

Let $T : A \cup B \to A \cup B$ be cyclic and satisfies

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y) + (1 - (\alpha + \beta + \gamma + \delta))d(A, B)$$

$\forall x \in A, y \in B$ with $\alpha + \beta + \gamma + \delta < 1$ where $0 \leq \alpha, \beta, \gamma, \delta < 1$. Then for any $x_0 \in A \cup B$ and $x_{n+1} = Tx_n, n = 0, 1, 2, \cdots$ the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.

**Proof.** Suppose $x_0 \in A$ (the proof when $x_0 \in B$ is similar) Since by Proposition 3.3.1 $d(x_{2n}, x_{2n+1})$ converges to $d(A, B)$. So it is enough to prove that $\{x_{2n+1}\}$ is bounded.

Suppose $\{x_{2n+1}\}$ is not bounded then there exists $N_0$ such that $d(T^2x_0, T^{2N_0+1}x_0) > M$ and $d(T^2x_0, T^{2N_0-1}x_0) \leq M$ where
\[ M > \max \left\{ \frac{2d(x_0, Tx_0)}{k^2 - 1} + d(A, B), d(T^2 x_0, Tx_0) \right\} \] and \[ k = \frac{\beta + \delta}{1 - (\alpha + \gamma)}. \]

\[ M < d(T^2 x_0, T^{2N_0+1} x_0) \]
\[ \leq kd(Tx_0, T^{2N_0} x_0) + (1 - k)d(A, B) \]
\[ \leq k[d(x_0, T^{2N_0-1} x_0) + (1 - k)d(A, B)] + (1 - k)d(A, B) \]
\[ = k^2 d(x_0, T^{2N_0-1} x_0) + (1 - k^2)d(A, B) \]

Therefore
\[ \frac{M - d(A, B)}{k^2} + d(A, B) < d(x_0, T^{2N_0-1} x_0) \]
\[ \leq d(x_0, T^2 x_0) + d(T^2 x_0, T^{2N_0} x_0) \]
\[ \leq d(x_0, T^2 x_0) + M \]
\[ \leq d(x_0, Tx_0) + d(Tx_0, T^2 x_0) + M \]
\[ \leq 2d(x_0, Tx_0) + M \]

Thus \( M < \frac{2d(x_0, Tx_0)}{k^2 - 1} + d(A, B) \) which is a contradiction.  

\textbf{Corollary 3.3.7.} Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Let \( T : A \cup B \rightarrow A \cup B \) be cyclic contraction map. Then for \( x_0 \in A \cup B \) and define \( x_{n+1} = Tx_n, n = 0, 1, 2 \cdots \) the sequences \( \{x_{2n}\} \) and \( \{x_{2n+1}\} \) are bounded.

\textbf{Theorem 3.3.8.} Let \( A \) and \( B \) be nonempty closed subsets of a metric space \( X \). Let \( T : A \cup B \rightarrow A \cup B \) be cyclic contraction map and satisfies
\[ d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y) \]
\[ + (1 - (\alpha + \beta + \gamma + \delta))d(A, B) \]
for all $x \in A, y \in B$ with $\alpha + \beta + \gamma + \delta < 1$ where $0 \leq \alpha, \beta, \gamma, \delta < 1$. If either $A$ or $B$ is boundedly compact then there exists $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$.

**Proof.** Suppose $A$ is boundedly compact. Let $x_0 \in A$ and $x_{n+1} = Tx_n$. By Proposition 3.3.6 $\{x_{2n}\}$ is bounded. Since $A$ is boundedly compact we have $\{x_{2n}\}$ has a subsequence converges to a point in $A$. By Proposition 3.3.3 there exists $x \in A$ such that $d(x, Tx) = d(A, B)$. Similarly we can prove when $B$ is boundedly compact. This completes the proof. \[\Box\]

The following result of Eldred et.al [24] is a special case of the above Theorem 3.3.8

**Corollary 3.3.9.** Let $A$ and $B$ be nonempty closed subsets of a metric space $X$. Let $T : A \cup B \to A \cup B$ be cyclic contraction map. If either $A$ or $B$ is boundedly compact then there exists $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$. 

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