Chapter 2

Best Proximity Points for Generalized Semi Cyclic Contraction pair in Banach Spaces

2.1 Introduction

In this chapter, by introducing the generalized semi-cyclic contraction pair \((S, T)\) on \(A \cup B\) we prove the existence of best proximity points theorems that generalize the results of M. Gabeleh and A. Abkar [29].

Let \((X, d)\) be a metric space and let \(A\) and \(B\) be nonempty subsets of \(X\). A self mapping \(T\) on \(A \cup B\) is called a cyclic mapping if \(T(A) \subseteq B\) and \(T(B) \subseteq A\). Assume further that there exists \(0 < \alpha < 1\) such that
\[
 d(Tx, Ty) \leq \alpha d(x, y), \quad x \in A, \; y \in B.
\]
It follows that if \(A \cap B \neq \emptyset\) then the
cyclic map $T$ has a unique fixed point (see [45]). Eldred and Veeramani [24] introduce a self map $T$ on $A \cup B$ is said to be a cyclic contraction if $T$ is cyclic and there exists $\alpha \in [0,1)$ such that $d(Tx,Ty) \leq \alpha d(x,y) + (1-\alpha)d(A,B)$ for all $x \in A, y \in B$.

Note that the above condition does not entail that $A \cap B \neq \emptyset$, therefore it makes no sense to ask for a fixed point for $T$. However, one may ask for a best proximity point, that is a point $x \in A$ such that $d(x,Tx) = d(A,B)$. Eldred and Veeramani [24] proved the following existence theorem.

**Theorem 2.1.1.** [24] Let $A$ and $B$ be nonempty closed convex subsets of a uniformly convex Banach space $X$. Suppose $T$ is a cyclic contraction on $A \cup B$, then $T$ has a unique best proximity point $z$ in $A$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

Recently M. Gabeleh and A. Abkar [29] introduced the concept of a semi-cyclic contraction pair and proved the following theorem which generalizes the result of A. A. Eldred and P.Veeramani [24].

**Theorem 2.1.2** ([29]). Let $A, B$ be two nonempty closed convex subsets of a uniformly convex Banach space $X$. Let $(S,T)$ be a semi-cyclic contraction pair

(i) if $d(A,B) = 0$, then $S,T$ have a unique common fixed point in $A \cap B$.

(ii) if $d(A,B) > 0$, then each mapping has a unique best proximity point.

Moreover either of fixed point or best proximity points can be approximated
In this thesis we introduce the new concept called generalized semi-cyclic contraction pair and prove some results that generalizes the results of M. Gabeleh and A. Abkar [29].

2.2 Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the context of our results.

Define

\[d(x, B) = \inf \{d(x, y) : y \in B \} \quad x \in X\]

\[d(A, B) = \inf \{d(x, y) : x \in A, y \in B \}\]

**Definition 2.2.1.** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(S, T\) be two self maps on \(A \cup B\). We call \((S, T)\) is generalized semi-cyclic contraction pair if it satisfies:

(i) \(S(A) \subseteq B\) and \(T(B) \subseteq A\);

(ii) \(d(Sx, Ty) \leq \alpha d(x, y) + \beta d(x, B) + \gamma d(A, y) + \delta d(A, B), \quad \forall x \in A, y \in B;\)

where \(0 \leq \alpha, \beta, \gamma, \delta \leq 1\) with \(\alpha + \beta + \gamma + \delta = 1\) and \(\delta \neq 0\).

Note that if \(\beta = \gamma = 0\), a generalized semi-cyclic contraction pair reduces to a semi-cyclic contraction pair. In addition if \(T = S\) it reduces to a cyclic contraction.
Example 2.2.2. Consider the space $X = \mathbb{R}^2$ with usual metric

Let $A = \{(x, 0) : 0 \leq x \leq 3\}$ and $B = \{(x, \frac{1}{2}) : 0 \leq x \leq 2\}.$

Define $S, T : A \cup B \to A \cup B$ by

$$S(x, y) = \begin{cases} 
(0, \frac{1}{2}) & \text{if } (x, y) \in A \\
(x, y) & \text{if } (x, y) \in B 
\end{cases}$$

and

$$T(x, y) = \begin{cases} 
(0, 0) & \text{if } (x, y) \in B \\
(x, y) & \text{if } (x, y) \in A 
\end{cases}$$

Clearly $S(A) \subseteq B$ and $T(B) \subseteq A.$ Note that neither $S$ nor $T$ is cyclic.

Let $a = (x, 0) \in A$ and $b = (x, \frac{1}{2}) \in B.$ $S(a) = (0, \frac{1}{2}),$ $T(b) = (0, 0),$ $d(Sa, Tb) = \frac{1}{2},$ $d(A, B) = \frac{1}{2},$ $d(a, B) = \frac{1}{2},$ and $d(a, b) = \frac{1}{2}.$

$$d(Sa, Tb) = \frac{1}{2} \leq \alpha d(a, b) + \beta d(a, B) + \gamma d(A, b) + \delta d(A, B)$$

$$\leq (\alpha + \beta + \gamma + \delta)\frac{1}{2} = \frac{1}{2}.$$

Hence $(S, T)$ is a generalized semi-cyclic contraction pair.

Definition 2.2.3 ([24]). A subset $K$ of a metric space $(X, d)$ is said to be boundedly compact if each bounded sequence in $K$ has a subsequence converging to a point in $K.$

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2.3 Best Proximity Theorems For Generalized Semi Cyclic Contraction Pair

First we give simple but very useful approximation result.

Proposition 2.3.1. Let \((S, T)\) be a generalized semi-cyclic contraction pair. Consider \(x_0 \in A\) and define:

\[ x_{n+1} = Ty_n \text{ and } y_n = Sx_n, \text{ for } n = 0, 1, 2 \cdots. \]  

(2.3.1)

then \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(A, B\), respectively. Moreover \(d(x_n, Sx_n) \rightarrow d(A, B)\) and \(d(y_n, Ty_n) \rightarrow d(A, B)\).

Proof.

\[
\begin{align*}
    d(x_n, Sx_n) &= d(Ty_{n-1}, Sx_n) \\
    &\leq \alpha d(y_{n-1}, x_{n-1}) + \beta d(x_n, B) + \gamma d(A, y_{n-1}) + \delta d(A, B) \\
    &\leq \alpha d(y_{n-1}, x_{n-1}) + \beta d(x_n, y_{n-1}) + \gamma d(x_n, y_{n-1}) + \delta d(A, B) \\
    &= (\alpha + \beta + \gamma)d(y_{n-1}, x_n) + (1 - (\alpha + \beta + \gamma))d(A, B) \\
    &= (\alpha + \beta + \gamma)d(Sx_{n-1}, Ty_{n-1}) + (1 - (\alpha + \beta + \gamma))d(A, B) \\
    &\leq (\alpha + \beta + \gamma)^2d(x_{n-1}, Sx_{n-1}) + (1 - (\alpha + \beta + \gamma)^2)d(A, B) \\
    &\vdots \\
    &\leq (\alpha + \beta + \gamma)^{2n}d(x_0, y_0) + (1 - (\alpha + \beta + \gamma)^{2n})d(A, B).
\end{align*}
\]

Since \(\alpha + \beta + \gamma < 1\) it follows that \(d(x_n, Sx_n) \rightarrow d(A, B)\) as \(n \rightarrow \infty\). Similarly, \(d(y_n, Ty_n) \rightarrow d(A, B)\).
The following result of M. Gabeleh and A. Abkar [29] is a special case of the above Proposition 2.3.1.

**Corollary 2.3.2.** Let \((S, T)\) be a semi-cyclic contraction pair.

Consider \(x_0 \in A\) and define:

\[ x_{n+1} = T y_n \quad \text{and} \quad y_n = S x_n, \quad \text{for } n = 0, 1, 2 \ldots \]

then \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(A, B\), respectively.

Moreover \(d(x_n, Sx_n) \to d(A, B)\) and \(d(y_n, Ty_n) \to d(A, B)\).

**Proposition 2.3.3.** Let \((S, T)\) be a generalized semi-cyclic contraction pair.

Consider the iterative sequences defined by (2.3.1). If both \(\{x_n\}\) and \(\{y_n\}\) have a convergent subsequence in \(A\) and \(B\), respectively, then there exist \(x \in A\) and \(y \in B\) such that

\[ d(x, Sx) = d(A, B) = d(y, Ty). \]

**Proof.** Let \(\{y_{n_k}\}\) be a subsequence of \(\{y_n\}\) such that \(y_{n_k} \to y\). Since

\[ d(A, B) \leq d(T y_{n_k}, y) \leq d(y, y_{n_k}) + d(y_{n_k}, Ty_{n_k}), \]

it follows from Proposition 2.3.1 that \(d(y, Ty_{n_k}) \to d(A, B)\). On the other hand

\[ d(A, B) \leq d(T y, y_{n_k}) = d(T y, Sx_{n_k}) \]

\[ \leq \alpha d(y, x_{n_k}) + \beta d(x_{n_k}, B) + \gamma d(A, y) + \delta d(A, B) \]

\[ \leq \alpha d(y, x_{n_k}) + \beta d(x_{n_k}, y) + \gamma d(x_{n_k}, y) + \delta d(A, B) \]

\[ = (\alpha + \beta + \gamma) d(y, Ty_{n_k-1}) + (1 - (\alpha + \beta + \gamma)) d(A, B) \]

\[ \to d(A, B) \text{ as } k \to \infty. \]
Hence \( d(Ty, y) = d(A, B) \). Similarly, \( d(x, Sx) = d(A, B) \).

The following result of M. Gabeleh and A. Abkar [29], is a special case of the above Proposition 2.3.3.

**Corollary 2.3.4.** Let \((S, T)\) be a semi-cyclic contraction pair. Consider the iterative sequences defined by (2.3.1). If both \(\{x_n\}\) and \(\{y_n\}\) have a convergent subsequence in \(A\) and \(B\), respectively, then there exist \(x \in A\) and \(y \in B\) such that

\[
d(x, Sx) = d(A, B) = d(y, Ty).
\]

**Proposition 2.3.5.** Let \((S, T)\) be a generalized semi-cyclic contraction pair. Then the iterative sequences defined by (2.3.1) are bounded.

**Proof.** Since \(d(x_n, Sx_n) \to d(A, B)\), it is enough to show that the sequence \(\{Sx_n\}\) is bounded in \(B\). Suppose not, then there exists \(N_0\) such that

\[
d(x_1, Sx_{N_0}) > M \text{ and } d(x_1, Sx_{N_0-1}) \leq M
\]

where \(M > \max\{ \frac{2d(x_0, Sx_0)}{1 - (\alpha + \beta + \gamma)^2} + \frac{1}{1 - (\alpha + \beta + \gamma)^2}d(A, B), d(x_1, Sx_0) \}\).

Since \((S, T)\) is a generalized semi-cyclic contraction pair, we have

\[
M < d(x_1, Sx_{N_0}) = d(Ty_0, Sx_{N_0})
\]

\[
\leq \alpha d(y_0, x_{N_0}) + \beta d(x_{N_0}, B) + \gamma d(A, y_0) + \delta d(A, B)
\]

\[
\leq (\alpha + \beta + \gamma) d(y_0, x_{N_0}) + (1 - (\alpha + \beta + \gamma))d(A, B)
\]

\[
\leq (\alpha + \beta + \gamma)^2 d(x_0, y_{N_0-1}) + (1 - (\alpha + \beta + \gamma)^2) d(A, B)
\]

Therefore, \((\frac{1}{(\alpha + \beta + \gamma)^2})(M - (1 - (\alpha + \beta + \gamma)^2)d(A, B)) < d(x_0, Sx_{N_0-1})\), and
consequently

\[
\left( \frac{1}{(\alpha + \beta + \gamma)^2} \right) (M - d(A, B)) + d(A, B) < d(x_0, Sx_{N_0-1})
\]

\[
\leq d(x_0, Ty_0) + d(Ty_0, Sx_{N_0-1}) \leq d(x_0, Ty_0) + M
\]

\[
\leq d(x_0, Sx_0) + d(Sx_0, Ty_0) + M
\]

\[
\leq d(x_0, Sx_0) + (\alpha + \beta + \gamma)d(x_0, y_0)
\]

\[
+ (1 - (\alpha + \beta + \gamma))d(A, B) + M
\]

\[
\leq 2d(x_0, Sx_0) + d(A, B) + M.
\]

This implies

\[
M \left( \frac{1}{(\alpha + \beta + \gamma)^2} - 1 \right) < 2d(x_0, Sx_0) + \frac{1}{(\alpha + \beta + \gamma)^2}d(A, B)
\]

or equivalently \(M < \frac{2d(x_0, Sx_0)}{(\alpha + \beta + \gamma)^2 - 1} + \frac{1}{1-(\alpha + \beta + \gamma)^2}d(A, B)\), which is a contradiction.

The following result of M. Gabeleh and A. Abkar [29] is a special case of the above Proposition 2.3.5.

**Corollary 2.3.6.** Let \((S, T)\) be a semi-cyclic contraction pair. Then the iterative sequences defined by (2.3.1) are bounded.

**Theorem 2.3.7.** Let \((S, T)\) be a generalized semi-cyclic contraction pair in a complete metric space \(X\). If either \(A\) or \(B\) is boundedly compact, then there exists \(z \in A \cup B\) such that either \(d(z, Sz) = d(A, B)\) or \(d(z, Tz) = d(A, B)\).

**Proof.** The result is an immediate consequence of Propositions 2.3.3 and 2.3.5.
Lemma 2.3.8. ([24]) Let $A$ be a nonempty closed convex subset, and $B$ be a nonempty closed subset of a uniformly convex Banach space $X$. Let $\{x_n\}, \{z_n\}$ be two sequences in $A$, and $\{y_n\}$ be a sequence in $B$ such that

(i) $\|z_n - y_n\| \to d(A, B)$,

(ii) $\forall \epsilon > 0, \exists N_0$ such that for all $m > n \geq N_0$, $\|x_m - y_n\| \leq d(A, B)$.

Then for every $\epsilon > 0$ there exists $N_1$ such that for all $m > n \geq N_1$ we have $\|x_m - z_n\| \leq \epsilon$.

Lemma 2.3.9. ([24]) Let $A$ be a nonempty closed convex subset, and $B$ be a nonempty closed subset of a uniformly convex Banach space $X$. Let $\{x_n\}, \{z_n\}$ be two sequences in $A$, and $\{y_n\}$ be a sequence in $B$ such that

(i) $\|x_n - y_n\| \to d(A, B)$,

(ii) $\|z_n - y_n\| \to d(A, B)$.

Then $\|x_n - z_n\| \to 0$.

Theorem 2.3.10. Let $A, B$ be two nonempty closed convex subsets of a uniformly convex Banach space $X$. Let $(S, T)$ be a generalized semi-cyclic contraction pair.

(i) if $d(A, B) = 0$, then $S, T$ have a unique common fixed point in $A \cap B$.

(ii) if $d(A, B) > 0$, then each mapping has a unique best proximity point.

Moreover either of fixed point or best proximity points can be approximated by some iterative sequences.
Proof. Assume that \( d(A, B) = 0 \). Then we have

\[
\|Sx - Ty\| \leq \alpha \|x - y\| + \beta d(x, B) + \gamma d(A, y) \forall x \in A, y \in B
\]

\[
\leq (\alpha + \beta + \gamma) \|x - y\|.
\]

Define a sequence \( \{z_n\} \) in \( A \cup B \) in the following way:

\[
z_n = \begin{cases} 
Ty_k, & n = 2k \\
Sx_k, & n = 2k - 1.
\end{cases}
\]

To show that \( \{z_n\} \) is a Cauchy sequence in \( A \cup B \). If \( n = 2k \) we have

\[
\|z_{n+1} - z_n\| = \|Sx_{k+1} - Ty_k\| \leq (\alpha + \beta + \gamma) \|x_{k+1} - y_k\|
\]

\[
\leq (\alpha + \beta + \gamma)^2 \|y_k - x_k\| \leq \cdots \leq (\alpha + \beta + \gamma)^{2k} \|y_1 - x_1\|
\]

\[
\to 0 \text{ as } k \to \infty.
\]

Similarly, for \( n = 2k - 1 \), we can get the same conclusion, so that for \( m > n \) we have

\[
\|z_m - z_n\| \leq \sum_{k=n}^{m-1} (\alpha + \beta + \gamma)^{2k} \|y_1 - x_1\| \to 0, \ n, m \to \infty.
\]

Then there exists \( z \in A \cup B \) such that \( z_n \to z \). Assume that \( z \in A \). Since \( \{z_{2k-1}\} \subseteq B \), it follows that \( z \in B \), and finally \( z \in A \cap B \). If \( z \in B \), the same argument again shows that \( z \in A \cap B \). On the other hand,

\[
\|z - Tz\| = \lim_{k \to \infty} \|y_k - Tz\| = \lim_{k \to \infty} \|Sx_k - Tz\|
\]

\[
\leq \lim_{k \to \infty} (\alpha + \beta + \gamma) \|x_k - z\| = 0.
\]

This implies that \( Tz = z \). Similarly, \( Sz = z \). Hence \( T, S \) have a common fixed point. We claim that the fixed point \( z \) is unique. In fact if \( Tw = w = Sw \)
for some \( w \in A \cap B \), then \( \|z - w\| = \|Tz - Sw\| \leq (\alpha + \beta + \gamma)\|z - w\| \). This implies that \( z = w \).

Assume that \( d(A, B) > 0 \). Since \((S, T)\) is a generalized semi-cyclic contraction pair, we have

\[
\|y_n - Ty_n\| = \|Sx_n - Ty_n\| \leq (\alpha + \beta + \gamma)\|x_n - y_n\| + (1 - (\alpha + \beta + \gamma))d(A, B).
\]

This together with Proposition 2.3.1 implies that \( \|y_n - x_{n+1}\| \to d(A, B) \). Similarly, \( \|y_{n+1} - x_{n+1}\| \to d(A, B) \). Now it follows from Lemma 2.3.9 that \( \|y_n - y_{n+1}\| \to 0 \). Similarly \( \|x_n - x_{n+1}\| \to 0 \). Now we claim that for every \( \epsilon > 0 \) there exists \( N_0 \) such that for all \( m > n > N_0 \) we have

\[
\|y_m - Ty_n\| = \|y_m - x_{n+1}\| \leq d(A, B) + \epsilon.
\]

Suppose not; then there exists \( \epsilon > 0 \) such that for all \( k \geq 1 \) there exist \( m_k > n_k \geq k \) for which \( \|y_{m_k} - Ty_{n_k}\| \geq d(A, B) + \epsilon \). This \( m_k \) can be chosen in such a way that it is the least integer greater than \( n_k \) to satisfy the above inequality. Now

\[
d(A, B) + \epsilon \leq \|y_{m_k} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_k-1}\| + \|y_{m_k-1} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_k-1}\| + d(A, B) + \epsilon.
\]

Hence

\[
\|y_{m_k} - Ty_{n_k}\| \to d(A, B) + \epsilon.
\]

Then

\[
\|y_{m_k} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_k+1}\| + \|y_{m_k+1} - Ty_{n_k+1}\| + \|Ty_{n_k+1} - Ty_{n_k}\|
\leq \|y_{m_k} - y_{m_k+1}\| + (\alpha + \beta + \gamma)^2\|y_{m_k} - Ty_{n_k}\| + (1 - (\alpha + \beta + \gamma)^2)d(A, B) + \|Ty_{n_k+1} - Ty_{n_k}\|.
\]
As \( k \to \infty \), we get,
\[
\begin{align*}
    d(A, B) + \epsilon &\leq (\alpha + \beta + \gamma)^2 (d(A, B) + \epsilon) + (1 - (\alpha + \beta + \gamma)^2) d(A, B) \\
    &= d(A, B) + (\alpha + \beta + \gamma)^2 \epsilon.
\end{align*}
\]
which is a contradiction. Therefore \( \{y_n\} \) is a Cauchy sequence by lemma 2.3.8. Therefore there exists \( y \in B \) such that \( \{y_n\} \) converges to \( y \). It follows from Proposition 2.3.3 that \( \|y - Ty\| = d(A, B) \). Similarly we can prove that the sequence \( \{x_n\} \) converges to some \( x \in A \) and \( \|x - Sx\| = d(A, B) \).

To prove the uniqueness let \( w \in A \) such that \( \|w - Sw\| = d(A, B) \). Since \( d(A, B) \leq \|TSx - Sx\| \)
\[
\leq (\alpha + \beta + \gamma) \|Sx - x\| + (1 - (\alpha + \beta + \gamma)) d(A, B) = d(A, B),
\]
it follows that \( \|TSx - Sx\| = \|x - Sx\| \). This implies \( TSx = x \).

Similarly, \( TSw = w \). Now if \( w \neq x \), then \( \|x - Sw\| > d(A, B) \). Therefore,
\[
\|Sx - w\| = \|Sx - TSw\| \leq (\alpha + \beta + \gamma) \|x - Sw\|
\]
\[
+ (1 - (\alpha + \beta + \gamma)) d(A, B)
\]
\[
< (\alpha + \beta + \gamma) \|x - Sw\|
\]
\[
+ (1 - (\alpha + \beta + \gamma)) \|x - Sx\|
\]
\[
= \|x - Sw\| = \|TSx - Sw\|
\]
\[
\leq (\alpha + \beta + \gamma) \|Sx - w\|
\]
\[
+ (1 - (\alpha + \beta + \gamma)) d(A, B)
\]
\[
\leq \|Sx - w\|
\]
which is a contradiction. \( \square \)

The following result of M. Gabeleh and A. Abkar [29] is a special case of the above Theorem 2.3.10.
Corollary 2.3.11. Let $A, B$ be two nonempty closed convex subsets of a uniformly convex Banach space $X$. Let $(S, T)$ be a semi-cyclic contraction pair. (i) if $d(A, B) = 0$, then $S, T$ have a unique common fixed point in $A \cap B$. (ii) if $d(A, B) > 0$, then each mapping has a unique best proximity point. Moreover either of fixed points or best proximity points can be approximated by some iterative sequences.