Chapter 5

Equilibrium Pair Results of
$S - KKM_0$ Multimap for Free $n$- Person Games

5.1 Introduction

In this chapter, we establish the existence theorem of equilibrium pair results of $S - KKM_0$ multimap for free $n$- person games in the setting of Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$ as an application.

We present some basic definitions and results to be used in the sequel. Throughout this chapter $E$ is a Hausdorff locally convex topological vector space with a continuous seminorm $p$. $A, B$ are nonempty subsets of $E$, $2^A$ is the family of all subsets of $A$, $Co(A)$ is the convex hull of $A$ in $E$, $int A$ is the interior of $A$ in $E$, $C(A, B)$ is the set of all continuous single valued maps
\[ d_p(x, A) = \inf \{ p(x-a) : a \in A \} \text{ and } d_p(A, B) = \inf \{ p(a-b) : a \in A, b \in B \} \]

A map \( T : A \to 2^B \) is called a multimap (multifunction or correspondence) if \( T(x) \) is non-empty for each \( x \in A \). A multimap \( T : A \to 2^A \) is said to have a fixed point \( a \in A \) if \( a \in T(a) \); the set of all fixed points of \( T \) is denoted by \( F(T) \). A multimap \( T : A \to 2^B \) is said to be

(a) upper semi-continuous if \( T^{-1}(D) = \{ x \in A : T(x) \cap D \neq \emptyset \} \) is closed in \( A \) whenever \( D \) is closed in \( B \).

(b) compact if \( \overline{T(A)} \) is compact in \( B \).

(c) closed if its graph \( G_r(T) = \{ (x, y) : x \in A \text{ and } y \in T(x) \} \) is closed in \( A \times B \) and

(d) compact valued (resp. convex) if \( T(x) \) is compact (resp. convex) in \( B \) for every \( x \in A \).

A map \( f : A \to B \) is proper if \( f^{-1}(K) \) is compact in \( A \) whenever \( K \) is compact in \( B \). A map \( f : A \to E \) is quasi \( p \)-affine if the set \( Q(x) = \{ a \in A : p(f(a) - x) \leq r \} \) is convex for every \( x \in E \) and \( r \in [0, \infty) \).

5.2 Preliminaries

**Definition 5.2.1** ([16]). Given a convex subset \( C \) of a topological vector space \( E \) with a seminorm \( p \). A single valued function \( g : C \to E \) is said to be almost \( p \)-affine if \( p((\lambda u + (1-\lambda)v) - x) \leq \lambda p(u - x) + (1-\lambda)p(v - x) \) for all \( u, v \in C \) and \( x \in E \).

Clearly any almost \( p \)-affine mapping is quasi \( p \)-affine but the converse is not true.
Example 5.2.2. Let $E = \mathbb{R}^2$ and the seminorm $p$ in $E$ be defined as $p(x, y) = \sqrt{x^2 + y^2}$. Let $g : E \to E$ be defined as follows

$$g(x, y) = \begin{cases} (2, e^x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $g$ is quasi $p$–affine but not almost $p$–affine.

Definition 5.2.3. Let $A$ be a non-empty subset of a topological vector space $E$ with a continuous seminorm $p$. A single valued function $g : A \to E$ is said to be $p$–continuous if $p[g(x_\alpha) - g(x)] \to 0$ for each $x$ in $A$ and every net $\{x_\alpha\}$ in $A$ converging to $x$.

It is apparent that $p$–continuity is in general weaker than continuity.

Example 5.2.4. Let $E = \mathbb{R}^2$ with the seminorm $p : E \to [0, \infty]$ defined as $p(x, y) = |x|$ for all $(x, y) \in E$. Let $g : E \to E$ be defined as

$$g(x, y) = \begin{cases} (0, 1) & \text{if } (x, y) \neq (0, 0) \\ (0, 0) & \text{otherwise} \end{cases}$$

Then $g$ is $p$–continuous but not continuous.

Definition 5.2.5 ([66]). Let $A$ be a non-empty subset of a Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$. A set $A$ is said to be approximately $p$–compact if for every $y \in E$ and each net $\{x_\alpha\}$ in $A$ satisfying the condition that $p(x_\alpha - y) \to d_p(y, A)$, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ converging to an element in $A$.

Note that approximately compact subsets are closed.
Evidently any compact subset of a Hausdorff locally convex topological vector space with a continuous seminorm is approximately $p$-compact. However, approximately compact sets which are not compact are available in great profusion. Indeed any non-empty closed and convex subset of a uniformly convex Banach space is approximately compact.

The set $P_A(x) = \{ a \in A : p(a - x) = d_p(x, A) \}$ is called the set of $p$-best approximation in $A$ to $x \in E$. Let $A, B$ be non-empty subsets of $E$. A polytope $P$ in $A$ is any convex hull of a non-empty finite subset $D$ of $A$. Whenever $\mathcal{X}$ is a class of maps, denote the set of all finite compositions of maps in $\mathcal{X}$ by $\mathcal{X}_c$ and denote the set of all multimaps $T : A \to 2^B$ in $\mathcal{X}$ by $\mathcal{X}(A, B)$. Let $\mathcal{U}$ be an abstract class of maps [66] satisfies the following properties.

1. $\mathcal{U}$ contains the class $\mathcal{C}$ of continuous single valued maps
2. each $T \in \mathcal{U}_c$ is upper semicontinuous with compact values
3. for any polytope $P$, each $T \in \mathcal{U}_c(P, P)$ has a fixed point.

**Definition 5.2.6.** Let $T : A \to 2^B$. We say that

(a) $T$ is an $\mathcal{U}^k_c$-multimap [66] if for every compact set $K$ in $A$ there exists an $\mathcal{U}_c$-multimap $f : K \to 2^B$ such that $f(x) \subseteq T(x)$ for each $x \in K$.

(b) $T$ is a $K$-multimap (or Kakutani multimap) [48] if $T$ is upper semicontinuous with compact and convex values

(c) $S : A \to 2^B$ is a generalized KKM-multimap with respect to $T$ [17] if $T(Co(D)) \subseteq S(D)$ for each finite subset $D$ of $A$.

(d) $T$ has the KKM property [17] if whenever $S : A \to 2^B$ is a generalized KKM-multimap with respect to $T$, the family $\{ S(x) : x \in A \}$ has a finite
intersection property.

(e) \( T \) is a PK-multipart \([59]\) if there exists a multipart \( g : A \to 2^B \) satisfying \( A = \bigcup \{ \text{int} g^{-1}(y) : y \in A \} \) and \( C_0(g(x)) \subseteq T(x) \) for each \( x \in A \).

Note that each \( U_k \)-multipart has the KKM property and each K-multipart (resp. \( U_c \)-multipart PK-multipart) is a \( U_k \)-multipart (see [49], [50], [66]). \( T : A \to 2^B \) is called a Kakutani factorization multipart \([48]\)if it can be expressed as a composition of finitely many K-multimaps.

The set of all K-multimaps (resp. Kakutani factorizable) \( T : A \to 2^B \) is denoted by \( K(A, B) \) (resp. \( K_c(A, B) \)).

The following lemma is needed in the sequel.

**Lemma 5.2.7** ([48]). If \( A \) is a non-empty, compact and convex subset of \( E \) and \( T \in K_c(A, A) \), then \( T \) has a fixed point in \( A \).

The following notations are essential in this sequel.

Let \( A, B_i \) be non-empty subsets of a topological vector space with a semi-norm \( p \), for each \( i \in I_n = \{1, 2, 3, \ldots, n\} \), define

\[
d_p(A, B_i) = \inf \{ p(a - b) : a \in A \text{ and } b \in B_i \}.
\]

\[
p_{\text{prox}}(A, B_i) = \{(a, b) \in A \times B_i : p(a - b) = d_p(A, B_i)\}
\]

\[
A_i^p = \{a \in A : p(a - b) = d_p(A, B_i) \text{ for some } b \in B_i\}
\]

\[
B_i^p = \{b \in B_i : p(a - b) = d_p(A, B_i) \text{ for some } a \in A\}
\]

\[
A^o = \bigcap_{i \in I_n} A_i^p \text{ for } n = 1, \text{ let } A_0 = A_1^o = A^o \text{ and } B_0 = B_1^o \text{ (see } [65]).
\]

Note that for each \( i \in I_n \), \( A_i^p \) is non-empty if and only if \( B_i^p \) is non-empty.

The following is essential in proving our results.
Lemma 5.2.8. Let $A$ and $B_i$ for each $i \in I_n$ be non-empty subsets of $E$. The following statements hold for each $i \in I_n$.

(a) If $A_i^\circ$ (resp. $A$) and $B_i$ are convex, then $B_i$ is (resp. $A_i^\circ$ and $B_i^\circ$) convex.

(b) If $A_i^\circ$ (resp. $A$) and $B_i$ are compact, then $B_i$ is (resp. $A_i^\circ$ and $B_i^\circ$) compact.

(c) $P_A(B_i^\circ) = P_{A_i^\circ}(B_i^\circ) = A_i^\circ$.

(d) If $A_i^\circ$ is non-empty, compact and convex and $B_i^\circ$ is convex, then $P_{A_i^\circ}\mid B_i^\circ$ is a $K$–multimap.

Remark 5.2.9. Note that from part (c) of the above theorem and the defi-
nition of $A_i^\circ$, $A_i^\circ$ and $B_i^\circ$, we have

$(C_1)$ $A_i^\circ$ is non-empty if and only if $B_i^\circ$ is non-empty.

$(C_2)$ $A^\circ \neq \emptyset$ is equivalent to $\bigcap_{i=1}^n P_A(b_i) \neq \emptyset$ for some $(b_1, b_2, \ldots, b_n) \in \prod_{i=1}^n B_i^\circ$.

$(C_3)$ $P_A(B_i^\circ) = A_i^\circ$ if and only if $A_i^\circ = A^\circ$; so by Kim and Lee[42], [43] Theorem 1,2 and 4] are valid only whenever $A_i^\circ = A^\circ$.

$(C_4)$ $\bigcap_{i=1}^n P_A(B_i^\circ) = \bigcap_{i=1}^n P_{A_i^\circ}(B_i^\circ) = \bigcap_{i=1}^n A_i^\circ = A^\circ$. So $A^\circ \neq \emptyset$ if and only if $\bigcap_{i=1}^n P_{A_i^\circ}(y_i) \neq \emptyset$ for some $(y_1, y_2, \ldots, y_n) \in \prod_{i=1}^n B_i^\circ$.

Lemma 5.2.10 ([66]). Let $A$ be a nonempty, approximately $p$-compact and convex subset of a Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$.

For every $y \in E$, let $P_A(y) = \{x \in A : p(x - y) = d_p(y, A)\}$ then the following statements hold good.
(a) The set $P_A(y)$ is a nonempty, compact and convex subset of $A$. (Every element of $P_A(y)$ is called a $p$–best approximation in $A$ to $y \in E$.)

(b) The multifunction $P_A : E \to 2^A$ is upper semicontinuous. (The multifunction $P_A$ is called a projection map).

**Definition 5.2.11** ([3]). Let $T : A \to 2^B$ be a multimap. $T$ is said to be a KKM$_0$ multimap if $T$ and $S \circ T : A \to 2^A$ are closed and have the KKM property for each K-multimap $S : B \to 2^A$.

**Lemma 5.2.12** ([72]). Let $A, B$ be notempty subsets of a normed space $E$. If $T : A \to 2^B$ is an upper semicontinuous multi map with compact values, then $T$ is closed.

**Lemma 5.2.13.** ([17], [49]) Let $A$ be a nonempty convex subset of normed space $E$. If $T : A \to 2^A$ is closed and compact multimap having the KKM property, then $T$ has a fixed point.

**Lemma 5.2.14** ([21]). For each $i \in I_n$, let $B_i$ be a nonempty, compact and convex subset of a normed space $E$ and $P_i : \prod_{j=1}^n B_j \to 2^B_i$ a map such that

(a) $x_i \notin C_0 P_i(x)$ for each $u = (x_1, \ldots, x_n) \in B = \prod_{j=1}^n B_j$.

(b) $P_i^{-1}(y)$ is open in $B$ for each $y \in B_i$.

Then there exists $b \in B$ such that $P_i(b) = \emptyset$ for each $i \in I_n$.

**Lemma 5.2.15.** ([11], [21], [42], [43]) Let $B$ be a nonempty, compact and convex subset of a normed space $E$ and $P : B \to 2^B$ a map such that
\((a)~x \notin C_0 P(x)\) for each \(x \in B\).

\((b_1)\) If \(z \in P^{-1}(y)\), then there exists some \(y' \in B\) such that \(z \in \text{int}P^{-1}(y')\).

\((b_2)\) \(P^{-1}(y')\) is open in \(B\) for each \(y \in B\).

Then there exists \(b \in B\) such that \(P(b) = \emptyset\).

Now we present best proximity results for \(KKM_0\)-multimaps.

**Lemma 5.2.16** ([41]). Let \(A\) and \(B_i\) be subsets of a Hausdorff locally convex topological vector space \(E\) with a continuous seminorm \(p\) such that \(A_i^0\) (resp. \(B_i^0\)) are nonempty approximately \(p\)-compact (resp. closed) and convex for each \(i \in I_n\). Assume that one of \(A_i^0\)'s is contained in some compact subset of \(A\). Let \(f : A^0 \to A^0\) be a \(p\)-continuous, proper, quasi \(p\)-affine and surjective self map and \(P : Y \to 2^{A^0}\) be a multimap defined by \(P(y_1, y_2, \ldots, y_n) = \bigcap_{i=1}^n P_{A_i^0}(y_i)\) for each \((y_1, y_2, \ldots, y_n) \in Y = \prod_{i=1}^n B_i^0\). Then \(f^{-1}P : Y \to 2^{A^0}\) is a \(K\)-multimap.

**Definition 5.2.17** ([4]). Let \(A, B_i\) be a nonempty subsets of a topological space \(E\) with a continuous seminorm \(p\). Let \(T_i : A \to 2^{B^i}\) be a multimap for each \(i \in I_n\), \(f : A' \to A'\) a self map of a nonempty subset \(A'\) of \(A\) and \(a \in A\).

If \(d_P(f(a), T_i(a)) = d_P(A, B_i)\), then we say that \((f(a), T_i(a))\) is a best proximity pair. The best proximity set for the pair \((f(a), T_i(a))\) is given below.

\[\Sigma_a^i(f) = \{b \in T_i(a) : d_P(f(a), T_i(a)) = p(f(a) - b) = d_P(A, B_i)\}.\]

**Theorem 5.2.18** ([41]). Let \(A, B_i\) be subsets of a Hausdorff locally convex
topological vector space $E$ with a continuous seminorm $p$ such that $A_i^o$ (resp. $B_i^o$) are nonempty approximately $p$–compact (resp. closed) and convex for each $i \in I_n$. Assume that one of $A_i^o$’s is contained in some compact subset of $A$. Suppose that $\bigcap_{i=1}^n P_{A_i^o}(y_i)$ is nonempty for each $(y_1, y_2, \ldots, y_n) \in Y = \prod_{i=1}^n B_i^o$ and $T : A^o \to 2^Y$ is a KKM$0$–multimap (resp. $\mathcal{U}_c^k$–multimap) where $T(x) = \prod_{i=1}^n T_i(x)$ for each $x \in A^o$. Then for each $p$–continuous, proper, quasi $p$–affine and surjective self map $f : A^o \to A^o$, there exists $a \in A^o$ such that the best proximity set $\mathfrak{T}^a_i(f)$ is nonempty and closed.

**Theorem 5.2.19** ([41]). Let $A$ and $B_i$ be subsets of a Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$, $A_i^o$ (resp. $B_i^o$) are nonempty approximately $p$–compact (resp. closed) and convex, $T_i : A^o \to 2^{B_i^o}$ be an upper semicontinuous multimap with compact values and $T_i(x) \cap B_i^o$ is nonempty for each $x \in A^o$ for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^o}(y_i)$ is nonempty for each $(y_1, y_2, \ldots, y_n) \in Y = \prod_{i=1}^n B_i^o$. Assume that one of $A_i^o$’s is contained in some compact subset of $A$. Then for each continuous, proper, quasi $p$–affine and surjective self map $f : A^o \to A^o$, there exists $a \in A^o$ such that the best proximity set $\mathfrak{T}^a_i(f)$ is nonempty and closed.

**Definition 5.2.20** ([4]). Let $X$ be a convex subset of a vector space and $Y$ be a topological space. A multimap $T : X \to 2^Y$ is said to have the KKM property if for any map $F : X \to 2^Y$ with closed values satisfying $T(\text{Co}(N)) \subset F(N)$ for each finite subset $N$ of $Y$. The family $\{F(x)\}_{x \in X}$ has the finite intersection property.

We denote $KKM(X, Y) = \{T : X \to 2^Y/T \text{ has the KKM property}\}$. 
Example 5.2.21. The set of all continuous function $C(X,Y)$ and the Kakutani map $K(X,Y)$ have KKM-property.

Definition 5.2.22. Let $X$ be a nonempty set, $Y$ a nonempty convex subset of a vector space and $Z$ a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$ and $F : X \to 2^Z$ are three multimaps satisfying $T(C \circ S(A)) \subseteq F(A)$ for each finite subset $A$ of $X$, then the family $\{F(x)\}_{x \in X}$ has the finite intersection property. We denote,

$$S - KKM(X,Y,Z) = \{T : Y \to 2^Z / T \text{ has the } S - KKM \text{ property}\}.$$ 

Remark 5.2.23. In the case where $X = Y$ and $S$ is the identity map $I_X$, then $S - KKM(X,Y,Z) = KKM(X,Z)$.

Proposition 5.2.24 ([41]). If $T \in KKM(Y,Z)$, then for any multi map $S : X \to 2^Y$, $T \in S - KKM(X,Y,Z)$.

Proposition 5.2.25 ([41]). Let $X$ be a nonempty set, $Y$ a nonempty convex subset of a vector space and $Z$ a topological space. For any surjective map $S : X \to Y$, $T \in KKM(Y,Z)$ if and only if $T \in S - KKM(X,Y,Z)$.

Definition 5.2.26 ([41]). Let $T : A \to 2^B$ be a multimap, $T$ is said to be a $S - KKM_0$-multimap if $T$ and $f \circ T : A \to 2^A$ are closed and have the $S - KKM$ property for each $K$-multimap $f : B \to 2^A$ and for any surjective map $S : A \to B$. 

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Theorem 5.2.27 ([41]). Let $E$ be a Locally convex Topological vector space with a continuous seminorm $p$. Let $A$ and $B_i$ for each $i \in I_n$ be nonempty subsets of $E$. Then the following statements hold for each $i \in I_n$.

(a), If $A_i^o$[resp.$A$] and $B_i$ are convex, then $B_i^o$ is[resp.$A_i^o$ and $B_i^o$ are ] convex

(b),If $A_i^o$[resp.$A$] and $B_i$ are compact, then $B_i^o$ is[resp.$A_i^o$ and $B_i^o$ are ] compact.

(c), $P_A(B_i^o) = P_{A_i^o}(B_i^o) = A_i^o$

(d), If $A_i^o$ is nonempty, compact and convex and $B_i^o$ is convex, then $P_{A_i^o} \in K(B_i^o, A_i^o)$

Theorem 5.2.28 ([41]). Let $A, B_i$ be subsets of a Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$ such that $A_i^o$ (resp. $B_i^o$) are nonempty approximately $p$–compact (resp. closed) and convex for each $i \in I_n$. Assume that one of $A_i^o$’s is contained in some compact subset of $A$. Suppose that $\bigcap_{i=1}^{n} P_A(y_i)$ is nonempty for each $(y_1, y_2, \ldots, y_n) \in Y = \prod_{i=1}^{n} B_i^o$ and $T : A^o \to 2^Y$ is a $S – KKM_0$–multimap (resp. $U_k^c$–multimap ) where $T(x) = \prod_{i=1}^{n} T_i(x)$ for each $x \in A^o$. Then for each $p$-continuous,proper, quasi $p$–affine and surjective self map $f : A^o \to A^o$, there exists $a \in A^o$ such that the best proximity set $\mathcal{T}_a^i(f)$ is nonempty and closed.
5.3 Equilibrium Pair Results for $S - KKM_0$

Multimap

A free $n$–person game is a family of ordered quadruples $(A, B_i, T_i, P_i)_{i \in I_n}$ such that $A$ and $B_i$ are nonempty subsets of $E$, $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap and $P_i : B \rightarrow 2^{B_i}$ is a preference map where $B := \prod_{j=1}^{n} B_j$ (see [43]). An equilibrium pair for $(A, B_i, T_i, P_i)_{i \in I_n}$ is a point $(a, b) \in A \times B$ such that $T_i(a) \cap P_i(b) = \emptyset$. For details on economic terminology see ([11], [43]).

Theorem 5.3.1. Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free $n$–person game such that $A$ and $B_i$ are nonempty subsets of a Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$, $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap and $P_i : B \rightarrow 2^{B_i}$ is a preference map where $B := \prod_{j=1}^{n} B_j$. Assume that $A^o$ is nonempty, $T(x) := \prod_{i=1}^{n} T_i(x)$ for each $x \in A^o$, $Y := \prod_{i=1}^{n} B_i^o$ and for each $i \in I_n$.

(a) $A^o_i$ is approximately $p$–compact and convex, $B_i$ is compact and convex. Also one of $A^o_i$’s is contained in some compact subset of $A$.

(b) $\bigcap_{i=1}^{n} P_{A^o_i}(y_i)$ is nonempty for each $(y_1, y_2, \ldots, y_n) \in \prod_{i=1}^{n} B_i^o$.

(c) $T : A^o \rightarrow 2^Y$ is a $S - KKM_0$–multi map.

(d) $x_i \notin C_0(P_i(x))$ for each $x = (x_1, x_2, \ldots, x_n) \in B$.

(e) $P_i^{-1}(y)$ is open for each $y \in B_i$.

Then there exists $b \in B$ such that $P_i(b) = \emptyset$ and for each $p$–continuous, proper, quasi $p$–affine and surjective self map $f : A^o \rightarrow A^o$, there exists $a \in A^o$ such that the best proximity set $\Sigma^i_a(f)$ is nonempty and closed. If in addition $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^{n} \Sigma^i_a(f)$, then $(a, b)$ is an equilibrium
Proof. Fix $i \in I_n$. Since $A_i^o$ is approximately $p$–compact, $A_i^o$ is closed for each $i \in I_n$ and hence $A^o$ is closed. Since one of $A_i^o$'s is contained in some compact subset of $A$, $A_i^o$ is compact. As $A_i^o$ and $B_i$ are compact and convex it follows from Theorem 5.2.27 that $B_i^o$ is compact and convex. By Theorem 5.2.18, there exists $a \in A^o$ such that the proximity set $\Sigma_i^o(f)$ is nonempty and closed. By Lemma 5.2.15, there exists $b = (b_1, b_2, \ldots, b_n) \in Y$ such that $P_i(b) = \emptyset$. As $P_i(z)$ is nonempty for each $z \in \prod_{i=1}^n \Sigma_i^o(f)$, we conclude that $b = (b_1, b_2, \ldots, b_n) \in \prod_{i=1}^n \Sigma_i^o(f)$. Thus $(a, b) \in A^o \times Y$, $b = (b_1, b_2, \ldots, b_n) \in \prod_{i=1}^n T_i(a)$, $T_i(a) \cap P_i(b) = \emptyset$ and $d_P(f(a), T_i(a)) = p(f(a) - b_i) = d_p(A, B_i)$. Hence $(a, b)$ is an equilibrium pair in $A_0 \times \prod_{i=1}^n \Sigma_i^o(f)$.

Theorem 5.3.2. Let $(A, B, T, P)$ be a free $1$–person game such that $A$ and $B$ are nonempty subsets of a Hausdorff locally convex topological vector space $E$, $T : A \to 2^B$ is a constraint multimap and $P : B \to 2^B$ is a preference map. Assume that

(a) $A_i^o$ is approximately $p$–compact and convex for each $i \in I_n$, $B$ is compact and convex and one of $A_i^o$'s is contained in some compact subset of $A$.

(b) $T : A^o \to 2^{B^o}$ is a $S$–$KKM_0$–multimap.

(c) $x_i \notin Co(P(x))$ for each $x \in B$.

(d) One of the following conditions is satisfied.

(1) If $z \in P^{-1}$ for some $y \in B$, then there exists some $y' \in B$ such
that \( z \in \text{int}P^{-1}(y') \).

\( (d_2) \) For each \( y \in B \), \( P^{-1}(y) \) is open in \( B \).

Then there exists \( b \in B \) such that \( P(b) = \emptyset \) and for each \( p \)-continuous, proper, quasi \( p \)-affine and surjective self map \( f : A_o \to A_o \), there exists \( a \in A_o \) such that the best proximity set \( \mathcal{T}^i_a(f) \) is nonempty and compact. If in addition \( P(z) \) is nonempty for each \( z \notin \mathcal{T}_a(f) \), then \( (a, b) \) is an equilibrium pair in \( A^0 \times \mathcal{T}_a^i(f) \).

**Proof.** Since \( A^i_o \) is approximately \( p \)-compact for each \( i \in I_n \).

\( A^i_o \) is closed for each \( i \in I_n \).

Hence \( A^o = \bigcap_{i=1}^{n} A^i_o \) is closed.

But \( A^o \subset A^i_o \).

Therefore \( A^o \) is compact.

That is \( A_0 \) is compact.

Since \( A_0 \) and \( B_0 \) are nonempty, compact and convex it follows from Theorem 5.2.18 that there exists \( (a, c) \in A_0 \times B_0 \) such that \( c \in T(a) \) and \( d(f(a), T(a)) = p(f(a) - c) = d(A, B) \) and so \( \mathcal{T}_a(f) \) is nonempty. By Theorem 5.2.19, there exists \( b \in B_0 \) such that \( P(b) = \emptyset \).

As \( P(z) \) is nonempty whenever \( z \in B \\setminus \mathcal{T}_a(f) \), we conclude that \( b \in \mathcal{T}_a(f) \).

So \( (a, b) \in A_0 \times B_0 \), \( b \in T(a) \) and \( d_P(f(a), T(a)) = p(f(a) - b) = d_P(A, B) \).

Thus \( (a, b) \) is an equilibrium pair in \( A_0 \times \mathcal{T}_a^i(f) \).