**THE EDGE-TO-VERTEX DETOUR NUMBER AND EDGE-TO-EDGE DETOUR NUMBER OF GRAPH**

**Theorem 3.21.** For a connected graph $G$ of size $q$, $2 \leq d_{ev}(G) \leq d_{ee}(G) \leq q$.

**Proof.** Any edge-to-vertex detour set needs at least two edges and therefore $d_{ev}(G) \geq 2$.

Let $S$ be an edge-to-edge detour set. Then every edge of $G$ is either an element or lies on a detour joining a pair of edge of $S$. Also every edge-to-edge detour set is an edge-to-vertex detour set of $G$ and then $d_{ev}(G) \leq d_{ee}(G)$. Clearly the set of all edges of $G$ is an edge-to-edge detour set of $G$ so that $d_{ee}(G) \leq q$. Thus $2 \leq d_{ev}(G) \leq d_{ee}(G) \leq q$. ■

**Remark 3.22.** The bounds in Theorem 3.21 are sharp. The set of the two end edges of a path $P_p$ ($p \geq 2$) is its unique edge-to-vertex detour set so that $d_{ev}(G) = 2$. For the cycle $C_p$, $d_{ev}(G) = d_{ee}(G) = 2$, for the star $G = K_1,q$, ($q \geq 2$), $d_{ee}(G) = q$. Also, the inequalities in the theorem can be strict. For the graph $G$, given in the Figure 3.7 $d_{ev}(G) = 2$, $d_{ee}(G) = 3$.

![Figure 3.7](image)

In the view of Theorem 3.21, we have the following realization result.
Theorem 3.23. For every two positive integers \( a \) and \( b \) with \( 2 \leq a \leq b \), there exists a connected graph \( G \) with \( d_{ev}(G) = a \) and \( d_{ee}(G) = b \).

Proof. Let \( G \) be a tree with \( a \) end edges. Then by Theorem 1.56, \( d_{ev}(G) = a \) and by Corollary 3.13, \( d_{ee}(G) = a \). Therefore by taking \( a = b \), the theorem is proved. For \( 2 \leq a < b \), let \( P: u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 \) be a path of order eight. Let \( G \) be the graph obtained from \( P \) by adding new vertices \( z_1, z_2, \ldots, z_{a-1} \) and \( w_1, w_2, \ldots, w_{b-a} \) and joining each \( z_i \) (\( 1 \leq i \leq a-1 \)) with \( u_6 \) and each \( w_i \) (\( 1 \leq i \leq b-a \)) with \( u_1 \) and \( u_7 \). The graph \( G \) is given in Figure 3.8. Let \( Z = \{u_6z_1, u_6z_2, \ldots, u_6z_{a-1}, u_7u_8\} \) be the set of all pendant edges of \( G \). By Theorem 1.55, \( Z \) is a subset of every edge-to-vertex detour set of \( G \). It is clear that \( Z \) is an edge-to-vertex detour set of \( G \) so that \( d_{ev}(G) = a \). By Corollary 3.11, \( Z \) is a subset of every edge-to-edge detour set of \( G \). It is easily verified that \( Z \) is not an edge-to-edge detour set of \( G \). It is easily observed that \( u_6w_i(1 \leq i \leq b-a) \) is a subset of every edge-to-edge detour set of \( G \) and so \( d_{ee}(G) \geq a + b - a = b \). Let \( W = \{u_6w_1, u_6w_2, \ldots, u_6w_{b-a}\} \). Now \( S = Z \cup W \) is an edge-to-edge detour set of \( G \) and so that \( d_{ee}(G) = b \).
Definition 3.24. An edge-to-edge detour set $S$ in a connected graph $G$ is called a minimal edge-to-edge detour set if no proper subset of $S$ is an edge-to-edge detour set of $G$. The upper edge-to-edge detour number is denoted by $d^{+}_{ee}(G)$ is maximum cardinality of a minimal edge-to-edge detour set of $G$.

Example 3.25. For the graph $G$ given in Figure 3.9, $S_1 = \{v_1, v_4\}$, $S_2 = \{v_4, v_6\}$, $S_3 = \{v_4, v_7\}$, $S_4 = \{v_2, v_4, v_5\}$ are the only minimal edge-to-edge detour sets of $G$ so that the $d^{+}_{ee}(G) = 3$. 
Remark 3.26. Every minimum edge-to-edge detour set of $G$ is a minimal edge-to-edge detour set of $G$. The converse is not true.

Theorem 3.27. For a connected graph $G$, $2 \leq d_{ee}(G) \leq d^+_{ee}(G) \leq q$.

Proof: It is enough to prove $d_{ee}(G) \leq d^+_{ee}(G)$. Since every minimal edge-to-edge detour set is also an edge-to-edge detour set, $d_{ee}(G) \leq d^+_{ee}(G)$. Thus $2 \leq d_{ee}(G) \leq d^+_{ee}(G) \leq q$.

Corollary 3.28. For any non-trivial tree $T$ with $k$ end-vertices, $d^+_{ee}(T) = k$ and the set of all $k$ end-edges of $T$ is the unique minimal edge-to-edge detour set of $T$.

Proof: Let $S$ be the set of all end edges of $T$. Then by Theorem 3.11, $S$ is a subset of every edge-to-edge detour set of $T$. Hence $d^+_{ee}(T) \leq k$, then it follows from the Theorem 3.11 that $d^+_{ee}(T) = k$.

Theorem 3.29. For the cycle $C_p (p \geq 6)$, $d^+_{ee}(C_p) = 3$.

Proof: Let $C_p$: $v_1, v_2, v_3, v_4, \ldots, v_p$ be the cycle. Let $S = \{v_1, v_2, v_3, v_4, v_{p-1}, v_p\}$. Then $S$ is an edge-to-edge detour set of $G$. Since no proper subset of $S$ is an edge-to-edge detour set of $G$, $S$ is a minimal edge-to-edge detour set of $G$ so that $d^+_{ee}(G) \geq 3$. We show that
\(d_{ee}^+(G) = 3\). Suppose that \(d_{ee}^+(G) > 3\), then there exists a minimal edge-to-edge detour set \(M\) such that \(|M| \geq 4\). Since any two adjacent edges of \(G\) is a detour set of \(G\), it follows that \(M\) contains no adjacent edges. Then \(M\) contains at least three non-adjacent edges. Hence \(M\) is not a minimal edge-to-edge detour set of \(G\). Therefore \(d_{ee}^+(G) = 3\).

\[\square\]

**Corollary 3.30.** For the graph \(K_p (p \geq 2)\), \(d_{ee}^+(K_p) = 2\).

**Proof:** Let \(K_p: v_1, v_2, v_k, v_{k+1}, \ldots, v_p\) be the graph. Since any two adjacent edges of \(G\) is an edge-to-edge detour set, it follows that \(d_{ee}^+(K_p) = 2\).

\[\square\]

**Theorem 3.31.** For a connected graph \(G\), \(d_{ee}(G) = q\) if and only if \(d_{ee}^+(G) = q\).

**Proof.** Let \(d_{ee}^+(G) = q\). Then \(S = E(G)\) is the unique minimal edge-to-edge detour set of \(G\). Since no proper subset of \(S\) is an edge-to-edge detour set, it is clear that \(S\) is the unique minimum edge-to-edge detour set of \(G\) and so \(d_{ee}(G) = q\). The converse follows from Theorem 2.51.

\[\square\]

**Corollary 3.32.** For a connected graph the following are equivalent.

\[i) \quad d_{ee}(G) = q\]
\[ii) \quad d_{ee}^+(G) = q\]
\[iii) \quad G\ is\ a\ star\]

**Proof:** This follows from Theorems 3.21, 3.31.

\[\square\]

**Theorem 3.33.** For any positive integers \(a\) and \(b\), \(2 \leq a \leq b\) there exists a connected graph \(G\) such that \(d_{ee}(G) = a\) and \(d_{ee}^+(G) = b\).
Proof: Let $P: v_1, v_2, v_3, v_4, v_5$ be path of length 4. Add new vertices $y, w_1, w_2, ..., w_{a-2}, u_1, u_2, ..., u_{b-a+1}$ to $P$ and join $w_1, w_2, ..., w_{a-2}$ to $v_5$ and joining $u_1, u_2, ..., u_{b-a+1}$ to both $v_2$ and $v_5$. Therefore by producing a graph $G$ of Figure 3.10. Let $S = \{v_1 v_2, v_5 w_1, v_5 w_2, ... v_5 w_{a-2}\}$. By Theorem 3.11, $S$ is a subset of every edge-to-edge detour set of $G$. It is clear that $S$ is not an edge-to-edge detour set of $G$ and so $d_{ee}(G) \geq a$. It is clear that $S \cup \{v_4 v_5\}$ is an edge-to-edge detour set of $G$ so that $d_{ee}(G) = a$. Now $S_1 = S \cup \{v_5 u_1, v_5 u_2, ..., v_5 u_{b-a}\}$ is an edge-to-edge detour set of $G$. Now we show that $S_1$ is a minimal edge-to-edge set of $G$. Assume, to the contrary, that $S_1$ is not a minimal edge-to-edge set of $G$. Then there is a proper subset $T$ of $S_1$ such that $T$ is an edge-to-edge set of $G$. Let $e \in S_1$ such that $e \notin T$. By Corollary 3.11, it is clear that $e \neq v_5 w_i$, $i = 1, 2, 3, ..., a - 2$. Therefore $e = v_5 u_i$ for some $i = 1, 2, 3, ..., u_{b-a}$. Clearly, this $e$ does not lie on a detour joining of a pair of edges of $T$ and so $T$ is not an edge-to-edge detour set of $G$, which is a contradiction. Thus $S_1$ is a minimal edge-to-edge detour set of $G$ so that $d_{ee}(G) \geq b$. Suppose that there exists a minimal edge-to-edge detour set $T'$ such that $|T'| \geq b + 1$. Then it is easily verified that $T$ containing either $S$ or $S_1$. Therefore $T'$ is not a minimal-edge-to-edge detour set of $G$, which is a contradiction. Therefore $d_{ee}(G) = b$. **
FORCING EDGE-TO-EDGE DETOUR NUMBER OF A GRAPH

The Forcing edge-to-edge detour number of a Graph

Even though every connected graph contains a minimum edge-to-edge detour set, some connected graph may contain several minimum edge-to-edge detour sets. For each minimum edge-to-edge detour set S in a connected graph G, there is always some subset T of S that uniquely determines S as the minimum edge-to-edge detour set containing T. Such “forcing subsets” will be considered in this section.

**Definition 3.34.** Let G be a connected graph and S a minimum edge-to-edge detour set of G. A subset T ⊆ S is called a forcing subset for S if S is the unique minimum edge-to-edge detour basis set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing edge-to-edge detour number of S, denoted by $f_{dee}(S)$, is the cardinality of a minimum forcing subset of S. The forcing edge-to-edge detour number of G, denoted by $f_{dee}(G)$ is defined as $f_{dee}(G) = \min\{ f_{dee}(S) \}$ where the minimum is taken over all minimum edge-to-edge detour sets S in G.
Example 3.35  For the graph $G$ given in Figure 3.11, $S_1=\{v_1v_2, v_2v_3\}$, $S_2=\{v_1v_2, v_2v_6\}$, $S_3=\{v_1v_2, v_4v_5\}$, are the only three edge-to-edge detour set of $G$ so that $d_{ee}(G) = 2$ such that $f_{dec}(S_1) = f_{dec}(S_2) = f_{dec}(S_3) = 1$ so that $f_{dec}(G) = 1$.

Theorem 3.36. Let $G$ be a connected graph. Then

a) $f_{dec}(G) = 0$ if and only if $G$ has a unique minimum edge-to-edge detour set

b) $f_{dec}(G) = 1$ if and only if $G$ has at least two minimum edge-to-edge detour sets, one of which is a unique minimum edge-to-edge detour set containing one of its elements, and

c) $f_{dec}(G) = d_{ee}(G)$ if and only if no minimum edge-to-edge detour set of $G$ is unique minimum edge-to-edge detour set containing any of its proper subsets.

Proof. (a) Let $f_{dec}(G) = 0$. Then, by definition, $f_{dec}(S) = 0$ for some minimum edge-to-edge detour set $S$ of $G$ so that the empty set $\phi$ is the minimum forcing subset for $S$.

Since the empty set $\phi$ is a subset of every set, it follows that $S$ is the unique minimum edge-to-edge detour set of $G$. The converse is clear.
(b) Let $f_{\text{det}}(G) = 1$. Then by Theorem 3.36(a) $G$ has at least two minimum edge-to-edge detour sets. Also, since $f_{\text{det}}(G) = 1$, there is a singleton subset $T$ of a minimum edge-to-edge detour set $S$ of $G$ such that $T$ is not a subset of any other minimum edge-to-edge detour set of $G$. Thus $S$ is the unique minimum edge-to-edge detour set containing one of its elements. The converse is clear.

(c) Let $f_{\text{det}}(G) = d_{ee}(G)$. Also by Theorem 3.6, $d_{ee}(G) \geq 2$ and hence $f_{\text{det}}(G) \geq 2$. Then by Theorem 3.36(a), $G$ has at least two minimum edge-to-edge detour sets and so the empty set $\emptyset$ is not a forcing subset for any edge-to-edge detour basis of $G$. Since $f_{\text{det}}(G) = d_{ee}(G)$, no proper subsets of $S$ is a forcing subset of $S$. Thus no minimum edge-to-edge detour set of $G$ is the unique minimum edge-to-edge detour set containing any of its proper subsets.

Conversely, the hypothesis implies that $G$ contains more than one edge-to-edge detour basis and no subset of any minimum edge-to-edge detour set $S$ other than $S$ is a forcing subset for $S$. Hence it follows that $f_{\text{det}}(G) = d_{ee}(G)$. ■

**Definition 3.37.** An edge $e$ of $G$ is said to be an edge-to-edge detour of edge of $G$, if $e$ belongs to every minimum edge-to-edge detour set of $G$.

**Example 3.38.** For the graph $G$ given in Figure 3.12, The sets $S_1 = \{v_4v_5, v_5v_6\}$ and $S_2 = \{v_2v_3, v_5v_6\}$ are the only two minimum edge-to-edge detour set of $G$ so that $v_5v_6$ is a minimum edge-to-edge detour set of $G$. 
Theorem 3.39. Let $G$ be a connected graph and let $\mathcal{F}$ be the set of relative complements of the minimum forcing subsets in their respective edge-to-edge detour bases in $G$. Then $\bigcap_{F \in \mathcal{F}} F$ is the set of edge-to-edge detour edges of $G$.

Proof. Proof is similar to that of Theorem 2.62

Corollary 3.40. Let $G$ be a connected graph and $S$ be a minimum edge-to-edge detour set of $G$. Then no edge-to-edge detour edge of $G$ belongs to any minimum forcing set of $S$.

Theorem 3.41. Let $G$ be a connected graph and $W$ be the set of all edge-to-edge detour edge of $G$. Then $f_{\text{de}}(G) \leq d_{\text{ee}}(G) - |W|$.

Proof. Let $S$ be any minimum edge-to-edge detour set of $G$. Then $d_{\text{ee}}(G) = |S|$, $W \subseteq S$ and $S$ is the unique minimum edge-to-edge detour set containing $S - W$. Thus $f_{\text{de}}(G) \leq |S - W| = |S| - |W| = d_{\text{ee}}(G) - |W|$.

Theorem 3.42.

i) For any cycle $C_p$ ($p \geq 4$), $f_{\text{de}}(C_p) = 1$.

ii) For any complete graph $G = K_p$ ($p \geq 2$), $f_{\text{de}}(G) = 2$.  

---
iii) For any non-trivial tree $G = T, f_{\text{dee}}(G) = 0$.

**Proof.** (i) Let $e, f$ be two adjacent edges of $G$. Then $S = \{e, f\}$ is a minimum *edge-to-edge detour* set of $C_p$ so that $f_{\text{dee}}(C_p) = 2$. Since $S$ is not a unique minimum *edge-to-edge detour* set of $C_p$ containing either $e$ or $f$, it follows that $f_{\text{dee}}(C_p) = 2$.

(ii) By the similar way as in the first part of Theorem 3.42 we can prove $f_{\text{dee}}(G) = 2$.

(iii) Let $G = T$ be any tree. Then the set of end edges of $G$ is the unique *edge-to-edge* detour set of $G$. Hence it follows from Theorem 3.36 (a) that $f_{\text{dee}}(G) = 0$. $\blacksquare$

**Theorem 3.43.** For every pair $a, b$ of integers with $0 \leq a < b$, $b \geq 2$ and $b - a - 1 > 0$, there exists a connected graph $G$ such that $f_{\text{dee}}(G) = a$ and $d_{ce}(G) = b$.

**Proof.** Let $P: x, v_1, v_2, v_3, v_4$ be a path of length 4. Let $H$ be a graph obtained from $P$ by adding new vertices $u_1, u_2, ..., u_a$ and joining $u_i$ with $v_1$ and $v_4$ $(1 \leq i \leq a)$. Let $G$ be a graph obtained from $H$ by adding new vertices $z_1, z_2, z_3, ..., z_{b-a-1}$ by joining each $z_i (1 \leq i \leq b - a - 1)$ with $v_4$. The graph $G$ is shown in Figure 3.13. Let $Z = \{xv_1, v_4z_1, v_4z_2, v_4z_3, ..., v_4z_{b-a-1}\}$ be the set of all end edges of $G$. Then $Z$ is not a *edge-to-edge* detour set of $G$. Let $H_i = \{v_1u_i, v_4u_i\}; (1 \leq i \leq a)$ Then it is easily observed that every *edge-to-edge* detour set of $G$ contains at least one edge from each $H_i (1 \leq i \leq a)$ and so $d_{ce}(G) \geq b - a + a = b$. Now $S = Z \cup \{u_1v_1, u_2v_1, ..., u_av_1\}$ is an *edge-to-edge* detour set of $G$ so that $d_{ce}(G) = b$.

Next we show that $f_{\text{dee}}(G) = a$. Since every $d_{ce}$-set of $G$ contains $Z$, it follows from Theorems 3.11 and 3.41 that $f_{\text{dee}}(G) \leq d_{ce}(G) - |Z| = b - (b - a) = a$. Now since $d_{ce}(G) = b$ and every $u_i, d_{ce}$-set of $G$ contains $Z$, it is easily seen that every $d_{ce}$-set $S$ is of the form $Z \cup \{c_1, c_2, c_3, ..., c_a\}$ where $c_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset
of $S$ with $|T| < a$. Then there is a vertex $c_j$ ($1 \leq j \leq a$) such that $c_j \notin T$. Let $f_j$ be an edge of $H_j$ distinct from $c_j$. Then $S_2 = [S-\{c_j\} \cup \{f_j\}]$ is a $d_{ee}$-set of $G$ properly containing $T$. Thus $S$ is not the unique $d_{ee}$-set containing $T$. Thus $T$ is not a forcing subset of $S$. This is true for all $d_{ee}$-sets of $G$ and it follows that $f_{d_{ee}}(G) = a$. 

**Figure 3.13**

**THE EDGE-TO-EDGE GEODETIC NUMBER AND THE EDGE-TO-EDGE DETOUR NUMBER OF A GRAPH**

**Remark 3.44.** There is no relationship between the edge-to-edge geodetic number and the edge-to-edge detour number of a graph.

**Example 3.45.** For the cycle $G = C_5$, $d_{ee}(G) = 2$ and $g_{ee}(G) = 3$ and so $d_{ee}(G) < g_{ee}(G)$. Also for the graph $G$ given in the Figure 3.14, $S_1 = \{v_1v_2, v_1v_9, v_4v_5\}$ is a minimum edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = 3$ and $S_2 = \{v_4v_5, v_4v_6, v_4v_7, v_4v_8\}$ is a minimum edge-to-edge detour set of $G$ so that $d_{ee}(G) = 4$ and so $d_{ee}(G) > g_{ee}(G)$. So we have the following realization results.
Theorem 3.46. For every pair of positive integers with \(2 \leq a \leq b\), there exists a connected graph \(G\) such that \(d_{ee}(G) = a\) and \(g_{ee}(G) = b\).

Proof. Case 1. \(a = b\)

Let \(G\) be a star with \(a\) pendant edges. Then by Corollary 3.13, \(d_{ee}(G) = a\). Also by Corollary 2.17, \(g_{ee}(G) = a\).

Case 2. \(2 \leq a < b\)

Let \(G\) be a graph obtained from the path on four vertices \(P : x_1, x_2, x_3, x_4\) by adding \(a - 1\) new vertices \(v_1, v_2, \ldots, v_{a-1}\) and joining each \(v_i (1 \leq i \leq a - 1)\) with \(x_4\).

Also adding \(b - a\) new vertices \(w_1, w_2, \ldots, w_{b-a}\) and joining each \(w_i (1 \leq i \leq b - a)\) with \(x_2, x_3\) and \(x_4\). The graph \(G\) is shown in Figure 3.15. First show that \(d_{ee}(G) = a\). Let \(Z = \{x_1x_2, v_1x_4, v_2x_4, \ldots, v_{a-1}x_4\}\) be the set of all pendant edges of \(G\). By Theorem 3.11, \(Z\) is a subset of every edge-to-edge detour set of \(G\) and so \(d_{ee}(G) \geq a\). It is clear that \(Z\) is an edge-to-edge detour set of \(G\) so that \(d_{ee}(G) = a\). Next show that \(g_{ee}(G) = b\). By Corollary 2.13, \(Z\) is a subset of every edge-to-edge geodetic set of \(G\). It is clear that \(Z\)
is not an edge-to-edge geodetic set of $G$. It is easily verified that every edge-to-edge geodetic set of $G$ contains $x_3 w_i$ ($1 \leq i \leq b - a$) and so $g_{ee}(G) \geq a + b - a = b$. Let $S = Z \cup \{x_3 w_1, x_3 w_2, \ldots, x_3 w_{b-a}\}$. Then $S$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = b$. Hence the proof.

\[\Box\]

**Theorem 3.47.** For every pair of positive integers with $4 \leq a < b$, there exists a connected graph $G$ such that $g_{ee}(G) = a$ and $d_{ee}(G) = b$.

**Proof.** Let $G$ be a graph obtained from the cycle $C_6 : x_1, x_2, x_3, x_4, x_5, x_6, x_1$ by adding $a - 2$ new vertices $z_1, z_2, \ldots, z_{a-2}$ and joining each $z_i$ ($1 \leq i \leq a - 2$) with $x_4$. Also adding $b - a + 2$ new vertices $w_1, w_2, \ldots, w_{b-a+2}$ and joining each $w_i$ ($1 \leq i \leq b - a + 2$) with $x_1$ and $x_4$. The graph $G$ is shown in Figure 3.16. First show that $g_{ee}(G) = a$. Let $Z = \{z_1 x_4, z_2 x_4, \ldots, z_{a-2} x_4\}$ be the set of all pendant edges of $G$. By Theorem 2.13, $Z$ is a subset of every edge-to-edge geodetic set of $G$ and so $g_{ee}(G) \geq a - 1$. It is clear that $Z$ is not an edge-to-edge geodetic set of $G$. It is easily verified that $Z \cup \{h\}$, where $h \notin Z$ is not an edge-to-edge geodetic set of $G$ and so $g_{ee}(G) \geq a$. However $Z \cup \{x_1 x_2, x_3 x_5\}$ is an
edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = a$. Next show that $d_{ee}(G) = b$. By Theorem 3.11, $Z$ is a subset of every edge-to-edge detour set of $G$. It is clear that $Z$ is not an edge-to-edge detour set of $G$. It is easily verified that every edge-to-edge detour set of $G$ contains $x_iw_i$ ($1 \leq i \leq b - a + 2$) and so $d_{ee}(G) \geq a - 2 + b - a + 2 = b$.

Let $W = Z \cup \{x_iw_1, x_iw_2, \ldots, x_iw_{b-a+2}\}$. Then $W$ is an edge-to-edge detour set of $G$ so that $d_{ee}(G) = b$. Hence the proof. 

Figure 3.16
CHAPTER – 4

THE EDGE FIXING EDGE-TO-EDGE GEODETIC NUMBER OF A GRAPH

In this Chapter we introduce the concept of edge fixing edge-to-edge geodetic number of a graph and some general properties satisfied by this concept are studied. The edge fixing edge-to-edge geodetic number of some standard graphs is determined. Connected graph $G$ of size $q$ with edge fixing edge-to-edge geodetic number $q - 1$ or $q - 2$ is characterized. It is shown that, for every positive integers $r$, $d$ and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph $G$ with $\text{rad}(G) = r$, $\text{diam}(G) = d$ and $g_{efee}(G) = l$ or $l - 1$ for all $e \in E(G)$. It is also shown that, for every positive integers $a$ and $b$ with $2 \leq a \leq b \leq q - 1$, there exists a connected graph $G$ of size $q$, $g_{ce}(G) = a$ and $g_{efee}(G) = b$ for some $e \in E(G)$.

**Definition 4.1.** Let $e$ be an edge of a graph $G$. A set $S(e) \subseteq E(G) - \{e\}$ is called an *edge fixing edge-to-edge geodetic set* of a connected graph $G$ if every edge of $G$ lies on an $e - f$ geodesic, where $f \in S(e)$. The *edge fixing edge-to-edge geodetic number* $g_{efee}(G)$ of $G$ is the minimum cardinality of its edge fixing edge-to-edge geodetic sets and any edge fixing edge-to-edge geodetic set of cardinality $g_{efee}(G)$ is an $g_{efee}$-set of $G$.

**Example 4.2.** For the graph $G$ given in Figure 4.1, the edge fixing edge-to-edge geodetic sets of each edge of $G$ is given in the following Table 4.1.
### Table 4.1

<table>
<thead>
<tr>
<th>Fixing Edge ($e$)</th>
<th>Minimum edge fixing edge-to-edge geodetic sets ($S(e)$)</th>
<th>$g_{e,\text{free}}(S(e))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1v_2$</td>
<td>${v_3v_6}$</td>
<td>1</td>
</tr>
<tr>
<td>$v_2v_3$</td>
<td>${v_1v_2, v_3v_6, v_6v_7}$</td>
<td>3</td>
</tr>
<tr>
<td>$v_3v_4$</td>
<td>${v_1v_2, v_6v_7, v_7v_8}$</td>
<td>3</td>
</tr>
<tr>
<td>$v_4v_5$</td>
<td>${v_1v_2, v_7v_8, v_2v_8}$</td>
<td>3</td>
</tr>
<tr>
<td>$v_3v_6$</td>
<td>${v_1v_2}$</td>
<td>1</td>
</tr>
<tr>
<td>$v_6v_7$</td>
<td>${v_1v_2, v_2v_3, v_3v_4}$</td>
<td>3</td>
</tr>
<tr>
<td>$v_7v_8$</td>
<td>${v_1v_2, v_4v_5, v_3v_4}$</td>
<td>3</td>
</tr>
<tr>
<td>$v_8v_2$</td>
<td>${v_1v_2, v_4v_5, v_3v_6}$</td>
<td>3</td>
</tr>
</tbody>
</table>
Remark 4.3. For a connected graph $G$, the edge $e$ of $G$ does not belong to the edge fixing edge-to-edge geodetic set $S(e)$.

Theorem 4.4. For an edge $e$ in $G$, the $g_{efee}$ set is unique.

Proof. Let $e$ be an edge of $G$. Suppose there are two $g_{efee}$ sets, say $S_1$ and $S_2$. Let $f$ be an edge of $G$ such that $f \in S_1$ and $f \notin S_2$. Since $S_2$ is a $g_{efee}$ set and $|S_1| = |S_2|$, hence there exists an edge $h \neq f$ in $G$ such that $h \notin S_2$ and $h \in S_1$. Since $S_1$ is a $g_{efee}$ set and $h \notin S_1$, there exists an edge $g$ in $S_1$ such that $h$ lies on an $e - g$ geodesic.

Case (i): Suppose $g \in S_2$.

Since $h$ is an internal edge of an $e - g$ geodesic and $S_2$ is a $g_{efee}$ set, $h \notin S_2$, which is a contradiction to $h \in S_2$.

Case (ii): Suppose $g \notin S_2$.

Since $S_2$ is a $g_{efee}$ set, there exists an edge $y \notin S_2$ such that $g$ lies on an $e - y$ geodesic, say $P$. From (1), $h$ lies on an $e - g$ geodesic, say $Q$. Then combining $Q$ and $P$, $e - g$ and $g - y$ section of the geodesic, $P$ is an $e - y$ geodesic. Thus $h$ is an internal edge of an $e - y$ geodesic. Since $S_2$ is a $g_{efee}$ set, $h \notin S_2$, which is a contradiction to $h \in S_2$. Therefore $g_{efee}$ set of $e$ of $G$ is unique.

Theorem 4.5. Let $e$ be an edge of $G$. Let $v$ be an extreme vertex of a connected graph $G$ such that $v$ is not incident with $e$. Then every edge fixing edge-to-edge geodetic set $e$ of $G$ contains at least one extreme edge that is incident with $v$.

Proof. Let $e$ be an edge of $G$ and let $v$ be an extreme vertex of $G$ such that $v$ is not incident with $e$. Let $e_1, e_2, \ldots, e_k$ be the edges incident with $v$ and let $S(e)$ be any edge
fixing edge-to-edge geodetic set of $e$ of $G$. We claim $e_i \in S(e)$ for some $i$ $(1 \leq i \leq k)$. If not, $e_i \notin S(e)$ for all $i$ $(1 \leq i \leq k)$. Since $S(e)$ is an edge fixing edge-to-edge geodetic set of $e$ of $G$, the vertex $v$ lies on the $e – f$ geodesic joining the edges $e$ and $f \in S(e)$. Since $v$ is an internal vertex of a geodesic $e – f$, $v$ is not an extreme vertex of $G$, which is a contradiction. Hence $e_i \in S(e)$ for some $i$ $(1 \leq i \leq k)$. □

**Corollary 4.6.** Every end–edge (other than the fixed edge) of a connected graph $G$ belongs to every edge fixing edge-to-edge geodetic set of an edge $e$ of $G$.

**Theorem 4.7.** Let $G$ be a connected graph and $S(e)$ be an edge fixing edge-to-edge geodetic set of $e$ of $G$. Let $f$ be a non–pendant cut edge of $G$ and let $G_1$ and $G_2$ be the two component of $G – \{f\}$.

If $e = f$, then each of the two component of $G – \{f\}$ contains an element of $S(e)$.

If $e \neq f$, then $S(e)$ contains at least one edge of component of $G – \{f\}$ where $e$ does not lie.

**Proof.** Let $f = uv$. Let $G_1$ and $G_2$ be the two component of $G – \{f\}$ such that $u \in V(G_1)$ and $v \in V(G_2)$.

Let $e = f$. Suppose that $S(e)$ does not contain any element of $G_1$. Then $S(e) \subseteq E(G_2)$. Let $h$ be an edge of $E(G_1)$. Then $h$ must lie in $e – f$ geodesic $P$. Since $e = f$, $P$ must be a cycle containing $h$ and $e (=f)$. This is a contradiction since $f$ is a cut edge that does not lie on any cycle. Hence each of the two component of $G – \{f\}$ contains an element of $S(e)$.

By similar argument, we can prove that if $e \neq f$, then $S(e)$ contains at least one edge of component of $G – \{f\}$ where $e$ does not lie. □
Theorem 4.8. Let $G$ be a connected graph and $S(e)$ be a minimum edge fixing edge-to-edge geodetic set of an edge $e$ of $G$. Then no non–pendant cut-edge of $G$ belongs to $S(e)$.

Proof. Let $S(e)$ be an edge fixing edge-to-edge geodetic set of an edge $e = uv$ of $G$. Let $f = u'v'$ be a pendant cut-edge of $G$ such that $f \in S(e)$. Then $e \neq f$. Let $G_1$ and $G_2$ be the two components of $G - \{f\}$ such that $u' \in V(G_1)$ and $v' \in V(G_2)$. By Theorem 4.7, $G_1$ contains an edge $xy$ and $G_2$ contains an edge $x'y'$ where $xy, x'y' \in S(e)$. Let $S'(e) = S(e) - \{f\}$. We claim that $S'(e)$ is an edge fixing edge-to-edge geodetic set of an edge $e$ of $G$.

Case 1. Suppose that $xy$ is an edge in $G_1$ and $x'y'$ is an edge in $G_2$. Let $z$ be any vertex of $G$. Assume without loss of generality that $z$ belongs to $G_1$. Since $u'v'$ is a cut-edge of $G$, every path joining a vertex of $G_1$ with a vertex of $G_2$ contains the edge $u'v'$. Suppose that $z$ is incident with $u'v'$ or the edge $xy$ of $S(e)$ or that lies on a geodesic joining $xy$ and $u'v'$. If $z$ is incident with $u'v'$, then $z = u'$. Let $P: y, y_1, y_2, ..., z = u'$ be a $xy - u'v'$ geodesic. Let $Q: v', v_1', v_2', ..., y'$ be a $u'v' - x'y'$ geodesic. Then, it is clear that $P \cup \{u'v'\} \cup Q$ is a $xy - x'y'$ geodesic. Thus $z$ lies on the $xy - x'y'$ geodesic. If $z$ is incident with $xy$, then there is nothing to prove. If $z$ lies on a $xy - x'y'$ geodesic, say $y, v_1, v_2, ..., z, ..., u'$, then let $v', v_1', v_2', ..., y'$ be $u'v' - x'y'$ geodesic. Then clearly $y, v_1, v_2, ..., z, ..., u', v', v_1', v_2', ..., y'$ is a $xy - x'y'$ geodesic. Thus $z$ lies on a geodesic joining $xy$ and an element of $S'(e)$. Thus we have proved that a vertex that is incident with $u'v'$ or an edge of $S(e)$ or that lies on a geodesic joining $xy$ and $u'v'$ of $S(e)$ also is incident with an edge of $S'(e)$ or lies on a geodesic joining $e$ and an edge of $S'(e)$. Hence it follows that $S'(e)$ is an edge fixing edge-to-edge geodetic set.
of an edge $e$ of $G$ such that $|S'(e)| = |S(e)| - 1$, which is a contradiction to $S(e)$ an
$geffe$ set of $G$.

**Case 2.** Suppose that $e = xy \in G_2$. The proof is similar to that of Case 1. Hence the
theorem follows.  

**Theorem 4.9.** For any non-trivial tree $T$ with $k$ end edges,

$$geffe(G) = \begin{cases} k - 1 & \text{if } e \text{ is an end edge of } G \\ k & \text{if } e \text{ is an internal edge of } G \end{cases}$$

**Proof.** This follows from Corollary 4.6 and Theorem 4.5. 

**Theorem 4.10.** For the graph $G = C_p$ ($p \geq 4$), $geffe(G) = \begin{cases} 1 & \text{if } p \text{ is even} \\ 2 & \text{if } p \text{ is odd} \end{cases}$ for any
edge $e$ of $E(G)$.

**Proof.** Suppose that $p$ is even. Let $p = 2k$ and let $C_p$: $v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_1$ be the cycle. Then $v_{k+1}$ is the antipodal vertex of $v_1$ and $v_{k+2}$ is the antipodal vertex of $v_2$. For the edge $e = v_1v_2$, let $S(e) = \{v_{k+1}v_{k+2}\}$. Since each vertex of $C_p$ lies on the $e-f$ geodesic where $f = v_{k+1}v_{k+2} \in S(e)$, it follows that $S(e)$ is an edge fixing edge-to-edge geodetic set of an edge $e$ of $C_p$. Hence $geffe(C_p) = |S(e)| = 1$.

Suppose that $p$ is odd. Let $p = 2k + 1$ and let $C_p$: $v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}$, $v_{2k+1}, v_1$ be the cycle. Let $e = v_1v_2$ be an edge of $C_p$. Let $S(e) = \{v_{k+1}v_{k+2}\}$. Clearly the
vertices $v_{k+3}, v_{k+4}, ..., v_{2k}$, $v_{2k+1}$ do not lie on the $e-f$ geodesic where $f = v_{k+1}v_{k+2} \in S(e)$.
Hence $S(e)$ is not an edge fixing edge-to-edge geodetic set of an edge $e$ of $C_p$. However $S(e) \cup \{v_{k+2}v_{k+3}\}$ is an edge fixing edge-to-edge geodetic set of an edge $e$ of $C_p$. Hence $geffe(C_p) = 2$.  

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Theorem 4.11. For the complete graph $K_p$ ($p \geq 4$) with $p$ even, $g_{efee}(G) = \frac{p-2}{2}$ for every edge in $E(G)$.

Proof. Let $G = K_p$ and $e$ be an edge of $G$. Let $S(e)$ be any set of $\frac{p-2}{2}$ independent edges of $K_p$ such that $e \notin S(e)$. Since each vertex of $K_p$ is either incident with an edge of $S(e)$ or incident with $e$, $S(e)$ is an edge fixing edge-to-edge geodetic set of an edge $e$ of $G$. Hence it follows that $g_{efee}(G) \leq \frac{p-2}{2}$. If $g_{efee}(G) < \frac{p-2}{2}$, then there exists an edge fixing edge-to-edge geodetic set $S'(e)$ of $K_p$ such that $|S'(e)| < \frac{p-2}{2}$. Therefore, there exists at least one edge $h$ of $K_p$ such that $h$ is not adjacent with any edge of $S'(e)$. Hence $h$ is not incident with any edge of $S'(e)$ nor lies on the $e-f$ geodesic where $f \in S'(e)$ and so $S'(e)$ is not an edge fixing edge-to-edge geodetic set of an edge $e$ of $G$, which is a contradiction. Thus $S(e)$ is an edge fixing edge-to-edge geodetic set of an edge $e$ of $K_p$. Hence $g_{efee}(K_p) = \frac{p-2}{2}$. □

Theorem 4.12. For the complete graph $G = K_p$ ($p \geq 5$) with $p$ odd, $g_{efee}(K_p) = \frac{p-1}{2}$, for every edge in $E(G)$.

Proof. Let $e$ be an edge of $G$ and let $M(e)$ consist of any set of $\frac{p-5}{2}$ independent edges of $K_p$ and such that $e \notin S(e)$ and $M'(e)$ consist of two adjacent edges of $K_p$, each of which is independent with the edges of $M(e)$. Let $S(e) = M(e) \cup M'(e)$. Since each edge of $K_p$ is either incident with an element of $S(e)$ or incident with $e$, $S(e)$ is an edge fixing edge-to-edge geodetic set of $e$ of $G$. Hence it follows that $g_{efee}(G) \leq \frac{p-1}{2}$. □
\[ \frac{p - 5}{2} + 2 = \frac{p - 1}{2} \]. If \( g_{ef}(G) < \frac{p - 1}{2} \), then there exists an edge fixing edge-to-edge geodetic set \( S'(e) \) of \( e \) of \( K_p \) such that \( |S'(e)| < \frac{p - 1}{2} \). Therefore, there exists at least one edge \( h \) of \( K_p \) such that \( h \) is not incident with any edge of \( S'(e) \). Hence the edge \( h \) is neither incident with any edge of \( S'(e) \) nor lies on \( e-f \) geodesic where \( f \in S'(e) \) and so \( S'(e) \) is not an edge fixing edge-to-edge geodetic set of \( e \) of \( G \), which is a contradiction. Thus \( S(e) \) is an edge fixing edge-to-edge geodetic set of an edge \( e \) of \( K_p \). Hence \( g_{ef}(G) = \frac{p - 1}{2} \).

**Theorem 4.13.** For the complete bipartite graph \( G = K_{m,n} (2 \leq m \leq n) \), \( g_{ef}(G) = n - 1 \), for every edge in \( E(G) \).

**Proof.** Let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) be a bipartition of \( G \). Let \( T \) consist of the set of \( m-1 \) independent edges \( x_1y_1, x_2y_2, \ldots, x_{m-1}y_{m-1} \) and \( T' \) consist of the \( n-m+1 \) adjacent edges \( x_my_m, x_my_{m+1}, \ldots, x_my_n \).

**Case 1.** Suppose that \( e \in T \). Then \( e = x_iy_i (1 \leq i \leq m - 1) \). Let \( S(e) = \{x_1y_1, x_2y_2, \ldots, x_{i-1}y_{i-1}, x_{i+1}y_{i+1}, \ldots, x_{m-1}y_{m-1}, x_my_m, x_my_{m+1}, \ldots, x_my_n\} \). It is easily verified that each vertex of \( G \) (except \( x_i, y_i \)) is incident on the \( e-f \) geodesic for some \( f \in S(e) \). Hence \( S(e) \) is an edge fixing edge-to-edge geodetic set of an edge \( e \) of \( G \), it follows that \( g_{ef}(G) \leq |S(e)| = n - 1 \). If \( g_{ef}(G) < n - 1 \), then there exists an edge fixing edge-to-edge geodetic set \( S'(e) \) of \( e \) of \( G \) such that \( |S'(e)| < n - 1 \). Therefore, there exists at least one edge \( h \) of \( G \) such that \( h \) is not incident with any edge of \( S'(e) \). Hence the edge \( h \) is neither incident with any edge of \( S'(e) \) nor lies on the geodesic \( e-f \) where