the edge $e$ does not lies on a geodesic joining a pair of edges of $S'$ and so $S'$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. Hence $g_{ee}(G) = \frac{p + 1}{2}$.

**Theorem 2.21.** For the cycle $C_p$ ($p \geq 4$), $g_{ee}(C_p) = \begin{cases} 2 & \text{if } p \text{ is even} \\ 3 & \text{if } p \text{ is odd} \end{cases}$.

**Proof.** Suppose that $p$ is even. Let $p = 2k$ and let $C_p : v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_1$ be the cycle. Then $v_{k+1}$ is the antipodal vertex of $v_1$ and $v_{k+2}$ is the antipodal vertex of $v_2$. Let $S = \{v_1v_2, v_{k+1}v_{k+2}\}$. Clearly, $S$ is an edge-to-edge geodetic set of $C_p$ so that $g_{ee}(C_p) = 2$. Suppose that $p$ is odd. Let $p = 2k + 1$ and let $C_p : v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_{2k+1}, v_1$ be the cycle. It is clear that no two element subset of edges is an edge-to-edge geodetic set of $C_p$. Let $S = \{v_1v_2, v_{k+1}v_{k+2}, v_{k+2}v_{k+3}\}$. Then $S$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(C_p) = 3$.

**Theorem 2.22.** A set $S$ of edges of $G = K_{n,n}$ ($n \geq 2$) is a minimum edge-to-edge geodetic of $G$ if and only if $S$ consists of $n$ independent edges.

**Proof.** Let $S$ be any set of $n$ independent edges of $G = K_{n,n}$ ($n \geq 2$). Since each edge of $G$ lies on a geodesic joining a pair of edges of $S$, it follows that $g_{ee}(G) \leq n$. If $g_{ee}(G) < n$, then there exists an edge-to-edge geodetic set $S'$ of $G$ such that $|S'| < n$. Therefore, there exists at least one edge $e$ of $S$ such that $e \notin S'$. Hence $e$ does not lies on a geodesic joining a pair of edges of $S'$ and so $S'$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. Hence $S$ is a minimum edge-to-edge geodetic set of $K_{n,n}$.

Conversely, let $S$ be a minimum edge-to-edge geodetic set of $G$. Let $S'$ be any set of $n$ independent edges of $G$. Then as in the first part of this theorem, $S'$ is a minimum edge-to-edge geodetic set of $G$. Therefore, $|S'| = n$. Hence $|S| = n$. If $S$ is not
independent, then there exists an edge $e$ of $G$ such that $e$ does not lie on a geodesic joining a pair of edges of $S$. Hence $S$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. Thus $S$ consists of $n$ independent edges. □

**Corollary 2.23.** For the complete bipartite graph $G = K_{n,n}$ ($n \geq 2$), $g_{ee}(G) = n$.

**Theorem 2.24.** For the complete bipartite graph $G = K_{m,n}$ ($2 \leq m < n$), $g_{ee}(G) = n$.

**Proof.** Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be a bipartition of $G$. Let $T$ consist of the set of $m - 1$ independent edges $x_1y_1, x_2y_2, \ldots, x_{m-1}y_{m-1}$ and $T'$ consist of the $n - m + 1$ adjacent edges $x_my_m, x_my_{m+1}, \ldots, x_my_n$. Let $S = T \cup T'$. Since each edge of $G$ lies on a geodesic joining a pair of edges of $S$, it follows that $g_{ee}(G) \leq m - 1 + n - m + 1 = n$. If $g_{ee}(G) < n$, then there exists an edge-to-edge geodetic set $S'$ of $G$ such that $|S'| < n$. Therefore there exists at least one edge $e$ of $S$ such that $e \notin S'$. Hence $e$ does not lie on a geodesic joining a pair of edges of $S'$ and so $S'$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. Hence $g_{ee}(G) = n$. □

The following theorem gives a realization result.

**Theorem 2.25.** For each pair of integers $k$ and $q$ with $2 \leq k \leq q$, there exists a connected graph $G$ of order $q + 1$ and size $q$ with $g_{ee}(G) = k$.

**Proof.** For $2 \leq k \leq q$, let $P$ be a path of order $q - k + 3$. Let $G$ be the graph obtained from $P$ by adding $k - 2$ new vertices to $P$ and joining them to any cut-vertex of $P$. Clearly, $G$ is a tree of order $q + 1$ and size $q$ with $k$ end-edges and so by Corollary 2.17, $g_{ee}(G) = k$. □
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We have seen that if $G$ is a connected graph of size $q \geq 2$, then $2 \leq g_{ee}(G) \leq q$. Indeed, by Theorem 2.25, for each pair $k, q$ of integers with $2 \leq k \leq q$, there is a tree of size $q$ with edge-to-edge geodetic number $k$. An improved upper bound for the edge-to-edge geodetic number of a graph can be given in terms of its size $q$ and diameter $d$.

**Theorem 2.26.** For a connected graph $G$ with $q \geq 2$, $g_{ee}(G) \leq q - d + 2$, where $d$ is the diameter of $G$.

**Proof.** Let $u$ and $v$ be vertices of $G$ for which $d(u, v) = d$, where $d$ is the diameter of $G$ and let $P: u = u_0, u_1, u_2, ..., u_d = v$ be a $u - v$ path of length $d$. Let $e_i = u_{i-1}u_i$ ($1 \leq i \leq d$). Let $S = E(G) - \{u_1u_2, u_2u_3, ..., u_{d-2}u_{d-1}\}$. Let $e$ be any edge of $G$. If $e = u_iu_{i+1}$ ($1 \leq i \leq d - 1$), then $e$ lies on the $e_1 - e_d$ geodesic $P_1$: $u_1, u_2, ..., u_{d-1}$. If $e \neq u_iu_{i+1}$ ($1 \leq i \leq d - 1$), then $e$ is an edge of $S$. Therefore, $S$ is an edge-to-edge geodetic set of $G$. Consequently, $g_{ee}(G) \leq |S| = q - d + 2$.

**Remark 2.28.** The bound in Theorem 2.26 is sharp. For the star $G = K_{1, q}$ ($q \geq 2$), $d = 2$ and $g_{ee}(G) = q$, by Corollary 2.18, so that $g_{ee}(G) = q - d + 2$.

We give below a characterization theorem for trees.

**Theorem 2.27.** For any nontrivial tree $T$ with $q \geq 2$, $g_{ee}(T) = q - d + 2$ if and only if $T$ is a caterpillar.

**Proof.** Let $P: u_0, u_1, ..., u_{d-1}, u_d = v$ be a diametral path of length $d$. Let $e_i = v_{i-1}v_i$ ($1 \leq i \leq d$) be the edges of the diametral path $P$. Let $k$ be the number of end edges of $T$ and $l$ be the number of internal edges of $T$ other than $e_i$ ($2 \leq i \leq d - 1$). Then $d - 2 + l + k = q$. By Corollary 2.17 $g_{ee}(T) = k$ and so $g_{ee}(T) = q - d + 2 - l$. 

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Hence $g_{ee}(T) = q - d + 2$ if and only if $l = 0$, if and only if all internal edges of $T$ lie on the diametral path $P$, if and only if $T$ is a caterpillar. 

The following Theorem gives a realization result.

**Theorem 2.29.** For each triple $d, k, q$ of integers with $3 \leq k \leq q - d + 2$, $d \geq 4$ and $q - d + k + 1 > 0$, there exists a connected graph $G$ of size $q$ with $\text{diam}(G) = d$ and $g_{ee}(G) = k$.

**Proof.** Let $3 \leq k = q - d + 2$. Let $G$ be the graph obtained from the path $P$ of length $d$ by adding $q - d$ new vertices to $P$ and joining them to any cut-vertex of $P$. Then $G$ is a tree of size $q$ and $\text{diam}(G) = d$. By Corollary 2.17, $g_{ee}(G) = q - d + 2 = k$. Now, let $2 \leq k < q - d + 2$.

**Case 1.** $q - d - k + 1$ is even. Let $(q - d - k + 1) \geq 2$. Let $n = \frac{q - d - k + 1}{2}$. Then $n \geq 1$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length $d$. Add new vertices $v_1, v_2, \ldots, v_{k-2}$ and $w_1, w_2, \ldots, w_n$ and join each $v_i (1 \leq i \leq k - 3)$ with $u_1$ and also join each $w_i (1 \leq i \leq n)$ with $u_1$ and $u_3$ in $P_d$. Now, join $w_1$ with $u_2$ and we obtain the graph $G$ in Figure 2.4(a). Then $G$ has size $q$ and diameter $d$. By Corollary 2.13, all the end-edges $u_1v_i (1 \leq i \leq k - 3)$, $u_0u_1$ and $u_{d-1}u_d$ lie on every edge-to-edge geodetic set of $G$. Let $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-3}, u_1u_0, u_{d-1}u_d\}$ be the set of all end-edges of $G$. Then it is clear that $S$ is not an edge-to-edge geodetic set of $G$ and so $g_{ee}(G) \geq k$. Now $S \cup \{u_2w_1\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = k$. 


Case 2. \( q - d - k + 1 \) is odd. Let \( q - d - k + 1 \geq 5 \). Let \( m = (q - d - k) / 2 \). Then \( m \geq 2 \). Let \( P_d: u_0, u_1, \ldots, u_d \) be a path of length \( d \). Add new vertices \( v_1, v_2, \ldots, v_{k-3} \) and \( w_1, w_2, \ldots, w_m \) and join each \( v_i \ (1 \leq i \leq k-2) \) with \( u_1 \) and also join each \( w_i \ (1 \leq i \leq m) \) with \( u_1 \) and \( u_3 \) in \( P_d \). Now join \( w_1 \) and \( w_2 \) with \( u_2 \) and we obtain the graph \( G \) in Figure 2.4(b). Then \( G \) has size \( q \) and diameter \( d \). Now, as in Case 1, \( S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d+1}u_d, u_2w_1, u_2w_2\} \) is an edge-to-edge geodetic set of \( G \) so that \( g_{ee}(G) = k \).
Let $q - d - k + 1 = 1$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length $d$. Add new vertices $v_1, v_2, \ldots, v_k, w_1$ and $w_1$ and join each $v_i$ ($1 \leq i \leq k - 2$) with $u_1$ and also join $w_1$ with $u_1$ and $u_3$ in $P_d$, thereby obtaining the graph $G$ in Figure 2.4(c). Then the graph is of size $q$ and diameter $d$. Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_d, u_d\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = k$. 

Now, let $q - d - k + 1 = 3$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length $d$. Add new vertices $v_1, v_2, v_3, \ldots, v_{k-2}, w_1$ and $w_2$ and join each $v_i$ ($1 \leq i \leq k - 2$) with $u_1$ and also join $w_1$ and $w_2$ with $u_1$ and $u_3$ and obtain the graph $G$ in Figure 2.4(d). Then $G$ has
size \( q \) and diameter \( d \). Now, as in Case 1, \( S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\} \) is an edge-to-edge geodetic set of \( G \) so that \( g_{ee}(G) = k \). \[\blacksquare\]

**Theorem 2.30.** For positive integers \( r, d \) and \( l \geq 2 \) with \( r \leq d \leq 2r \), there exists a connected graph \( G \) with \( \text{rad}(G) = r \), \( \text{diam}(G) = d \) and \( g_{ee}(G) = l \).

**Proof.** When \( r = 1 \), we let \( G = K_{2, l} \) or \( G = K_{1, l} \) according to whether \( d = 1 \) or \( d = 2 \) respectively. Then the result follows from Corollary 2.20 and Corollary 2.18 respectively. Let \( r \geq 2 \). If \( r = d \) and \( l = 2 \), let \( G = C_{2r} \). Then by Theorem 2.22, \( g_{ee}(G) = 2 = l \). Let \( l \geq 3 \). Let \( C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1 \) be the cycle of order \( 2r \). Let \( G \) be the graph obtained by adding the new vertices \( y_1, y_2, \ldots, y_{l-2} \) and joining each \( y_i (1 \leq i \leq l - 2) \) with \( u_1 \) and \( u_2 \) of \( C_{2r} \). The graph \( G \) is shown in Figure 2.5.
It is easily verified that the eccentricity of each vertex of $G$ is $r$ so that $\text{rad}(G) = \text{diam}(G) = r$. It is clear that each $y_i$ $(1 \leq i \leq l-2)$ is an extreme vertex of $G$. By Theorem 2.11, every edge-to-edge geodetic set of $G$ contains at least one edge incident on $x_i$ $(1 \leq i \leq l-2)$. It is easily verified that $S \cup \{e\}$, where $e \not\in S$ is not an edge–to-edge geodetic set of $G$ and so $g_{ee}(G)=l$. Let $S = \{u_1y_1, u_1y_2, \ldots, u_1y_{l-2}, u_2y_{l-1}\}$. It is clear that $S$ is not an edge-to-edge geodetic set of $G$. However, $S \cup \{u_1u_2, u_{r+1}u_{r+2}\}$ is an edge-to-edge geodetic set of $G$. Since $y_1, y_2, \ldots, y_{l-1}$ are the only extreme vertices of $G$, it follows from Corollary 2.13 that $g_{ee}(G) = l$.

Let $r < d$. If $l = 2$, then take $G$ to be any path on at least three vertices. Let $l \geq 3$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $p_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let $H$ be the graph obtained from $C_{2r}$ and $u_0$ in $P_{d-r+1}$ by identifying $v_1$ in $C_{2r}$ and $u_0$ in $P_{d-r+1}$. Now, add $(l-3)$ new vertices $z_1, z_2, \ldots, z_{l-3}$ to $H$ and join each vertex $z_i$ $(1 \leq i \leq l-3)$ to the vertex $u_{d-r-1}$ and obtain the graph $G$ of Figure 2.6. Then $\text{rad}(G) = r$ and $\text{diam}(G) = d$. Let $S = \{u_{d-r}z_1, u_{d-r}z_2, \ldots, u_{d-r}z_{l-3}, u_{d-r}u_{d-r}\}$ be the set of end-edges of $G$. By Corollary 2.13, $S$ is contained in every edge-to-edge geodetic set of $G$. It is clear that $S$ is not an edge-to-edge geodetic set of $G$. It is also seen that $S \cup \{e\}$, where $e \in E(G) - S$ is not an edge-to-edge geodetic set.
of $G$. However, the set $S_1 = S \cup \{v_r, v_{r+1}, v_{r+2}\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = l - 2 + 2 = l$.

![Diagram](image)

Figure 2.6

In the following we characterize graphs $G$ for which $g_{ee}(G) = q$ or $q - 1$.

**Theorem 2.31.** If $G$ is a connected graph such that it is not a star, then $g_{ee}(G) \leq q - 1$.

**Proof.** Let $G$ be a tree and let $e$ be an internal edge of $G$. Then $S = E(G)$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) \leq q - 1$. If $G$ is not a tree, then $G$ contains a cycle, say $C$. If $v_1, v_2, \ldots, v_k$ be the vertices of $C$. Let $v$ be the vertex of $G - C$ such that $v$ is adjacent to $v_1$ say. Then $S = E(G) - \{v_1v_2\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) \leq q - 1$.

**Remark 2.32.** The bound in Theorem 2.35 is sharp. For the complete graph $G = K_3$, $g_{ee}(G) = 2 = q - 1$.

**Theorem 2.33.** For any connected graph $G$, $g_{ee}(G) = q$ if and only if $G$ is a star.

**Proof.** Let $G$ be a star. Then by Corollary 2.19, $g_{ee}(G) = q$. Conversely, let $g_{ee}(G) = q$. If $G$ is not a star, then by Theorem 2.31, $g_{ee}(G) \leq q - 1$, which is a contradiction. Therefore, $G$ is a star.
Theorem 2.34. Let $G$ be a connected graph which is not a tree. Then
\[ g_{ee}(G) \leq q - 2 \quad (q \geq 4). \]

Proof. If the graph $G$ is a cycle $C_p$ ($p \geq 4$), then by Theorem 2.22, $g_{ee}(G) \leq q - 2$. If the graph $G$ is not a cycle, let $C : v_1, v_2, v_3, ..., v_k, v_1 (k \geq 3)$ be a smallest cycle in $G$. Without loss of generality let us assume that $d(v_1) \geq 3$. Now, $S = E(G) - \{v_1v_2, v_1v_k\}$ is an edge-to-edge geodetic set so that $g_{ee}(G) \leq q - 2$. 

Remark 2.35 The bound in Theorem 2.34 is sharp. For the graph $G$ given in Figure 2.7, $S = \{v_1v_2, v_3v_4\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = 2 = q - 2$.

![Figure 2.7](image)

Theorem 2.36. For any connected graph $G$ with $q \geq 3$, $g_{ee}(G) = q - 1$ if and only if $G$ is either $C_3$ or a double star.

Proof. If $G$ is $C_3$, then $g_{ee}(G) = 2 = q - 1$. If $G$ is a double star, then by Corollary 2.19, $g_{ee}(G) = q - 1$. Conversely, let $g_{ee}(G) = q - 1$. Let $q = 3$. If $G$ is a tree, then $G = P_4$ or $K_{1,3}$. For $G = K_{1,3}$, by Corollary 2.19, $g_{ee}(G) = 3 = q$, which is a contradiction. If $G = P_4$, it is a double star and by Corollary 2.19, $g_{ee}(G) = 2 = q - 1$ which satisfies the requirement of the theorem. If $G$ is not a tree, then $G = C_3$, which satisfies the requirements of the theorem. Thus the theorem follows.

Let $q \geq 4$. If $G$ is not a tree, then by Theorem 2.34, $g_{ee}(G) \leq q - 2$, which is a contradiction. Hence $G$ is a tree. Now it follows from Theorem 2.26 that $d \leq 3$. 

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Therefore $d = 2$ or $3$. If $d = 2$, then $G$ is the star $K_{1,q}$. By Corollary 2.18, $g_{ce}(G) = q$, which is a contradiction to the hypothesis. If $d = 3$, then $G$ is a double star, which satisfies the requirements of the theorem. Thus the theorem is proved.

**Theorem 2.37.** Let $G$ be a connected graph with $q \geq 4$, which is not a cycle and not a tree and let $C(G)$ be the length of a smallest cycle. Then $g_{ce}(G) \leq q - C(G) + 1$ if $C(G)$ is odd, and $g_{ce}(G) \leq q - C(G) + 2$ if $C(G)$ is even.

**Proof.** Let $C(G)$ denote the length of a smallest cycle in $G$ and let $C$ be a cycle of length $C(G)$. We consider two cases.

**Case 1.** $C(G)$ is odd. First suppose that $C(G) = 3$. Let $C : v_1, v_2, v_3, v_1$ be a cycle of length 3. Since $G$ is not a cycle, there exists a vertex $v$ in $G$ such that $v$ is not on $C$ and $v$ is adjacent to $v_1$, say. Let $S = E(G) - \{v_1v_2, v_1v_3\}$. Then every edge of $G$ lies on a geodesic joining a pair of edge of $S$ and so $S$ is an edge-to-edge geodetic set of $G$. Thus $g_{ce}(G) \leq q - 2 = q - C(G) + 1$.

Next suppose that $C(G) \geq 5$. Let $C : v_1, v_2, \ldots, v_k, v_{k+1}, v_k, \ldots, v_{2k+1}, v_1$ be a cycle of least length $C(G) = 2k + 1$. Since $G$ is not a cycle, there exists a vertex $v$ in $G$ such that $v$ is not on $C$ and $v$ is adjacent to $v_1$, say. We claim that $d(vv_1, v_{k+1}v_{k+2}) = k$.

Since $P : v_1, v_2, v_3, \ldots, v_{k+1}$ is a path of length $k$ on $C$, it follows that $d(vv_1, v_{k+1}v_{k+2}) \geq k$. If $d(vv_1, v_{k+1}v_{k+2}) \leq k - 1$, then at least one of $d(vv_1, v_i)$ and $d(v, v_i)$ for $i = k + 1, k + 2$ is less than or equal to $k - 1$. First suppose that $d(vv_1, v_{k+1}) \leq k - 1$. Let $Q$ be a $v_1 - v_{k+1}$ shortest path of length at most $k - 1$ different from $P$. Hence there exists at least one vertex of $Q$ that is not on $P$ and since the length of $Q$ is at most $k - 1$, it follows that a cycle of length at most $2k - 1$ is formed. This is a contradiction to $C(G) = 2k + 1$. Thus $d(vv_1, v_{k+1}) = k$. Similarly we can prove that $d(vv_1, v_{k+2}) = k$. 


Next, suppose that \( d(v, v_{k+1}) \leq k - 1 \). Since \( P': v, v_1, v_2, \ldots, v_{k+1} \) is a path of length \( k + 1 \), it follows that \( d(v, v_{k+1}) \leq k + 1 \). Then, as above, a cycle of length at most \( 2k \) is formed and this is a contradiction. Hence \( d(v, v_{k+1}) = k \) or \( k + 1 \). Similarly we can prove that \( d(v, v_{k+2}) = k \) or \( k + 1 \). Since \( d(v_1, v_{k+1}) = d(v_1, v_{k+2}) = k \), it follows that \( d(vv_1, v_{k+1}v_{k+2}) = k \).

Now, let \( S = (E(G) - E(C)) \cup \{v_{k+1}v_{k+2}\} \). Then \( S \) is an edge-to-edge geodetic set of \( G \) and so \( g_{ee}(G) \leq q - C(G) + 1 \).

**Case 2.** \( C(G) \) is even. First suppose that \( C(G) = 4 \). Let \( C = v_1, v_2, v_3, v_4, v_1 \) be a cycle of length 4. Since \( G \) is not a cycle, there exists a vertex \( v \) in \( G \) such that \( v \) is not on \( C \) and \( v \) is adjacent to \( v_1 \), say. Let \( S = E(G) - \{v_1v_2, v_1v_4\} \). Then \( S \) is an edge-to-edge geodetic set of \( G \). Thus \( g_{ee}(G) \leq q - 2 = q - C(G) + 2 \).

Next suppose that \( C(G) \geq 6 \). Let \( C = v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_{2k}, v_1 \) be a cycle of least length \( C(G) = 2k \). Since \( G \) is not a cycle, there exists a vertex \( v \) in \( G \) such that \( v \) is not on \( C \) and \( v \) is adjacent to \( v_1 \), say. We claim that \( d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) = k - 1 \). Since \( Q: v_1, v_2, v_3, \ldots, v_k \) and \( Q': v_1, v_2k, v_2k-1, \ldots, v_k+3, v_k+2 \) are paths of length \( k - 1 \) each on \( C \), it follows that \( d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) \leq k - 1 \). If \( d(vv_1, v_kv_{k+1}) \leq k - 2 \) or \( d(vv_1, v_{k+1}v_{k+2}) \leq k - 2 \), then proceeding as in Case 1, a cycle of length at most \( 2k - 3 \) or \( 2k - 2 \) or \( 2k - 1 \) is formed as the case may be, contradicting that the least length of a cycle is \( 2k \). Thus \( d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) = k - 1 \).

Let \( S = (E(G) - E(C)) \cup \{v_kv_{k+1}, v_{k+1}v_{k+2}\} \). Then \( S \) is an edge-to-edge geodetic set of \( G \) and so \( g_{ee}(G) \leq q - C(G) + 2 \).

**Theorem 2.38.** If \( G \) is a connected graph of size \( q \geq 4 \) and not a tree such that \( g_{ee}(G) = q - 2 \), then \( G \) is unicyclic.
Proof. Suppose that $G$ is not unicyclic. Let $C(G)$ denote the length of a smallest cycle in $G$. It follows from Theorem 2.37 that $C(G) \leq 4$.

**Case 1.** $C(G) = 3$. Let $C' : u, v, w, u$ be a cycle of length 3. Let $C''$ be any other cycle in $G$.

**Subcase 1a.** Suppose that $C'$ and $C''$ have exactly one vertex, say $u$ in common. Then $\deg(u) \geq 4$. Let $ux$ be an edge of $C'$ and let $S = E(G) – \{ux, uv, uw\}$. Then $S$ is an edge-to-edge geodetic set of $G$. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction.

**Subcase 1b.** Suppose that $C$ and $C'$ have exactly two vertices, say $u$ and $v$ in common. Then $\deg u \geq 3$ and $\deg v \geq 3$. Let $ux$ be an edge incident at $u$ on $C'$ such that $ux \neq uv$ and let $S = E(G) – \{ux, uv, vw\}$. Then $S$ is an edge-to-edge geodetic set of $G$. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction.

**Subcase 1c.** Suppose that $C$ and $C'$ have no vertex in common. Since $G$ is connected, there is a path $P$ connecting the vertex $u$ on $C$ to a vertex $z$ on $C'$. Let $zx$ be an edge of $C'$ incident at $z$ on $C'$. Then the set $S = E(G) – \{uv, uw, zx\}$ is an edge-to-edge geodetic set of $G$, as earlier. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction.

**Case 2.** $C(G) = 4$. Let $C : u, v, w, x, u$ be a cycle of length 4. Let $C'$ be any other cycle in $G$.

**Subcase 2a.** Suppose that $C$ and $C'$ have exactly one vertex, say $u$ in common. Then $\deg(u) \geq 4$. Let $uv$ be an edge of $C'$ and let $S = E(G) – \{ux, uy, uv\}$. Since every edge of $G$ is incident with an element of $S$, it is clear that $S$ is an edge-to-edge geodetic set of $G$. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction.
Subcase 2b. Suppose that $C$ and $C'$ have exactly two vertices, say $u$ and $v$ in common. Then $\deg u \geq 3$ and $\deg v \geq 3$. Let $uy \neq uv$ be an edge of $C'$ incident at $u$ on $C'$ and let $S = E(G) - \{uy, uv, xw\}$. Since every edge of $G$ is incident with an element of $S$, it is clear that $S$ is an edge-to-edge geodetic set of $G$. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction.

Subcase 2c. Suppose that $C$ and $C'$ have exactly three vertices say $u$, $v$ and $w$ in common. Then at least two vertices, say $u$ and $w$ have degree at least 3. Let $uy$ and $wz$ be edges on $C'$ such that $uy \neq uv$ and $wz \neq wv$. Let $S = E(G) - \{ux, uv, wz\}$. Since every edge of $G$ is incident with an element of $S$, $S$ is an edge-to-edge geodetic set of $G$. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction.

Subcase 2d. Suppose that $C$ and $C'$ have no vertex in common. Since $G$ is connected, there is a path $P$ connecting the vertex $u$ on $C$ to a vertex $z$ on $C'$. Let $zy$ be an edge of $C'$ incident at $z$ on $C'$. Then the set $S = E(G) - \{zy, uv, xw\}$ is an edge-to-edge geodetic set of $G$, as earlier. Hence $g_{ee}(G) \leq q - 3$, which is a contradiction. Thus $G$ is unicyclic.

THE EDGE-TO-VERTEX GEODETIC NUMBER AND EDGE-TO-EDGE GEODETIC NUMBER OF GRAPH.

Theorem 2.39. For a connected graph $G$ of size $q$, $2 \leq g_{ev}(G) \leq g_{ee}(G) \leq q$.

Proof. Any edge-to-edge geodetic set needs at least two edges and therefore $g_{ee}(G) \geq 2$. Let $S$ be an edge-to-edge geodetic set. Then every edge of $G$ is either an element of $S$ or lies on a geodesic joining a pair of edge of $S$. Also every edge-to-edge geodetic set is an edge-to-vertex geodetic set of $G$ and then $g_{ev}(G) \leq g_{ee}(G)$. Clearly

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the set of all edges of $G$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) \leq q$. Thus

$$2 \leq g_{ev}(G) \leq g_{ee}(G) \leq q.$$  

\[\blacksquare\]

**Remark 2.40.** The bounds in Theorem 2.39 are sharp. The set of the two end edges of a path $P_p$ ($p \geq 2$) is its unique edge-to-vertex geodetic set so that $g_{ev}(G) = 2$. For the cycle $C_4$, $g_{ev}(G) = g_{ee}(G) = 2$, for the star $G = K_{1,q}$ ($q \geq 2$), $g_{ee}(G) = q$. Also, the inequalities in the theorem can be strict. For the graph $G$, given in the Figure 2.2

$g_{ev}(G) = 2$, $g_{ee}(G) = 3$.

In the view of Theorem 2.39, we have the following realization result.

**Theorem 2.41.** For every two positive integers $a$ and $b$ with $2 \leq a \leq b$, there exists a connected graph $G$ with $g_{ev}(G) = a$ and $g_{ee}(G) = b$.

**Proof.** Let $G$ be a tree with $a$ end edges. Then by Theorem 1.50, $g_{ev}(G) = a$ and by Corollary 2.16, $g_{ee}(G) = a$. Therefore by taking $a = b$, the theorem is proved. For $2 \leq a < b$, let $P$: $x, y, z$ be a path of order three. Let $G$ be the graph obtained from $P$ by adding new vertices $w, z_1, z_2, ..., z_{a-1}$ and $w_1, w_2, ..., w_{b-a}$ and joining $w$ with $x$, each $z_i$ ($1 \leq i \leq a - 1$) with $z$ and each $w_i$ ($1 \leq i \leq b - a$) with $x, y$ and $z$. The graph $G$ is given in Figure 2.8. Let $Z = \{wx, zz_1, zz_2, ..., zz_{a-1}\}$ be the set of all end edges of $G$. By Theorem 1.49, $Z$ is a subset of every edge-to-vertex geodetic set of $G$ so that $g_{ev}(G) \geq a$. It is clear that $Z$ is an edge-to-vertex geodetic set of $G$ so that $g_{ev}(G) = a$.

By Corollary 2.13, $Z$ is a subset of every edge-to-edge geodetic set of $G$. It is easily verified that $Z$ is not an edge-to-edge geodetic set of $G$. It is easily observed that $yw_i$ ($1 \leq i \leq b-a$) is a subset of every edge to edge geodesic set of $V$ and so $g_{ee}(G) \geq a + b - a = b$. Let $W = \{yw_1, yw_2, ..., yw_{b-a}\}$. Now $S = Z \cup W$ is an edge-to-edge geodetic set of $G$ and so that $g_{ee}(G) = b$. 

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**THE UPPER EDGE-TO-EDGE GEODETIC NUMBER OF A GRAPH**

**Definition 2.42.** An edge-to-edge geodetic set $S$ in a connected graph $G$ is called a **minimal edge-to-edge geodetic set** if no proper subset of $S$ is an edge-to-edge geodetic set of $G$. The **upper edge-to-edge geodetic number** $g_{ee}^+(G)$ of $G$ is the maximum cardinality of a minimal edge-to-edge geodetic set of $G$.

**Example 2.43.** For the graph $G$ given in Figure 2.9, $S = \{v_1v_6, v_3v_4\}$ is a minimum edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = 2$. The set $S_1 = \{v_1v_2, v_3v_4, v_5v_6\}$ is an edge-to-edge geodetic set of $G$ and it is clear that no proper subset of $S_1$ is an edge-to-edge geodetic set of $G$ and so $S_1$ is a minimal edge-to-edge geodetic set of $G$. Also it is easily verified that no four element or five element subset of edge set is a minimal edge-to-edge geodetic set of $G$, it follows that $g_{ee}^+(G) = 3$. 
Remark 2.44. Every minimum edge-to-\textit{edge} geodetic set of $G$ is a minimal edge-to-edge geodetic set of $G$ and the converse is not true. For the graph $G$ given in Figure 2.9, $S_1 = \{v_1v_2, v_3v_4, v_5v_6\}$ is a minimal edge-to-edge geodetic set but not a minimum edge-to-edge geodetic set of $G$.

Theorem 2.45. Let $G$ be a connected graph with cut-edges and let $S$ be a minimal edge-to-edge geodetic set of $G$. There is no non-pendant cut-edge $e$ of $G$, which is not belongs to $S$.

\textbf{Proof.} Proof follows from Theorem 2.16

In the following we determine the upper edge-to-edge geodetic number of some standard graphs.
Theorem 2.46. For any non-trivial tree $T$ with $k$ end-edges, $g_{ee}^+(T) = k$.

Proof. By Corollary 2.13, any edge-to-edge geodetic set contains all the end-edges of $T$. By Theorem 2.45, no cut-edge of $T$ belongs to any minimal edge-to-edge geodetic set of $G$. Hence it follows that the set of all end-edges of $T$ is the unique minimal edge-to-edge geodetic set of $T$ so that $g_{ee}^+(T) = k$. Thus the proof is complete. □

Theorem 2.47. For a complete graph $G = K_p (p \geq 4)$, $g_{ee}^+(G) = p - 1$.

Proof. Let $S$ be any set of $p - 1$ adjacent edges of $K_p$ incident at a vertex, say $v$. Since each edge of $K_p$ is incident with an edge of $S$, it follows that $S$ is an edge-to-edge geodetic set of $G$. If $S$ is not a minimal edge-to-edge geodetic set of $G$, then there exists a proper subset $S'$ of $S$ such that $S'$ is an edge-to-edge geodetic set of $G$. Therefore there exists at least one vertex, say $u$ of $K_p$ such that $u$ is not incident with any edge of $S'$. Hence $u$ is neither incident with any edge of $S'$ nor lies on a geodesic joining a pair of edges of $S'$ and so $S'$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. Hence $S$ is a minimal edge-to-edge geodetic set of $G$. Therefore $g_{ee}^+(G) \geq p - 1$. Suppose that there exists a minimal edge-to-edge geodetic set $M$ such that $|M| \geq p$. Since $M$ contains at least $p$ edges, $< M >$ contains at least one cycle. Let $M' = M - \{e\}$, where $e$ is an edge of a cycle which lies in $< M >$. It is clear that $M'$ is an edge-to-edge geodetic set with $M' \subset M$, which is a contradiction. Therefore, $g_{ee}^+(G) = p - 1$. □

Theorem 2.48. For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$, $g_{ee}^+(G) = n + m - 2$. 49
**DISTANCES RELATED PARAMETERS IN GRAPHS WITH RESPECT TO EDGES**

Proof. Let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) be a bipartition of \( G \). Let \( S_i = \{x_1 y_1, x_i y_2, \ldots, x_{i-1} y_{m-1}, x_i y_{m}, x_{i+1} y_{m}, \ldots, x_m y_n\} \), \((1 \leq i \leq m)\), \( M_j = \{x_1 y_j, x_2 y_j, \ldots, x_{m-1} y_j, x_m y_{j-1}, x_m y_j, \ldots, x_m y_{n}\} \), \((1 \leq j \leq n)\) and \( N_k = \{x_1 y_1, x_2 y_2, \ldots, x_{m-1} y_{m-1}, x_m y_m, x_{m+1} y_1, \ldots, x_m y_n\} \) with \(|S_i| = |M_j| = n + m - 2\) and \(|N_k| = n\). It is easily verified that any minimal edge-to-edge geodetic set of \( G \) is of the form either \( S_i \) or \( M_j \) or \( N_k \). Since no proper subset of \( S_i \) \((1 \leq i \leq m)\), \( M_j \) \((1 \leq j \leq n)\) and \( N_k \) is an edge-to-edge geodetic set of \( G \), it follows that, \( g_{ee}^+(G) = n + m - 2 \).  

**THE EDGE-TO-EDGE GEODETIC NUMBER AND UPPER EDGE-TO-EDGE GEODETIC NUMBER OF A GRAPH**

In this section, connected graphs \( G \) of size \( q \) with upper edge-to-edge geodetic number \( q \) or \( q-1 \) are characterized.

**Theorem 2.49.** For a connected graph \( G \), \( 2 \leq g_{ee}(G) \leq g_{ee}^+(G) \leq q \).

Proof. Any edge-to-edge geodetic set needs at least two edges and so \( g_{ee}(G) \geq 2 \). Since every minimal edge-to-edge geodetic set is an edge-to-edge geodetic set, \( g_{ee}(G) \leq g_{ee}^+(G) \). Also, since \( E(G) \) is an edge-to-edge geodetic set of \( G \), it is clear that \( g_{ee}^+(G) \leq q \). Thus \( 2 \leq g_{ee}(G) \leq g_{ee}^+(G) \leq q \).  

**Remark 2.50.** The bounds in Theorem 2.49 are sharp. For any non-trivial path \( P \), \( g_{ee}(P) = 2 \). For any tree \( T \), \( g_{ee}(T) = g_{ee}^+(T) \) and \( g_{ee}^+(K_{1,q}) = q \) for \( q \geq 2 \). Also, all the inequalities in the theorem are strict. For the complete graph \( G = K_5 \), \( g_{ee}(G) = 3 \), \( g_{ee}^+(G) = 4 \) and \( q = 10 \) so that \( 2 < g_{ee}(G) < g_{ee}^+(G) < q \).

**Theorem 2.51.** For a connected graph \( G \), \( g_{ee}(G) = q \) if and only if \( g_{ee}^+(G) = q \).

Proof. Let \( g_{ee}^+(G) = q \). Then \( S = E(G) \) is the unique minimal edge-to-edge geodetic set of \( G \). Since no proper subset of \( S \) is an edge-to-edge geodetic set, it is clear that \( S \)
is the unique minimum edge-to-edge geodetic set of $G$ and so $g_{ee}(G) = q$. The converse follows from Theorem 2.49.

**Corollary 2.52.** For a connected graph $G$ of size $q$, the following are equivalent:

1. $g_{ee}(G) = q$
2. $g_{ee}^+(G) = q$
3. $G = K_{1,q}$.

**Proof.** This follows from Theorems 2.32 and 2.51.

**Theorem 2.53.** For every two positive integers $a$ and $b$ with $2 \leq a \leq b$, there exists a connected graph $G$ such that $g_{ee}(G) = a$ and $g_{ee}^+(G) = b$.

**Proof.** If $a = b$, let $G = K_{1,a}$. Then by Corollary 2.52, $g_{ee}(G) = g_{ee}^+(G) = a$. So, let $2 \leq a < b$. Let $P$: $x, y$ be a path on two vertices. Let $G$ be the graph in Figure 2.10 obtained from $P$ by adding new vertices $z, x_1, x_2, \ldots, x_{b-a+1}, y_1, y_2, \ldots, y_{a-1}$ and joining each vertex $y_i$ $(1 \leq i \leq a - 1)$ and each vertex $x_i$ $(1 \leq i \leq b - a + 1)$ with $z$, each vertex $x_i$ $(2 \leq i \leq b - a + 1)$ with $x$ and $x_1$ with $y$. Let $S = \{zy_1, zy_2, \ldots, zy_{a-1}\}$ be the set of end-edges of $G$. By Corollary 2.13, $S$ is contained in every edge-to-edge geodetic set of $G$. It is clear that $S$ is not an edge-to-edge geodetic set of $G$ and so $g_{ee}(G) \geq a$. However $S' = S \cup \{xy\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = a$.

Now, $T = S \cup \{yx_1, xx_2, \ldots, xx_{b-a+1}\}$ is an edge-to-edge geodetic set of $G$. We show that $T$ is a minimal edge-to-edge geodetic set of $G$. Let $W$ be any proper subset of $T$. Then there exists at least one edge say $e \in T$ such that $e \notin W$. First assume that $e = zy_i$ for some $i$ $(1 \leq i \leq a - 1)$. Then the edge $zy_i$ is neither incident with an edge of $W$ nor lies on any geodesic joining a pair of edges of $W$ and so $W$ is not an edge-to-
edge geodetic set of $G$. Now, assume that $e = x x_j$ for some $j \ (2 \leq j \leq b – a + 1)$. Then the edge $x x_j$ is neither incident with an edge of $W$ nor lies on a geodesic joining any pair of edges of $W$ and so $W$ is not an edge-to-edge geodetic set of $G$. Next, assume that $e = y x_1$. Then the edge $y x_1$ is neither incident with an edge of $W$ nor lies on a geodesic joining any pair of edges of $W$ and so $W$ is not an edge-to-edge geodetic set of $G$. Hence $T$ is a minimal edge-to-edge geodetic set of $G$ so that $g_{ee}^+(G) \geq b$. Now, we show that there is no minimal edge-to-edge geodetic set $X$ of $G$ with $|X| \geq b + 1$. Suppose that there exists a minimal edge-to-edge geodetic set $X$ of $G$ such that $|X| \geq b + 1$. Then by Corollary 2.13, $S \subseteq X$. Since $S'$ is an edge-to-edge geodetic set of $G$, it follows that $y x \notin X$. Let $M_1 = \{y x_1, x x_2, x x_3, \ldots, x x_{b-a+1}\}$ and $M_2 = \{z x_1, z x_2, z x_3, \ldots, z x_{b-a+1}\}$. Let $X = S \cup S_1 \cup S_2$, where $S_1 \subseteq M_1$ and $S_2 \subseteq M_2$. First we show that $S_1 \subseteq M_1$ and $S_2 \subseteq M_2$. Suppose that $S_1 = M_1$. Then $T \subseteq X$ and so $X$ is not a minimal edge-to-edge geodetic set of $G$, which is a contradiction. Suppose that $S_2 = M_2$. If $y x_1 \notin X$, then $y$ is neither incident with an edge of $X$ nor lies on a geodesic joining any pair of edges of $X$ and so $X$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. If $y x_1 \in X$ and if $x y_i$ do not belong to $S_1$ for all $i \ (2 \leq i \leq b – a + 1)$, then $x$ is neither incident with an edge of $X$ nor lies on a geodesic joining any pair of edges of $X$ and so $X$ is not an edge-to-edge geodetic set of $G$, which is a contradiction. Therefore $x x_i$ belong to $S_1$ for some $i \ (2 \leq i \leq b – a + 1)$. Without loss of generality let us assume that $x y_2 \in S_1$. Then $X' = X - \{z x_2\}$ is an edge-to-edge geodetic set of $G$ with $X' \subseteq X$, which is a contradiction. Therefore, $S_1 \subseteq M_1$ and $S_2 \subseteq M_2$. Next we show that $V(<S_1>) \cap V(<S_2>)$ contains no $x_i \ (1 \leq i \leq b – a + 1)$. Suppose that $V(<S_1>) \cap V(<S_2>)$ contains $v_i$ for some $i \ (1 \leq i \leq b – a + 1)$. Without loss of generality let us assume that $y_2 \in V(<S_1>) \cap V(<S_2>)$. Then $X'' = X - \{z x_2\}$ is an edge-to-edge geodetic set of $G$.
with \( X'' \subset X \), which is a contradiction. Therefore \(|S_1 \cup S_2| = b - a + 1\). Hence it follows that \(|X| = a - 1 + b - a + 1 = b\), which is a contradiction to \(|X| \geq b + 1\). Therefore \(g_{ee}^+(G) = b\).

\[ \text{Figure 2.10} \]

**THE FORCING EDGE-TO-EDGE GEODETIC NUMBER OF A GRAPH**

**Definition 2.54** Let \( G \) be a connected graph and let \( S \) be a minimum edge-to-edge geodetic set of \( G \). A subset \( T \subseteq S \) is called a **forcing subset** for \( S \) if \( S \) is the unique minimum edge-to-edge geodetic set containing \( T \). A forcing subset for \( S \) of minimum cardinality is a **minimum forcing subset** of \( S \). The **forcing edge-to-edge geodetic number** of \( S \), denoted by \( f_{gee}(S) \), is the cardinality of a minimum forcing subset of \( S \).

The **forcing edge-to-edge geodetic number** of \( G \), denoted by \( f_{gee}(G) \), is \( f_{gee}(G) = \min \{f_{gee}(S)\} \), where the minimum is taken over all minimum edge-to-edge geodetic sets \( S \) in \( G \).

**Example 2.55.** For the graph \( G \) given in Figure 2.11, \( S = \{v_1v_2, v_4v_5\} \) is the unique minimum edge-to-edge geodetic set of \( G \) so that \( f_{gee}(G) = 0 \). For the graph \( G \) given in
Figure 2.12, $S_1 = \{v_1v_2, v_5v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only $g_{ee}$-sets of $G$ such that $f_{gee}(S_1) = 2, f_{gee}(S_2) = f_{gee}(S_3) = 1$ so that $f_{gee}(G) = 1$.

The next theorem follows immediately from the definition of the edge-to-edge geodetic number and the forcing minimum edge-to-edge geodetic number of a connected graph $G$.

**Theorem 2.56.** For every connected graph $G$, $0 \leq f_{gee}(G) \leq g_{ee}(G)$.

**Remark 2.57.** The bounds in Theorem 2.56 are sharp. For the graph $G$ given in Figure 2.11, $f_{gee}(G) = 0$ and for the graph $G = K_3, f_{gee}(G) = g_{ee}(G) = 2$. Also, all the
inequalities in the Theorem 2.56 are strict. For the graph $G$ given in Figure 2.12, $f_{gee}(G) = 1$ and $g_{ee}(G) = 3$ so that $0 < f_{gee}(G) < g_{ee}(G)$.

In the following, we characterize graphs $G$ for which bounds in the Theorem 2.56 attained and also graph for which $f_{gee}(G) = 1$.

**Theorem 2.58.** Let $G$ be a connected graph. Then

- $f_{gee}(G) = 0$ if and only if $G$ has a unique minimum edge-to-edge geodetic set.
- $f_{gee}(G) = 1$ if and only if $G$ has at least two minimum edge-to-edge geodetic sets, one of which is a unique minimum edge-to-edge geodetic set containing one of its elements, and
- $f_{gee}(G) = g_{ee}(G)$ if and only if no minimum edge-to-edge geodetic set of $G$ is the unique minimum edge-to-edge geodetic set containing any of its proper subsets.

**Proof.** (a) Let $f_{gee}(G) = 0$. Then, by definition, $f_{gee}(S) = 0$ for some minimum edge-to-edge geodetic set $S$ of $G$ so that the empty set $\phi$ is the minimum forcing subset for $S$. Since the empty set $\phi$ is a subset of every set, it follows that $S$ is the unique minimum edge-to-edge geodetic set of $G$. The converse is clear.

(b) Let $f_{gee}(G) = 1$. Then by Theorem 2.58(a), $G$ has at least two minimum edge-to-edge geodetic sets. Also, since $f_{gee}(G) = 1$, there is a singleton subset $T$ of minimum edge-to-edge geodetic set $S$ of $G$ such that $T$ is not a subset of any other minimum edge-to-edge geodetic set of $G$. Thus $S$ is the unique minimum edge-to-edge geodetic set containing one of its elements. The converse is clear.
(c) Let $f_{gee}(G) = g_{gee}(G)$. Then $f_{gee}(S) = g_{gee}(G)$ for every minimum edge-to-edge geodetic set $S$ in $G$. Also, by Theorem 2.6, $g_{gee}(G) \geq 2$ and hence $f_{gee}(G) \geq 2$.

Then by Theorem 2.58(a), $G$ has at least two minimum edge-to-edge geodetic sets and so the empty set $\emptyset$ is not a forcing subset for any minimum edge-to-edge geodetic sets of $G$. Since $f_{gee}(S) = g_{gee}(G)$, no proper subset of $S$ is a forcing subset of $S$. Thus no minimum edge-to-edge geodetic set of $G$ is the unique minimum edge-to-edge geodetic set containing any of its proper subsets. Conversely, the hypothesis implies that $G$ contains more than one minimum edge-to-edge geodetic set and no subset of any minimum edge-to-edge geodetic set $S$ other than $S$ is a forcing subset for $S$.

Hence it follows that $f_{gee}(G) = g_{gee}(G)$. □

**Definition 2.59.** An edge $e$ of a connected graph $G$ is an *edge-to-edge geodetic edge* of $G$ if $e$ belongs to every minimum edge-to-edge geodetic set of $G$. If $G$ has a unique minimum edge-to-edge geodetic set $S$, then every edge of $S$ is an edge-to-edge geodetic edge of $G$.

**Example 2.60.** For the graph $G$ given in Figure 2.11, $S = \{v_1v_2, v_4v_5\}$ is the unique minimum edge-to-edge geodetic set of $G$ so that both the edges in $S$ are edge-to-edge geodetic edges of $G$.

**Remark 2.61.** By Corollary 2.13, each end-edge of $G$ is an edge-to-edge geodetic edge of $G$. In fact there are certain edge-to-edge geodetic edges, which are not end-edges of $G$ is evident from Figure 2.12 as mentioned above.

**Theorem 2.62.** Let $G$ be a connected graph and let $\mathcal{F}$ be the set of relative complements of the minimum forcing subsets in their respective minimum edge-to-edge geodetic set of $G$. Then $\bigcap_{F \in \mathcal{F}} F$ is the set of edge-to-edge geodetic edges of $G$. 
**Proof.** Let \( W \) be the set of all edge-to-edge geodetic edges of \( G \). We are to show that \( W = \bigcap_{e \in F} F \). Let \( v \in W \). Then \( e \) belongs to every minimum edge-to-edge geodetic set \( S \) of \( G \). Let \( T \subseteq S \) be the minimum forcing subset for any minimum edge-to-edge geodetic set \( S \) of \( G \). We claim that \( e \in T \). If \( e \in T \), then \( T' = T - \{ e \} \) is a proper subset of \( T \) such that \( S \) is the unique minimum edge-to-edge geodetic set containing \( T' \) so that \( T' \) is a forcing subset for \( S \) with \( |T'| < |T| \), which is a contradiction to \( T \) is a minimum forcing subset for \( S \). Thus \( e \notin T \) and so \( e \in F \), where \( F \) is the relative complement of \( T \) in \( S \). Hence \( e \in \bigcap_{e \in F} F \) so that \( W \subseteq \bigcap_{e \in F} F \).

Conversely, let \( e \in \bigcap_{e \in F} F \). Then \( e \) belongs to every relative complement of minimum forcing subset \( T \) for \( S \). Since \( F \) is the relative complement of \( T \) in \( S \), we have \( F \subseteq S \) and thus \( e \in S \) for every \( S \), which implies that \( e \) is an edge-to-edge geodetic edge of \( G \). Thus \( e \in W \) and so \( \bigcap_{e \in F} F \subseteq W \). Hence \( W = \bigcap_{e \in F} F \). \( \blacksquare \)

**Corollary 2.63.** Let \( S \) be a minimum edge-to-edge geodetic set of a graph \( G \). Then no edge-to-edge geodetic edge of \( G \) belongs to any minimum forcing set of \( S \).

**Theorem 2.64.** Let \( G \) be a connected graph and \( W \) be the set of all edge-to-edge geodetic edges of \( G \). Then \( f_{gee}(G) \leq g_{gee}(G) - |W| \).

**Proof.** Let \( S \) be a minimum edge-to-edge geodetic set of \( G \). Then \( g_{gee}(G) = |S|, \ W \subseteq S \) and \( S \) is the unique minimum edge-to-edge geodetic set containing \( S - W \). Thus \( f_{gee}(G) \leq |S - W| \leq |S| - |W| = g_{gee}(G) - |W| \). \( \blacksquare \)

**Corollary 2.65.** If \( G \) is a connected graph with \( k \) end edges, then \( f_{gee}(G) \leq g_{gee}(G) - k \).
Proof. This follows from Corollary 2.13 and 2.64.

Remark 2.66. The bound in Theorem 2.64 is sharp. For the graph $G$ given in Figure 2.13, $S_1 = \{v_1v_2, v_2v_3, v_4v_5, v_4v_6\}$, $S_2 = \{v_1v_2, v_3v_4, v_4v_5, v_4v_6\}$, $S_3 = \{v_1v_2, v_2v_3, v_4v_5, v_2v_6\}$ and $S_4 = \{v_1v_2, v_3v_4, v_4v_5, v_2v_6\}$ are the only four minimum edge-to-edge geodetic sets of $G$ such that $f_{gee}(S_1) = f_{gee}(S_2) = f_{gee}(S_3) = f_{gee}(S_4) = 2$ so that $f_{gee}(G) = 2$ and $g_{ee}(G) = 4$. Also, $W = \{v_1v_2, v_4v_5\}$ is the set of all edge-to-edge geodetic edges of $G$ and so $f_{gee}(G) = g_{ee}(G) - |W|$. Also, the inequality in Theorem 2.64 is strict. For the graph $G$ given in Figure 2.12, $g_{ee}(G) = 3$ and $f_{gee}(S_2) = f_{gee}(S_3) = 1$ and $f_{gee}(S_1) = 2$ so that $f_{gee}(G) = 1$. Here, $v_1v_2$ is the only edge-to-edge geodetic edge of $G$ and so $f_{gee}(G) < g_{ee}(G) - |W|$.

In the following we determine the forcing edge-to-edge geodetic number of some standard graphs.

Theorem 2.67. For a non-trivial tree $G = T$ of size $q \geq 2$, $f_{gee}(G) = 0$.

Proof: Since set of all end edges of $G$ is the unique edge-to-edge geodetic set of $G$, the result follows from Theorem 2.58(i).
Theorem 2.68. For an even cycle $C_p$ ($p \geq 4$), a set $S \subseteq E(G)$ is a minimum edge-to-edge geodetic set if and only if $S$ consists of a pair of eccentric edges.

Proof. Let $p = 2k$ and let $C_p : v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_1$ be the cycle. Then the edges $v_1v_2$ and $v_{k+1}v_{k+2}$ are eccentric edges. Let $S = \{v_1v_2, v_{k+1}v_{k+2}\}$. Clearly, $S$ is a minimum edge-to-edge geodetic set of $C_p$. Conversely, let $S$ be a minimum edge-to-edge geodetic set of $C_p$. Then $g_{ee}(C_p) = |S|$. Let $S'$ be any set of pair of eccentric edges of $C_p$. Then as in the first part of this theorem, $S'$ is a minimum edge-to-edge geodetic set of $C_p$. Hence $|S'| = |S|$. Let $S = \{uv, xy\}$. If $uv$ and $xy$ are not eccentric, then any edge that is not in $uv$ – $xy$ geodesic does not lie on the $uv$ – $xy$ geodesic. Thus $S$ is not a minimum edge-to-edge geodetic set, which is a contradiction. \[\blacksquare\]

Theorem 2.69. For any cycle $C_p$, $f_{ee}(C_p) = \begin{cases} 1 & \text{if } p \text{ is even} \\ 2 & \text{if } p \text{ is odd} \end{cases}$

Proof. If $p$ is even, then by Theorem 2.68, every minimum edge-to-edge geodetic set of $C_p$ consists of pair of eccentric edges. Hence $C_p$ has $p/2$ independent minimum edge-to-edge geodetic sets and it is clear that each singleton set is the minimum forcing set for exactly one minimum edge-to-edge geodetic set of $C_p$. Hence it follows from Theorem 2.68 that $f_{ee}(C_p) = 1$.

Let $p$ be odd. Let $p = 2n+1$. Let the cycle be $C_p : v_1, v_2, v_3, ..., v_{2n+1}, v_1$. If $S = \{uv, xy\}$ is any set of two edges of $C_p$, then no edge of the $uv$ – $xy$ longest path lies on the $uv$ – $xy$ geodesic in $C_p$ and so no two element subset of $C_p$ is an edge-to-edge geodetic set of $C_p$. Now, it clear that the sets $S_1 = \{v_1v_2, v_{n+1}v_{n+2}, v_{2n+1}v_1\}$, $S_2 = \{v_1v_2, v_{n+1}v_{n+2}, v_{2n+1}v_1\}$, $S_3 = \{v_2v_3, v_{n+2}v_{n+3}, v_{2n+1}v_1\}$, $S_4 = \{v_2v_3, v_{n+2}v_{n+3}, v_{2n+1}v_1\}$, $S_{2n} = \{v_nv_{n+1}, v_{2n}v_{2n+1}, v_{n-1}v_n\}$, $S_{2n+1} = \{v_{n+1}v_{n+2}, v_{2n+1}v_1, v_{n-1}v_n\}$ are the minimum edge-to-edge geodetic sets.
of \( C_\text{p} \). (Note that there are more minimum edge-to-edge geodetic sets of \( C_\text{p} \), for example \( S’ = \{ v_{n+2}, v_{n+3}, v_1v_2, v_nv_{n+1} \} \) is a minimum edge-to-edge geodetic set different from these). It is clear from the minimum edge-to-edge geodetic sets \( S_i (1 \leq i \leq 2n+1) \) that each \( \{ v_iv_{i+1} \} (1 \leq i \leq 2n) \) and \( v_{2n+1}v_1 \) is a subset of more than one minimum edge-to-edge geodetic set \( S_i (1 \leq i \leq 2n+1) \). Hence it follows from Theorem 2.68 that \( f_{\text{gee}}(C_\text{p}) \geq 2 \). Since \( S_1 \) is the unique minimum edge-to-edge geodetic set containing \( T = \{ v_{n+1}v_{n+2}, v_{2n}v_{2n+1} \} \), it follows that \( f_{\text{gee}}(S_1) = 2 \). Thus \( f_{\text{gee}}(C_\text{p}) = 2 \). □

**Theorem 2.70.** For the complete graph \( G = K_\text{p} (p \geq 4) \) with \( p \) even, \( f_{\text{gee}}(G) = \frac{p-2}{2} \).

**Proof.** Let \( S \) be a minimum edge-to-edge geodetic set of \( G \) such that \( |S| = p/2 \). Then by Theorem 2.18, every element of \( S \) is independent. We show that \( f_{\text{gee}}(G) = \frac{p-1}{2} \).

Suppose that \( f_{\text{gee}}(G) \leq \frac{p}{2} - 2 \). Then there exists a forcing subset \( T \) of \( S \) such that \( S \) is the unique minimum edge-to-edge geodetic set of \( G \) containing \( T \) and \( |T| \leq \frac{p}{2} - 2 \).

Hence there exists at least two edges \( u\mu_j, u\mu_m \in S \) such that \( u\mu_j, u\mu_m \notin T \) and \( i \neq l, j \neq m \). Then \( S_1 = S - \{ u\mu_j, u\mu_m \} \cup \{ u\mu_m, u\mu_l \} \) is a set of \( p/2 \) independent edges of \( G \) containing \( T \). By Theorem 2.16, \( S_1 \) is a minimum edge-to-edge geodetic set of \( G \) which is a contradiction to \( T \) is a forcing subset of \( S \). Hence \( f_{\text{gee}}(G) = \frac{p}{2} - 1 = \frac{p-2}{2} \).

**Theorem 2.71.** For the complete graph \( G = K_\text{p} (p \geq 5) \) with \( p \) odd, \( f_{\text{gee}}(G) = \frac{p-1}{2} \).
Proof. Let $S$ be a minimum edge-to-edge geodetic set of $G$. Then by Theorem 2.20, $S = S_1 \cup S_2$, where $S_1$ consists of $\frac{p-3}{2}$ independent edges and $S_2$ consists of two adjacent edges and $|S| = \frac{p+1}{2}$. We show that $f_{gee}(G) = \frac{p+1}{2}-1$. Suppose that $f_{gee}(G) \leq \frac{p+1}{2}-2$. Then there exists a forcing subset $T$ of $S$ such that $S$ is the unique minimum edge-to-edge geodetic set of $G$ containing $T$ and $|T| \leq \frac{p+1}{2}-2$. Hence there exists at least two edges $x, y \in S$ such that $x, y \notin T$. Let us assume that $S_2 = \{x, y\}$. Suppose that $x, y \in S_1$. Then $x = v_i v_j$ and $y = v_l v_m$ such that $i \neq l, j \neq m$. Now, $S_3 = S - \{x, y\} \cup \{v_i v_m, v_j v_l\}$ consists of $\frac{p-3}{2}$ independent edges and two adjacent edges of $G$ containing $T$. By Theorem 2.18, $S_3$ is a minimum edge-to-edge geodetic set of $G$ containing $T$, which is a contradiction to $T$ is a forcing subset of $G$. Suppose that $x, y \in S_2$. Let $x = v_i v_s$ and $y = v_r v_k$. Let $v_i v_j$ be an edge of $S_1$. Now, join the vertices $v_i, v_j$ and $v_s, v_k$. Now $S_4 = S_1 - \{v_i v_j\} \cup \{v_i v_k\} \cup \{v_i v_j, v_s v_k\}$ consists of $\frac{p-3}{2}$ independent edges and two adjacent edges of $G$. By Theorem 2.18, $S_4$ is a minimum edge-to-edge geodetic set of $G$ containing $T$, which is a contradiction. Suppose that $x \in S_1$ and $y \in S_2$. Let $x = v_i v_j$ and $y = v_i v_s$. $S_5 = S_1 - \{v_i v_j\} \cup \{v_i v_j\} \cup \{v_i v_s, v_i v_j\}$ consists of $\frac{p-3}{2}$ independent edges and two adjacent edges of $G$ containing $T$. By Theorem 2.18, $S_5$ is a minimum edge-to-vertex geodetic set of $G$, which is a contradiction to that $T$ is a forcing subset of $G$. Hence $f_{gee}(G) = \frac{p+1}{2}-1 = \frac{p-1}{2}$. 

Theorem 2.72 For the complete bipartite graph $G = K_{n,n}$ $(n \geq 2)$, $f_{gee}(G) = n - 1$. 

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**Proof.** The proof is similar to the proof of Theorem 2.71.

**Theorem 2.73.** For the complete bipartite graph $G = K_{m,n}$ $(2 \leq m < n)$, $f_{ge}(G) = n - 1$.

**Proof.** The proof is similar to the proof of Theorem 2.71.

We have the following realization theorem.

**Theorem 2.74.** For every pair $a, b$ of integers with $0 \leq a < b$ and $b > a + 1$, there exists a connected graph $G$ such that $f_{gee}(G) = a$ and $g_{ee}(G) = b$.

**Proof.** Suppose $a = 0$. Let $G = K_{1,b}$. Then by Theorem 2.67, $f_{gee}(G) = 0$ and from Corollary 2.17, $g_{ee}(G) = b$. Suppose that $b = a + 1$. Let $G = K_{2,b}$. Then by Theorem 2.24, $g_{ee}(G) = b$ and from Theorem 2.72, $f_{gee}(G) = b - 1 = a$. Thus, we assume that $0 < a < b$. Let $P: x, y, z$. Let $G$ be the graph obtained from $P$ by adding new vertices $z_1, z_2, \ldots, z_{b-a-1}, w_1, w_2, \ldots, w_a$ by joining each $z_i$ $(1 \leq i \leq a)$ with $z$ and joining each $w_i$ $(1 \leq i \leq a)$ with $y$ and $z$. The graph $G$ is given in Figure 2.14.

![Figure 2.14](image-url)
Let \( Z = \{zz_1, zz_2, \ldots, zz_{a-1}, xy\} \) be the set of all end-edges of \( G \). By Corollary 2.3, \( Z \) is a subset of every edge-to-edge geodetic set of \( G \). Let \( H_i = \{h_i, k_i\} \) (1 \( \leq \) \( i \) \( \leq \) \( a \)), where \( h_i = zw_i \) and \( k_i = yw_i \). First we show that \( g_{ee}(G) = b \). By Theorem 2.11, every edge-to-edge geodetic set of \( G \) must contain at least one vertex from \( H_i \) (1 \( \leq \) \( i \) \( \leq \) \( a \)). Thus \( g_{ee}(G) \geq b - a + a = b \). On the other hand, since the set \( S = Z \cup \{h_1, h_2, \ldots, h_a\} \) is a minimum edge-to-edge geodetic set of \( G \), it follows that \( g_{ee}(G) \leq |S| = b \). Thus \( g_{ee}(G) = b \). Next we show that \( f_{gee}(G) = a \). Since every \( g_{ee} \)-set of \( G \) contains \( Z \), it follows from Theorem 2.64 that \( f_{gee}(G) \leq g_{ee}(G) - |Z| = b - (b - a) = a \).

Now, since \( g_{ee}(G) = b \) and every minimum edge-to-edge geodetic set of \( G \) contains \( S \), it is easily seen that every minimum edge-to-edge geodetic set \( W \) is of the form \( W \cup \{e_1, e_2, \ldots, e_a\} \), where \( e_i \in H_i \) (1 \( \leq \) \( i \) \( \leq \) \( a \)). Let \( T \) be any proper subset of \( S \) with \( |T| < a \). Then there exists an edge \( e_j \) (1 \( \leq \) \( j \) \( \leq \) \( a \)) such that \( e_j \notin T \). Let \( f_j \) be an edge of \( H_j \) distinct from \( e_j \). Then \( W_j = (S - \{e_j\}) \cup \{f_j\} \) is a \( g_{ee} \)-set properly containing \( T \). Thus \( W \) is not the unique \( g_{ee} \)-set containing \( T \). Thus \( T \) is not a forcing subset of \( S \). This is true for all minimum edge-to-edge geodetic sets of \( G \) and so it follows that \( f_{gee}(G) = a \). \( \blacksquare \)
CHAPTER 3

THE EDGE-TO-EDGE DETOUR NUMBER OF A GRAPH

In this chapter we introduce the edge-to-edge detour number $d_{ee}(G)$ of a connected graph with at least 3 vertices and study some of its general properties. We also determine the edge-to-edge detour number of certain classes of graphs. For each pair of integers $k$ and $q$ with $2 \leq k \leq q$, there exists a connected graph $G$ of order $q + 1$ and size $q$ with $d_{ee}(G) = k$. For each triple $d$, $k$, $q$ of integers with $2 \leq k \leq q - d + 2$, $d \geq 4$ and $q - d - k + 1 > 0$, there exists a connected graph $G$ of size $q$ with $diam(G) = D$ and $d_{ee}(G) = k$. For positive integers $R$, $D$ and $l \geq 2$ with $R \leq D < 2R$ there exists a connected graph $G$ with $rad(G) = R$, $diam(G) = D$ and $d_{ee}(G) = l$. Connected graphs of size $q \geq 4$ with edge-to-edge detour number $q$ or $q - 1$ are characterized. The upper edge-to-edge detour number $d_{ee}^+(G)$ of a graph is studied and is determined for certain classes of graphs. It is shown that, for every pair $a$, $b$ of integers with $2 \leq a \leq b$, there exists a connected graph $G$ such that $d_{ee}(G) = a$ and $d_{ee}^+(G) = b$. The forcing fixing edge-to-edge detour number $f_{d_{ee}}(G)$ of a graph is studied and is determined for certain classes of graphs. It is shown that, for every pair $a$, $b$ of integers with $0 \leq a < b$, there exists a connected graph $G$ such that $f_{d_{ee}}(G) = a$ and $d_{ee}(G) = b$. 
Definition 3.1. Let $G = (V, E)$ be a connected graph with at least two edges. For subsets $A$ and $B$ of $V(G)$, the detour distance $D(A, B)$ is defined as $D(A, B) = \max \{ D(x, y) : x \in A, y \in B \}$. A $u - v$ path of length $D(A, B)$ is called an $A - B$ detour joining the sets $A, B$, where $u \in A$ and $v \in B$. An edge $e$ is said to lie on an $A - B$ detour if $e$ is an edge of an $A - B$ detour. For $A = \{u, v\}$ and $B = \{z, w\}$ with $uv$ and $zw$ edges, we write an $A - B$ detour as $uv - zw$ detour and $D(A, B)$ as $D(uv, zw)$.

Example 3.2. For the graph $G$ given in Figure 3.1 with $A = \{v_4, v_5\}$ and $B = \{v_1, v_6\}$, the path $P : v_5, v_4, v_3, v_2, v_1, v_6$ is the only one $A - B$ detour so that $D(A, B) = 5$.

![Figure 3.1](image)

Definition 3.3. Let $G = (V, E)$ be a connected graph with at least 2 edges. A set $S \subseteq E$ is called an edge-to-edge detour set of $G$ if every edge of $G$ is an element of $S$ or lies on a detour joining a pair of edges of $S$. The edge-to-edge detour number $d_{ee}(G)$ of $G$ is the minimum cardinality of its edge-to-edge detour sets and any edge-to-edge detour set of cardinality $d_{ee}(G)$ is said to be a $d_{ee}$-set of $G$. 
Example 3.4. For the graph $G$ given in Figure 3.2, the two $v_1v_5 - v_3v_4$ detours are $P: v_1, v_6, v_5, v_4, v_3, v_2$; $Q: v_2, v_3, v_4, v_5, v_7, v_1$ with each of length 5 so that $D(v_1v_5, v_3v_4) = 5$. Since all the edges of $G$ lie on $v_1v_2 - v_2v_3$ detour, $S = \{v_1v_2, v_2v_3\}$ is an edge-to-edge detour set of $G$ so that $d_{ee}(G) = 2$.

![Figure 3.2](image)

Example 3.5. For the graph $G$ given in Figure 3.2, $S_1 = \{v_1v_2, v_3v_4\}$ is another $d_{ee}$-set of $G$. Thus there can be more than one $d_{ee}$-set of $G$.

Theorem 3.6. For a connected graph $G$ of size $q \geq 2$, $2 \leq d_{ee}(G) \leq q$.

Proof. A $d_{ee}$-set needs at least two edges and therefore $d_{ee}(G) \geq 2$. Also, the set of all edges of $G$ is an edge-to-edge detour set of $G$ so that $d_{ee}(G) \leq q$. Thus $2 \leq d_{ee}(G) \leq q$. □

Remark 3.7. The bounds in Theorem 3.6 are sharp. For the star $G = K_{1,q} (q \geq 2)$, it is clear that the set of all edges is the unique edge-to-edge detour set of $G$ so that $d_{ee}(G) = q$. The set of two end-edges of a path $P$ of length at least two is its unique minimum edge-to-edge detour so that $d_{ee}(P) = 2$. Thus the star $K_{1,q}$ has $E(K_{1,q})$ as the only possible edge-to-edge detour set and the detour number is $q$. If $G$ is a path of
length 2, the detour number is 2. Also the bounds in Theorem 3.6 is strict for the graph \( G \) given in the Figure 3.2, \( d_{ee}(G) = 3 \), \( q = 7 \). Thus \( 2 < d_{ee}(G) < q \).

**Definition 3.8.** An edge \( e \) in a graph \( G \) is an edge-to-edge detour edge if \( e \) belongs to every minimum edge-to-edge detour set of \( G \). If \( G \) has a unique minimum edge-to-edge detour set \( S \), then every edge in \( S \) is an edge-to-edge detour edge.

**Example 3.9.** For the graph \( G \) in Figure 3.3 \( v_2v_5 \) is the detour edge which is in every minimum edge-to-edge detour set.

**Remark 3.10.** Every edge-to-edge detour edges are not the end edges of \( G \).

**Theorem 3.11.** Every end-edge of a nontrivial connected graph \( G \) belongs to every edge-to-edge detour set of \( G \). Moreover if the set \( S \) of all end edges of \( G \) is a edge-to-edge detour set, then \( S \) is the unique minimum edge-to-edge detour set for \( G \).
**Proof:** Let $G$ be a connected nontrivial graph. Since every end edge $e$ in $G$ is either the initial edge or the terminal edge of a detour, it follows that $e$ belongs to every edge-to-edge detour set of $G$. Thus $d_{ee}(G) \leq |S|$. So $d_{ee}(G) = |S|$ and $S$ is the unique minimum edge-to-edge detour set for $G$.

![Figure 3.5](image)

**Theorem 3.12** Let $G$ be a connected graph with cut edges and let $S$ be an edge-to-edge detour set of $G$. Then every branch of $G$ contains an element of $S$, if $S$ is a minimum detour set then no cut-vertex of $G$ belongs to $S$.

**Proof.** Assume that there is a branch $B$ of $G$ at a cut-vertex $v$ such that $B$ contains no element of $S$. Then by Theorem 3.11, $B$ does not contain any end-edge of $G$. Hence it follows that no vertex of $B$ is an end vertex of $G$. Let $z = ux$ be any edge of $B$ such that $v \neq u$ and $v \neq x$ (such a vertex exists since $|V(B)| \geq 2$). Then $z$ is not an edge of $S$ and so $z$ lies on an $e-f$ detour $P : u_1, u_2, \ldots, u, x, \ldots, u_i$, where $u_1$ is an end of $e$ and $u_i$ is an end of $f$ with $e, f \in S$. Since $v$ is a cut-vertex of $G$, the $u_1 - u$ and $u - u_i$ subpaths of $P$ both contain $v$ and so $P$ is not a path, which is a contradiction. Hence every branch of $G$ contains an element of $S$. 
Corollary 3.13. For any non-trivial tree $T$ with $k$ end-vertices, $d_{ee}(G) = k$ and the set of all $k$ end-edges of $T$ is the unique minimum edge-to-edge detour set of $T$.

Proof: Let $S$ be the set of all end edges of $T$. Then by Theorem 3.11, $S$ is a subset of every edge-to-edge detour set of $T$. Hence $d_{ee}(T) \geq k$. Now $S$ is an edge-to-edge detour set of $G$ so that $d_{ee}(T) = k$. □

Theorem 3.14. For the cycle $C_p$ ($p \geq 4$), $d_{ee}(C_p) = 2$.

Proof: Let $C_p : v_1, v_2, v_k, v_{k+1}, \ldots, v_p$ be the cycle. The set $S = \{v_1v_2, v_2v_3\}$ is an edge-edge detour set of $G$ so that $d_{ee}(C_p) = 2$. □

Theorem 3.15. For the graph $K_p$, $p \geq 2$, $d_{ee}(K_p) = 2$.

Proof: Let $K_p : v_1, v_2, v_k, v_{k+1}, \ldots, v_p$ be the complete graph. Since every edge of $G$ lies on $v_1v_2- v_3v_4$ detour, $S = \{v_1v_2, v_3v_4\}$ is an edge-to-edge detour set of $G$ so that $d_{ee}(K_p) = 2$. □

Theorem 3.16. Let $G$ be a connected graph with diameter $D$ and size $q \geq 2$. Then $d_{ee}(G) \leq q - D + 2$.

Proof. Since $q \geq 2$, we have $D \geq 2$. Let $u$ and $v$ be two vertices of $G$ such that $D(u, v) = D \geq 2$. Let $P: u = v_0, v_1, v_2, \ldots, v_{D-1}, v_D = v$ be a detour diametral path. Let $S = E(G) - \{v_1v_2, v_2v_3, \ldots, v_{D-2}v_{D-1}\}$. Then the edge $v_i v_j$ ($1 \leq i \leq j \leq D - 1$) lies on the $v_0v_1- v_{D-1}v_D$ detour so that $S$ is an edge-edge detour set of $G$. Hence $d_{ee}(G) \leq q - D + 2$. □

Theorem 3.17. For every non-trivial tree $T$, $d_{ee}(G) = q - D + 2$ if and only if $T$ is a caterpillar.
Proof. Let $T$ be any non-trivial tree. Let $D = D(u, v)$ and let $P : u = v_0, v_1, \ldots v_{D-1}, v_D = v$ be a detour diametral path. Let $k$ be the number of end-edges of $T$ and $l$ be the number of internal edges of $T$ other than $v_1v_2, v_2v_3, \ldots, v_{D-2}v_{D-1}$. Then $D - 2 + l + k = q$. By Corollary 3.13, $d_{ee}(T) = k$ and so $d_{ee}(T) = q - D + 2 - l$. Hence $d_{ee}(T) = q - D + 2$ if and only if $l = 0$, if and only if all the internal edges of $T$ lie on the detour diametral path $P$, if and only if $T$ is a caterpillar.

Theorem 3.18. For each triple $D, R, l$ of integers with $R \leq D < 2R$, there is a connected graph $G$ of detour radius $R$, detour diameter $D$, and detour number $l$.

Proof. When $R = 1$, we let $G = K_{1,l}$. Then the result follows from Corollary 3.13. Let $R \geq 2$. Let $C_{R+1} : v_1, v_2, \ldots, v_{R+1}$ be a cycle of length $R + 1$ and let $P_{D,R} : u_0, u_1, u_2, \ldots, u_{D,R}$ be a path of length $D - R$. Let $H$ be a graph obtained from $C_{R+1}$ and $P_{D,R}$ by identifying $v_1$ in $C_{R+2}$ and $u_0$ in $P_{D,R}$. Now add $l - 2$ new vertices $w_1, w_2, \ldots, w_{l-2}$ to $H$ and join each $w_i (1 \leq i < l - 2)$ to the vertex $u_{D,R-1}$ and obtain the graph $G$ as shown in Figure 3.6. Then $\text{rad}_D(G) = R$ and $\text{diam}_D(G) = D$. Let $S = \{ u_{D,R-1}u_{D,R}, u_{D,R-1}w_1, u_{D,R-1}w_2, \ldots, u_{D,R-1}w_{l-2} \}$ be the set of end-edges of $G$. By Theorem 3.11, $S$ is a subset of every edge-to-edge detour set of $G$. It is clear that $S$ is not a edge-to-edge detour set of $G$ and so $d_{ee}(G) \geq 1$. However $S \cup \{ v_1, v_2 \}$ is a edge-to-edge detour set of $G$ and so that $d_{ee}(G) = l$. ■
Theorem 3.19. Let $G$ be a connected graph with $q \geq 4$, which is not a cycle and not a tree and let $C(G)$ be the length of the longest cycle. Then $d_{ee}(G) \leq q - C(G) + 1.$

Proof. Let $C(G)$ denote the length of the longest cycle in $G$. Let $C = v_1,v_2,v_3,\ldots,v_k$. Since $G$ is not a cycle, there exists a vertex $v$ in $G$, such that $v$ is not on $C$ and $v$ is adjacent to $v$, say. Then $S = E(G) - \{vv_1\}$ is an edge-to-edge detour set of $G$. Hence $d_{ee}(G) \leq q - e(G) + 1.$

Theorem 3.20. Let $G$ be a connected graph with size $q \geq 3$. Then $d_{ee}(G) = q$ if and only if $G = K_{1,q}$.

Proof. If $G = K_{1,q}$ then the result follows from Corollary 3.13 that $d_{ee} = q$. Conversely let $d_{ee}(G)=q$. By Theorem 3.16, $D \leq 2$. If $D = 1$, then $G = K_2$, which is a contradiction to $q \geq 3$. Suppose that $D = 2$. If $G$ is a tree, then $G = K_{1,q}$ we have done. If $G$ is not a tree, then since $D = 2$, $G$ is not a cycle. Hence by Theorem 3.20, $d_{ee}(G) \leq q - 2$. Since $C(G) \geq 3$, we get a contradiction. Hence $G = K_{1,q}.$